# Continuity in type theory

Martín Hötzel Escardó

University of Birmingham, UK

Semantics of proofs and programs, IHP, Paris, 12 June 2014

# A wish that can't be fulfilled literally

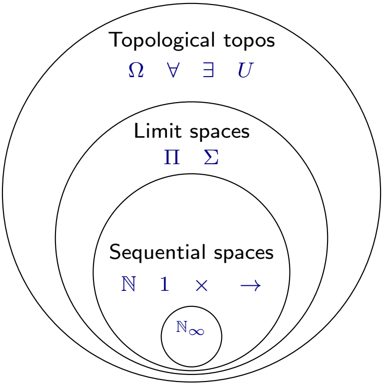
1. Types are interpreted as topological spaces.
2. Terms are interpreted as points of spaces.
3. Functions are interpreted as continuous maps.

The category of continuous maps of topological spaces is not even cartesian closed (it doesn't have exponentials (function spaces)).

Hence it can't interpret Gödel's system T or Martin-Löf type theory.

However, there are natural continuous models of type theory.

# Johnstone's topological topos (1979)




1. The site is the category of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ with the canonical coverage.
2. Taking colimits of $\mathbb{N}_\infty$ in topological spaces gives sequential spaces.
3. The limit spaces arise as the subobjects of sequential spaces.

# Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.

# Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.

# Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.
3. The interpretation of the simple types gives the Kleene–Kreisel continuous functionals. (Start from $\mathbb{N}$ and close under $\to$.)

# Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.
3. The interpretation of the simple types gives the Kleene–Kreisel continuous functionals. (Start from $\mathbb{N}$ and close under $\to$.)
4. The interpretation of the type

$$\mathbb{N}_\infty \stackrel{\mathsf{def}}{=} \left( \sum_{\alpha : \mathbb{N} \to 2} \quad \prod_{n : \mathbb{N}} \quad \alpha_n = 0 \to \alpha_{n+1} = 0 \right)$$

gives the one-point compactification of $\mathbb{N}$, with $\infty \stackrel{\mathsf{def}}{=} (\lambda i.1, -)$.

Here "$=$" is the identity type, interpreted as an equalizer.

# Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.
3. The interpretation of the simple types gives the Kleene–Kreisel continuous functionals. (Start from $\mathbb{N}$ and close under $\to$.)
4. The interpretation of the type

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \left( \sum_{\alpha : \mathbb{N} \to 2} \prod_{n : \mathbb{N}} \alpha_n = 0 \to \alpha_{n+1} = 0 \right)$$

   gives the one-point compactification of $\mathbb{N}$, with $\infty \stackrel{\text{def}}{=} (\lambda i.1, -)$.

   Here "$=$" is the identity type, interpreted as an equalizer.
5. The interpretation of the type

$$\sum_{x : \mathbb{N}_\infty} 2^{x = \infty}$$

   is a $T_1$, non-Hausdorff, but compact, space with two points at infinity,

$$\infty_0 \stackrel{\text{def}}{=} (\infty, \lambda p.0), \quad \infty_1 \stackrel{\text{def}}{=} (\infty, \lambda p.1).$$

# The topological topos validates continuity axioms

### Continuity axiom (Cont)

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\forall f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha : \mathbb{N}^{\mathbb{N}}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

### Uniform continuity axiom (UC)

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n : \mathbb{N}.\ \forall \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

# The topological topos validates continuity axioms

Continuity axiom (Cont)

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\forall f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha : \mathbb{N}^{\mathbb{N}}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Uniform continuity axiom (UC)

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n : \mathbb{N}.\ \forall \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

- This assumes a classical meta-theory.
- Towards the end I discuss another topological topos developed within a constructive meta-theory by Chuangjie Xu and myself.

  (Also formalized in Agda by Chuangjie.)
- For the moment ignore constructivity issues until further notice.

## Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has $\Pi$ and $\Sigma$.

If we apply the BHKCH interpretation:

Continuity axiom (Cont):

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$\Pi f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \Pi \alpha : \mathbb{N}^{\mathbb{N}}.\ \Sigma n : \mathbb{N}.\ \Pi \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \to f\alpha = f\beta.$

Uniform continuity axiom (UC):

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$

## Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has $\Pi$ and $\Sigma$.

If we apply the BHKCH interpretation:

Continuity axiom (Cont): $\times$

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$\Pi f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \Pi\alpha : \mathbb{N}^{\mathbb{N}}.\ \Sigma n : \mathbb{N}.\ \Pi\beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \to f\alpha = f\beta.$

Uniform continuity axiom (UC):

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi\alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$

## Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has $\Pi$ and $\Sigma$.

If we apply the BHKCH interpretation:

Continuity axiom (Cont): × Moreover, no topos can validate this.

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$\Pi f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \Pi \alpha : \mathbb{N}^{\mathbb{N}}.\ \Sigma n : \mathbb{N}.\ \Pi \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \to f\alpha = f\beta.$

Uniform continuity axiom (UC):

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$

## Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has $\Pi$ and $\Sigma$.

If we apply the BHKCH interpretation:

Continuity axiom (Cont): ✗ Moreover, no topos can validate this.

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$\Pi f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \Pi \alpha : \mathbb{N}^{\mathbb{N}}.\ \Sigma n : \mathbb{N}.\ \Pi \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \to f\alpha = f\beta.$

Uniform continuity axiom (UC): ✓

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$

# Theorem of intensional Martin-Löf type theory

If all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous then $0 = 1$.

$$\left( \prod_{f \,:\, \mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \; \sum_{n : \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \; \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

# Theorem of intensional Martin-Löf type theory

$$\left( \prod_{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \sum_{n : \mathbb{N}} \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

I could instead say "not all functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous", but:

1. This would give the false impression that there might exist a non-continuous function to be found by looking hard enough.
   (In the topological topos all functions are continuous, and yet this holds.)

2. It is $0 = 1$ that our proof actually does give from the assumption.
   (A technicality that leads to the next item.)

3. We would need a universe to map the type $0 = 1$ to the type $\emptyset$, and our proof doesn't require universes.
   (So we are more general.)

# Theorem of intensional Martin-Löf type theory

$$\left( \prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \; \sum_{n : \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

Proof sketch. Let

$$\phi : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \; \sum_{n : \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta.$$

Using $\phi$ and the projections and choosing $\alpha = 0^\omega$, we get

$$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$$

and

$$\gamma : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} 0^\omega =_{Mf} \beta \to f0^\omega = f\beta.$$

Now define $m = M(\lambda\alpha.0)$ and consider

$$f\beta = M(\lambda\alpha.\beta(\alpha_m)).$$

Then argue $Mf = 0$ and $Mf > 0$ give $0 = 1$, using $f0^\omega = m$.
(Induction on $Mf$ not needed). Q.E.D.

# Proof discussion

This is an adaptation of a well known argument (due to Kreisel?).

1. Continuity, choice and extensionality are together impossible.

2. No *extensional* modulus-of-continuity functional $M$.

3. But here we are working in *intensional* Martin-Löf type theory.

4. No *continuous* modulus-of-continuity functional $M$.

5. We used our hypothetical $M$ to define a non-continuous function $f$ and hence prove $M$ wrong.

6. And this is exactly what is happening in the topological topos:
   - All functions are continuous.
   - But there is no continuous way of finding moduli of continuity.
   - No finite amount of information about $f$ suffices to determine its modulus.

# $\Sigma$ versus $\exists$

Fix an object $X$.

1. $\Sigma$ is understood in slices $\mathcal{E}/X$.
   If we have an object classifier $U$ (universe), we can understand it as

   $$\Sigma : (X \to U) \to U.$$

   Given a family of objects we get an object.

2. $\exists$ is understood as a function

   $$\exists : (X \to \Omega) \to \Omega.$$

3. They are related via a reflection of $U$ into $\Omega$:

   $$U \underset{\hookleftarrow}{\xrightarrow{\|-\|}} \Omega.$$

   $$(\exists x : X.P(x)) = \|\Sigma x : X.P(x)\|.$$

   (Used in Homotopy Type Theory to define $\exists$ from $\Sigma$.)

# Continuity in type theory extended with $\|-\|$

Add a universal map $|-| : X \to \|X\|$ into types with at most one element.

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type $P$ with at most one element.

(We are quotienting $X$ by the relation that identifies any two points.)

$$\prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \left\| \sum_{n : \mathbb{N}} \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right\|.$$

- In a sheaf topos, this means we can find $n$ locally but not globally.
- In a realizability topos, we can find $n$ intensionally but not extensionally.
- In other toposes this of course acquires other meanings.
- In type theory, it seems difficult to give a direct *meaning-explanation*.

# Another well-known example

If you try to say that $f : X \to Y$ is a surjection by saying

$$\prod_{y:Y} \sum_{x:X} fx = y,$$

you are actually saying that $f$ has a section $Y \to X$.

You should instead say

$$\prod_{y:Y} \left\| \sum_{x:X} fx = y \right\|.$$

A similar distinction arises in the definition of the image of a function, and many other definitions and theorems and proofs.

# Disclosing secrets

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type $P$ with at most one element.

We can disclose a secret $\|X\|$ to $P$ provided we have a map $X \to P$.

Example. If $A(n)$ is decidable then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Proof sketch. If we have any $n$ with $A(n)$, we can find the minimal $n$, using the decidability of $A(n)$, but "having a minimal $n$ such that $A(n)$" is a type with at most one element.

# More general lemma

From now on everything in the talk is joint work with Chuangjie Xu.

Assume that $A(n)$ has at most one element for every $n : \mathbb{N}$.

If for any given $n$ we have that $A(n)$ implies that $A(m)$ is decidable for all $m < n$, then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

# Theorem of MLTT extended with $\| - \|$

$$\Pi f \colon 2^{\mathbb{N}} \to \mathbb{N}.\ \big\|\Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big\| \\ \to \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Proof. Set $A(n) = \big(\Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big)$ in the lemma.

Corollary. The topological topos validates the uniform-continuity axiom

$$\Pi f \colon 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Because the premise of the theorem is validated.

(In the topological topos, the theorem can be seen as getting global existence from local existence by compactness.)

# Getting constructive

1. Kleene–Kreisel functionals constructively.
2. Another topological topos for that.
3. If all functions $2^{\mathbb{N}} \to \mathbb{N}$ are continuous, then the Kleene–Kreisel hierarchy agrees with the full-type hierarchy.
4. A model of type theory that constructively validates the uniform-continuity axiom.
5. Implemented in Agda.

# Kleene–Kreisel continuous functionals

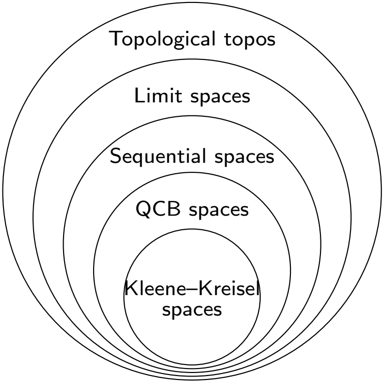
Identified in the 1950's as

- Kleene's countable functionals.
- Kreisel's continuous functionals.

Start from $\mathbb{N}$ and close under exponentiation.

This is automatically closed under finite products, excluding the empty product $1$.

Fully abstract model of Gödel's system $T$.
(By the Kleene-Kreisel density theorem.)



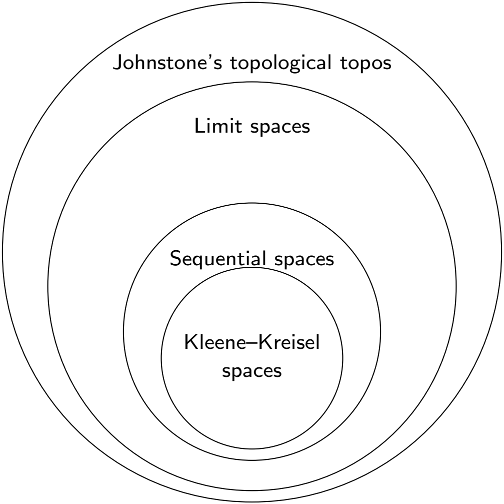

The set-theoretical full type hierarchy is not fully abstract (Kreisel).

# Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.

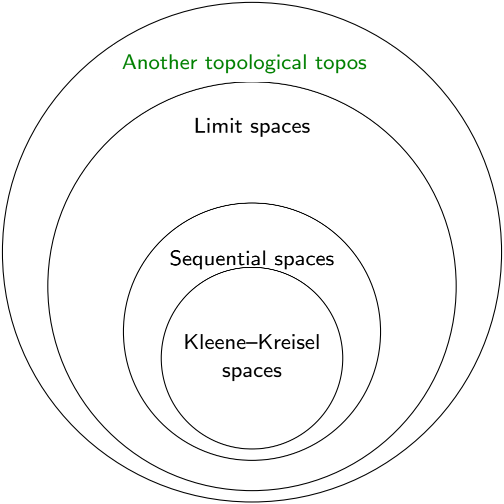


# Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.

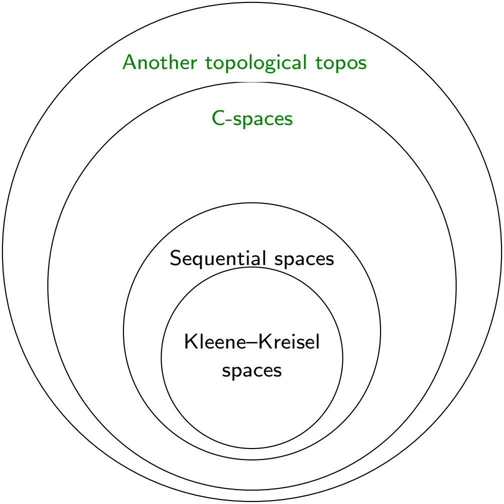


# Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.

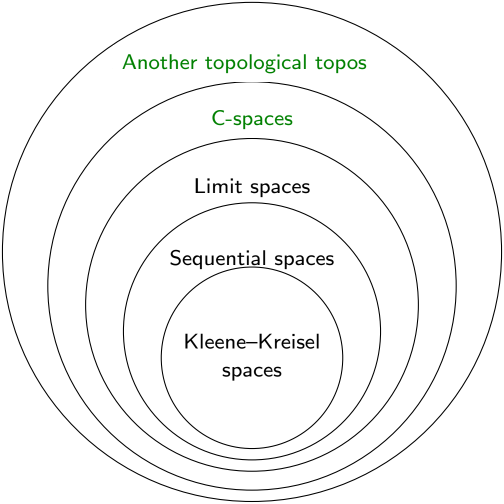


# Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.



# The uniform-continuity coverage

1. Let $2^n$ denote the set of binary strings of length $n$.
2. For $s \in 2^n$, let $\mathrm{cons}_s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the concatenation map
$$\mathrm{cons}_s(\alpha) = s\alpha.$$
3. A function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is uniformly continuous iff
$$\forall m : \mathbb{N}.\, \exists n : \mathbb{N}.\, \forall s : 2^n.\, \exists f' : 2^{\mathbb{N}} \to 2^{\mathbb{N}}.\, \exists s' : 2^m.\, f \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ f'.$$
4. This shows that the countable collection $\{(\mathrm{cons}_s)_{s:2^n} \mid n : \mathbb{N}\}$ satisfies the coverage axiom.
5. This coverage is subcanonical.
6. Moreover, crucially: $y\left(2^{\mathbb{N}}\right)$ has the universal property of the exponential $2^{\mathbb{N}}$ in the resulting topos, where of course $2$ is $1+1$ and $\mathbb{N}$ is the natural numbers object of the topos.

# What we get

1. A constructive treatment of sheaves and $\mathrm{C}$-spaces suitable for development in Martin–Löf type theory.

   Definitions, theorems and proofs implemented in Agda.

   We don't need $\| - \|$.

   We need $\neg\neg$(function extensionality).

2. $\mathrm{C}$-Spaces give a constructive model of dependent types with the uniform continuity axiom.

   At the moment we haven't modelled the universe.

   The amalgamation property for the "naive" version of the Hofmann–Streicher universe holds only up to isomorphism.

   We want to avoid sheafification.

3. If we assume that all functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then we can show constructively that the full type hierarchy is equivalent to the Kleene–Kreisel continuous hierarchy.

# End