Constructive decidability of classical continuity

Martín H. Escardó

Version of January 9, 2013

Abstract

We show that the following instance of the principle of excluded middle holds: any function on the one-point compactification of the natural numbers with values on the natural numbers is either classically continuous or classically discontinuous. The proof doesn't require choice and can be understood in any of the usual varieties of constructive mathematics. Classical (dis)continuity is a weakening of the notion of (dis)continuity, where the existential quantifiers are replaced by negated universal quantifiers. We also show that the classical continuity of all functions is equivalent to the negation of WLPO. We use this to relate uniform continuity and searchability of the Cantor space.

1 Introduction

Brouwer understood that all functions are continuous in his intuitionistic mathematics [2, 11], but Bishop rejected Brouwer's analysis [4]. What does it take to accept continuity principles in constructive mathematics?

To identify the essence of the problem, we consider $\mathbb{N}$-valued functions on the simplest non-trivial space: the generic convergent sequence $\mathbb{N}_\infty$. This is the one-point compactification of the discrete set of natural numbers $\mathbb{N}$, which adds a new point ∞ as the limit of the sequence of points $n \in \mathbb{N}$. It can be constructed as the set of decreasing binary sequences, with the number n represented by the sequence consisting of n ones followed by infinitely many zeros, and with ∞ represented by the sequence consisting of infinitely many ones.

We consider discrete-valued functions $\mathbb{N}_\infty \to \mathbb{N}$, for which continuity and uniform continuity are known to coincide [7, Proposition 5.3]. A result of Ishihara's [10] implies [7, Lemma 6.3] that

any strongly extensional $f \colon \mathbb{N}_\infty \to \mathbb{N}$ *is continuous or discontinuous.*

The disjunction amounts to

$$\exists n \in \mathbb{N}\, \forall m \geq n(f(m) = f(\infty)) \vee \forall n \in \mathbb{N}\, \exists m \geq n(f(m) \neq f(\infty)).$$

Ishihara's proof, given in the context of Bishop mathematics, and for spaces more general than $\mathbb{N}_\infty$ and $\mathbb{N}$, relies on countable choice.

In this work we avoid the axiom of choice and remove the assumption of strong extensionality (that is, that the function reflects apartness). We avoid choice by exploiting the fact that the set $\mathbb{N}_\infty$ is searchable [7, Section 3] (see Section 2 below). To remove the assumption of strong extensionality, we need to weaken the notion of (dis)continuity. Define the *classical existential quantifier* by

$$\tilde{\exists}\, x \in X(P(x)) \iff \neg\forall x \in X(\neg P(x)).$$

A possible informal reading of this is that there "must" exist some $x \in X$ satisfying the property P, but we don't know how to construct any.

We say that a function $f\colon \mathbb{N}_\infty \to \mathbb{N}$ is

1. *classically continuous* iff $\tilde{\exists}\, m \in \mathbb{N}\, \forall n \geq m (f(n) = f(\infty))$,
2. *classically discontinuous* iff $\forall m \in \mathbb{N}\, \tilde{\exists}\, n \geq m (f(n) \neq f(\infty))$.

Notice that classical continuity is the negation of classical discontinuity, because a doubly negated equality in $\mathbb{N}$ is equivalent to an equality. We prove the following constructive instance of the principle of excluded middle:

> *Every function $\mathbb{N}_\infty \to \mathbb{N}$ is either classically continuous or classically discontinuous.*

We also show that:

> *The existence of some classically discontinuous function $\mathbb{N}_\infty \to \mathbb{N}$ is equivalent to* WLPO*.*

Recall WLPO (the weak limited principle of omniscience) asserts that it is decidable whether any given binary sequence is constantly one, which is equivalent to the assertion that $u = \infty$ is decidable for any $u \in \mathbb{N}_\infty$. The fact that classical continuity implies WLPO may be slightly surprising, because the notion of classical discontinuity is devoid of constructive content, but the conclusion of WLPO is a disjunction, which amounts to the decidability of the condition $u = \infty$ for $u \in \mathbb{N}_\infty$. The above two facts together give that the negation of WLPO can be understood as a continuity principle:

> *The classical continuity of all maps $\mathbb{N}_\infty \to \mathbb{N}$ is equivalent to* $\neg$WLPO*.*

We also show that classically continuous functions have moduli of continuity in $\mathbb{N}_\infty$:

> *If $f\colon \mathbb{N}_\infty \to \mathbb{N}$ is classically continuous, then there is $v \in \mathbb{N}_\infty$ distinct from ∞ such that $f(u) = f(\infty)$ for all $u \geq v$.*

We use this to conclude that if Markov's Principle (MP) holds, then classical continuity implies continuity, and hence to deduce the following version of the KLST Theorem [10]:

> *If* MP *and* $\neg$WLPO *hold, then all functions $\mathbb{N}_\infty \to \mathbb{N}$ are continuous.*

Further analysis of (classical and constructive and uniform) continuity is contained in the technical development.

Foundations. Our proofs can be understood in Bishop mathematics without any choice axiom, in Martin-Löf type theory (either assuming that all quantifications over functions implicitly refer to extensional functions, as in Bishop mathematics, or, more conveniently, assuming the propositional axiom of function extensionality), and indeed in any of the usual varieties of constructive mathematics [5], and also in topos theory.

Related work. Hannes Diener [6] has independently and before us established complementary results, published in the volume as this paper, in the generality of functions of metric spaces rather than functions $\mathbb{N}_\infty \to \mathbb{N}$, but using choice, and with methods of proof closer to those of Ishihara's [10] rather than the searchability of the set $\mathbb{N}_\infty$.

Acknowledgements. I benefited from fruitful discussions with Hannes Diener, Paulo Oliva and Chuangjie Xu, and from valuable feedback from the two anonymous referees.

2 Preliminaries

We avoid the axiom of choice by exploiting the fact, proved in [7, Theorem 3.5], that there is a functional

$$\varepsilon\colon (\mathbb{N}_\infty \to 2) \to \mathbb{N}_\infty,$$

called a *selection function* for the set $\mathbb{N}_\infty$, such that for all $p\colon \mathbb{N}_\infty \to 2$,

$$p(\varepsilon(p)) = 1 \iff \forall u \in \mathbb{N}_\infty (p(u) = 1). \tag{1}$$

Equivalently,

$$\exists u \in \mathbb{N}_\infty (p(u) = 0) \iff p(\varepsilon(p)) = 0. \tag{2}$$

We say that $\mathbb{N}_\infty$ is *searchable* [7, Section 2]. This implies that, for every $p\colon \mathbb{N}_\infty \to 2$,

$$\exists u \in \mathbb{N}_\infty (p(u) = 0) \vee \forall u \in \mathbb{N}_\infty (p(u) = 1),$$

which is the principle of omniscience for the set $\mathbb{N}_\infty$, and that

$$\tilde{\exists}\, u \in \mathbb{N}_\infty (p(u) = 0) \implies \exists u \in \mathbb{N}_\infty (p(u) = 0), \tag{3}$$

which is Markov's Principle for the set $\mathbb{N}_\infty$. Searchability also implies the existence of functionals

$$E, A\colon (\mathbb{N}_\infty \to 2) \to 2 \tag{4}$$

such that

$$\begin{aligned} E(p) = 0 &\iff \exists u \in \mathbb{N}_\infty (p(u) = 0), \\ A(p) = 0 &\iff \forall u \in \mathbb{N}_\infty (p(u) = 0), \end{aligned}$$

constructed as

$$E(p) = p(\varepsilon(p)), \qquad A(p) = 1 - E(1 - p).$$

The generic convergent sequence $\mathbb{N}_\infty$ can be sensibly constructed in many isomorphic ways that are equally good, for example as the set of binary sequences with at most one non-zero element. As discussed in the introduction, we adopt the following construction:

$$\mathbb{N}_\infty = \{u \in 2^{\mathbb{N}} \mid \forall i \in \mathbb{N}(u_i \geq u_{i+1})\}.$$

Then $\mathbb{N}_\infty$ has points

$$\underline{n} = 1^n 0^\omega,$$

the sequence of n ones followed by infinitely many zeros, usually written simply n by an abuse of notation, and

$$\infty = 1^\omega,$$

the constantly one sequence. The sequence n converges to ∞ in the usual metric on the Cantor space $2^{\mathbb{N}}$, and $\mathbb{N}_\infty$ is a closed subspace of $2^{\mathbb{N}}$. In fact, it is the closure of $\underline{\mathbb{N}} \cup \{\infty\}$ where $\underline{\mathbb{N}} = \{\underline{n} \mid n \in \mathbb{N}\}$. The set $\underline{\mathbb{N}} \cup \{\infty\}$ has empty complement in $\mathbb{N}_\infty$, but is equal to $\mathbb{N}_\infty$ if and only if LPO holds [7, Section 3] (and hence not equal if WLPO fails). However, $\mathbb{N}_\infty \setminus \underline{\mathbb{N}} = \{\infty\}$ always holds [7, Lemma 3.3]:

$$\forall u \in \mathbb{N}_\infty (\forall n \in \mathbb{N}(u \neq n) \implies u = \infty). \tag{5}$$

Here $(\neq)$ denotes the negation of equality, rather than apartness as is customary in Bishop mathematics. We use these and several other facts about $\mathbb{N}_\infty$ from [7], recalled below on demand. The selection function $\varepsilon\colon (\mathbb{N}_\infty \to 2) \to \mathbb{N}_\infty$ is given by

$$\varepsilon(p) = (i \mapsto \min_{n<i} p(n)),$$

but we don't need to work explicitly with this definition in calculations.

3 Classical continuity constructively

Our first lemma invokes twice the following instance of the principle of excluded middle [7, Theorem 8.2], which holds for any $p\colon \mathbb{N}_\infty \to 2$:

$$\forall n \in \mathbb{N}(p(n) = 1) \vee \neg\forall n \in \mathbb{N}(p(n) = 1). \tag{6}$$

This amounts to $\forall n \in \mathbb{N}(p(n) = 1) \vee \tilde{\exists} n \in \mathbb{N}(p(n) = 0)$. The point is that, perhaps surprisingly, the quantifications are over $\mathbb{N}$ rather than $\mathbb{N}_\infty$. In order to emphasize the distinction, we use m, n, i, j, k to range over $\mathbb{N}$ and u, v, w to range over $\mathbb{N}_\infty$.

Lemma 3.1. *For any* $q\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to 2$*,*

$$\forall m \in \mathbb{N}\,\tilde{\exists} n \in \mathbb{N}(q(m, n) = 0) \vee \tilde{\exists} m \in \mathbb{N}\forall n \in \mathbb{N}(q(m, n) = 1). \tag{7}$$

Proof. For given $u \in \mathbb{N}_\infty$, define $p_u(v) = q(u, v)$. By (6) applied to p_u, there is $p\colon \mathbb{N}_\infty \to 2$ such that

$$p(u) = 1 \iff \neg\forall n \in \mathbb{N}(p_u(n) = 1).$$

By (6) applied to p, we conclude that

$$\forall m\neg\forall n(q(m, n) = 1) \vee \neg\forall m\neg\forall n(q(m, n) = 1),$$

which amounts to (7). □

The definition and basic properties of the natural order of $\mathbb{N}_\infty$ and the function $\max\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ extending $\max\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ can be found in [7, Section 5]. In the technical development we reduce quantifications of the form $\exists u \geq v(P(u))$ or $\forall u \geq v(P(u))$ to their equivalents $\exists u(P(\max(u, v)))$ or $\forall u(P(\max(u, v)))$.

Theorem 3.2. *Every function* $f\colon \mathbb{N}_\infty \to \mathbb{N}$ *is either classically continuous or classically discontinuous.*

Proof. By the decidability of equality of $\mathbb{N}$, there is $q\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to 2$ such that

$$q(u, v) = 0 \iff f(\max(u, v)) \neq f(\infty),$$

and hence we conclude, by (7), that

$$\forall m \in \mathbb{N}\,\tilde{\exists} n \in \mathbb{N}(f(\max(m, n)) \neq f(\infty)) \vee \tilde{\exists} m \in \mathbb{N}\forall n \in \mathbb{N}(f(\max(m, n)) = f(\infty)),$$

which amounts to the conclusion of Theorem 3.2. □

If a proposition P is decidable, that is $P \vee \neg P$ holds, then there is a number $[P] \in 2$ such that

$$[P] = 0 \iff P$$

by considering the cases P and $\neg P$, as already used in the above proof. Hence if $P(x)$ is a decidable propositional function of $x \in X$, then we can define a function $p\colon X \to 2$ such that

$$p(x) = 0 \iff P(x)$$

by

$$p(x) = [P(x)].$$

In general the construction of the function p requires choice, because for each x the value $[P(x)]$ is defined by (logical) case analysis, but in general $[P(x)]$ is not defined uniformly from x by a (mathematical) rule, unless one is working in a system such as Martin-Löf type theory (in which choice is a theorem anyway).

But our simple uses don't require choice, so that the construction can be regarded as a convenient notational device. For example, if

$$P(v) = \forall u \in \mathbb{N}_\infty (f(\max(u, v) = f(\infty))),$$

then

$$p(v) = A(u \mapsto e(f(\max(u, v), f(\infty)))),$$

where $A\colon (\mathbb{N}_\infty \to 2) \to 2$ is defined in (4) and $e\colon \mathbb{N} \times \mathbb{N} \to 2$ is the characteristic function of equality.

We showed in [7, Theorem 9.4] that, for any $p\colon \mathbb{N}_\infty \to 2$,

$$\varepsilon(p) = \inf\{u \in \mathbb{N}_\infty \mid p(u) = 0\},$$

where the infimum of the empty set is of course ∞. It follows that, for any decidable propositional function $P(u)$ of a variable $u \in \mathbb{N}_\infty$,

$$\varepsilon(u \mapsto [P(u)]) = \inf\{u \in \mathbb{N}_\infty \mid P(u)\}.$$

Hence, by (2), if this set is inhabited then

$$\varepsilon(u \mapsto [P(u)]) = \text{least } u \in \mathbb{N}_\infty \text{ such that } P(u).$$

Lemma 3.3. *There is a functional $F\colon (\mathbb{N}_\infty \to \mathbb{N}) \to \mathbb{N}_\infty$ such that for any map $f\colon \mathbb{N}_\infty \to \mathbb{N}$, the number $F(f)$ is the least $v \in \mathbb{N}_\infty$ with $f(u) = f(\infty)$ for all $u \geq v$.*

Proof. Because, for any $v \in \mathbb{N}_\infty$, the proposition $\forall u \in \mathbb{N}_\infty (f(\max(u, v) = f(\infty))$ is decidable by the omniscience of $\mathbb{N}_\infty$ and the decidability of equality of $\mathbb{N}$, we can define, using ε as above,

$$F(f) = \inf\{v \in \mathbb{N}_\infty \mid \forall u \in \mathbb{N}_\infty (f(\max(u, v) = f(\infty)))\}.$$

Because the universally quantified equation holds for at least one v, namely $v = \infty$, the number $F(f)$ has the required property. □

Notice that:

1. f is classically discontinuous iff $F(f) = \infty$.

   (Because f is classically discontinuous $\iff$ $F(f) \neq n$ for every $n \in \mathbb{N}$.)

2. f is classically continuous iff $F(f) \neq \infty$.

   (Because this is the contra-positive of the previous equivalence.)

Hence the number $F(f)$ can be thought of as the *discontinuity degree* of f. This shows that we can replace the classical existential quantifier by the constructive one, in the notion of classical continuity, provided we also replace the set $\mathbb{N}$ by $\mathbb{N}_\infty \setminus \{\infty\}$:

Corollary 3.4. *A map $f\colon \mathbb{N}_\infty \to \mathbb{N}$ is classically continuous if and only if there is $v \neq \infty$ such that $f(u) = f(\infty)$ for all $u \geq v$.*

Hence the discontinuity degree of a classically continuous function gives its $\mathbb{N}_\infty$-*valued modulus of continuity.* We observed in [7, Section 2] that MP is equivalent to $\mathbb{N}_\infty \setminus \{\infty\} = \underline{\mathbb{N}}$, that is,

$$\forall u \in \mathbb{N}_\infty (u \neq \infty \implies \exists n \in \mathbb{N}(u = n)).$$

It follows that:

Corollary 3.5. *If* MP *holds, then any classically continuous map $f\colon \mathbb{N}_\infty \to \mathbb{N}$ is continuous.*

The following lemma is applied to use any discontinuous function as an oracle to decide the conclusion of WLPO, but it holds for arbitrary functions:

Lemma 3.6. *For any function* $f\colon \mathbb{N}_\infty \to \mathbb{N}$ *there is* $G = G(f)\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ *such that, for all* $v \in \mathbb{N}_\infty$*,*

$$G(v) \geq v, \tag{8}$$

$$\tilde{\exists}\, u \geq v (f(u) \neq f(\infty)) \implies f(G(v)) \neq f(\infty). \tag{9}$$

We refer to $G(f)$ as the *modulus of discontinuity* of f.

Proof. First define $g\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ by

$$g(v) = \varepsilon(u \mapsto [f(\max(u, v)) \neq f(\infty)]).$$

By (2), for any $v \in \mathbb{N}_\infty$ we have that

$$\exists u \in \mathbb{N}_\infty (f(\max(u, v)) \neq f(\infty)) \implies f(\max(g(v), v)) \neq f(\infty). \tag{10}$$

If we define

$$G(v) = \max(g(v), v),$$

then (8) holds, and so does (9) by (10) combined with (3). □

It follows that $f\colon \mathbb{N}_\infty \to \mathbb{N}$ is classically discontinuous iff for every $n \in \mathbb{N}$ there is $u \geq n$ in $\mathbb{N}_\infty$ with $f(u) \neq f(\infty)$, namely $u = G(n)$. We observed in [7, Section 2] that WLPO amounts to $\mathbb{N}_\infty = \{\infty\} \cup (\mathbb{N}_\infty \setminus \{\infty\})$, that is

$$\forall u \in \mathbb{N}_\infty (u = \infty \vee u \neq \infty).$$

Theorem 3.7. *The existence of a classically discontinuous function* $f\colon \mathbb{N}_\infty \to \mathbb{N}$ *is equivalent to* WLPO*.*

Proof. (⇒) If f is classically discontinuous then

$$\forall n \in \mathbb{N} (f(G(n)) \neq f(\infty)). \tag{11}$$

By (8), we have $G(\infty) = \infty$ and hence $f(G(\infty)) = f(\infty)$. This shows that $u = \infty$ implies $f(G(u)) = f(\infty)$. Conversely, if $f(G(u)) = f(\infty)$ then $u \neq n$ for every $n \in \mathbb{N}$, because $u = n$ contradicts (11), and hence $u = \infty$ by (5). This shows that, for any $u \in \mathbb{N}_\infty$,

$$u = \infty \iff f(G(u)) = f(\infty).$$

Hence $u = \infty$ is decidable, because the right-hand side is decidable, as $\mathbb{N}$ has decidable equality, which shows that WLPO holds.

(⇐): If WLPO holds then one can define $f(u) = 0$ if $u = \infty$, and $f(u) = 1$ if $x \neq \infty$, which is clearly discontinuous. □

Corollary 3.8. *All functions* $f\colon \mathbb{N}_\infty \to \mathbb{N}$ *are classically continuous if and only if* WLPO *fails.*

Proof. (⇒): If WLPO holds, then not every function is continuous by Theorem 3.7, which contradicts the hypothesis, and hence WLPO must fail.

(⇐). Assume that WLPO fails and let $f\colon \mathbb{N}_\infty \to \mathbb{N}$. Then f is either classically continuous or classically discontinuous by Theorem 3.2. The second case is ruled out because it contradicts Theorem 3.7, and so the first must hold. □

Is Weak Markov's Principle (WMP) [10] enough to deduce the same conclusion in the following corollary?

Corollary 3.9. *If* MP *and* $\neg$ WLPO *hold, then all functions* $f\colon \mathbb{N}_\infty \to \mathbb{N}$ *are continuous.*

We finish this section with a brief discussion of this in connection with sequence convergence in the *intrinsic* topology, related to [1, Section 4]. Let X be any set, and say that a sequence $x\colon \mathbb{N} \to X$ converges to a limit x_∞ if it can be extended to a function $\boldsymbol{x}\colon \mathbb{N}_\infty \to X$ that maps ∞ to x_∞ (cf. e.g. [7, Lemma 5.5]). We refer to the collection of sequences $\mathbb{N}_\infty \to X$ as the *intrinsic (sequential) topology* of the set X. A function $\boldsymbol{x}\colon \mathbb{N}_\infty \to X$ can be thought of as a *converging sequence*, that is, a sequence $\boldsymbol{x}_n$ given together with its limit $\boldsymbol{x}_\infty$. With this terminology, a sequence is *convergent* if it can be extended to a converging sequence. All functions of any two sets are automatically continuous in this topology, without postulating any continuity axiom, in the sense that, for any $f\colon X \to Y$, from a converging sequence $\boldsymbol{x}\colon \mathbb{N}_\infty \to X$ with limit $\boldsymbol{x}_\infty$ we get a converging sequence $f \circ \boldsymbol{x}\colon \mathbb{N}_\infty \to Y$ with limit $f(\boldsymbol{x}_\infty)$.

If excluded middle holds, or more generally if WLPO holds, then every sequence $\mathbb{N} \to X$ converges to any point of X, which amounts to saying that the intrinsic topology of any set is *indiscrete*. On the other hand, if $\neg$ WLPO holds, then the convergent sequences in $\mathbb{N}$ are precisely the classically eventually constant ones, and if additionally MP holds, then they are precisely the eventually constant ones, so that we get the *discrete* sequential topology on $\mathbb{N}$.

Conversely, if $\mathbb{N}$ is indiscrete, one can define a discontinuous map $\mathbb{N}_\infty \to \mathbb{N}$, for example from the convergence of an alternating binary sequence to zero, and hence WLPO holds. On the other hand, if $\mathbb{N}$ is intrinsically discrete then all functions $\mathbb{N}_\infty \to \mathbb{N}$ are continuous and hence WLPO fails.

The last two paragraphs show that if WLPO and MP are left undecided (neither they or they negation are postulated), then whether $\mathbb{N}$ is intrinsically (in)discrete also remains undecided, and that the precise nature of the intrinsic topology of $\mathbb{N}$ is tightly related to which way WLPO and MP are decided.

The intrinsic topology of the Baire space $\mathbb{N}^\mathbb{N}$ can be reduced to that of $\mathbb{N}$. From a converging sequence $\alpha\colon \mathbb{N}_\infty \to \mathbb{N}^\mathbb{N}$, we get a sequence of converging sequences $\hat{\alpha}\colon \mathbb{N} \to (\mathbb{N}_\infty \to \mathbb{N})$ defined by transposition as $\hat{\alpha}(i)(u) = \alpha(u)(i)$. Hence if the natural numbers have the discrete sequential topology, then the converging sequences $\alpha\colon \mathbb{N}_\infty \to \mathbb{N}^\mathbb{N}$ are those that satisfy $\forall i\, \exists n\, \forall j, k \geq n(\alpha(i)(j) = \alpha(i)(k))$, which constitute the usual sequential topology of the Baire space.

If S is a subset of a set X, from any converging sequence $\mathbb{N}_\infty \to S$ we get a converging sequence $\mathbb{N}_\infty \to X$ by composition with the inclusion map $S \to X$. One may ask whether, conversely, any given converging sequence $\mathbb{N}_\infty \to X$ that happens to have values in S necessarily correstricts to a converging sequence $\mathbb{N}_\infty \to S$. This amounts to asking whether the intrinsic topology of S coincides with the intrinsic topology of X relativized to S. This would mean that every subset is a subspace in the relative intrinsic topology. Perhaps counter-intuitively, this is not the case in general. We give two examples. (1) Consider $X = \mathbb{R}$ and $S = \mathbb{Q}$. Because $\mathbb{Q}$ has decidable equality, if $\neg$ WLPO and MP then $\mathbb{Q}$ is sequentially discrete by the above discussion. But the rational sequence $1/2^n$ intrinsically converges to the rational number 0 and is not eventually constant. (2) Consider $X = \mathbb{N}^\mathbb{N}$ and S the subset of eventually constant sequences of natural numbers. Then S is intrinsically discrete if *neg* WLPO, and hence they don't form a *subspace* of the Baire space.

An application of these ideas is developed in [8], which identifies the intrinsic topology of a Martin-Löf universe à la Russell and uses it to show that the universe satisfies the conclusion of Rice's Theorem: it has no non-trivial decidable extensional properties. More precisely, from a hypothetical such property, we derive WLPO, by a reduction to discontinuity.

4 Searchability of $2^{\mathbb{N}}$ and uniform continuity

As is well known, Brouwer derived the following *uniform continuity principle* in his conception of intuitionistic mathematics:

$$\forall f\colon 2^{\mathbb{N}} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall \alpha, \beta \in 2^{\mathbb{N}} (\alpha =_n \beta \implies f(\alpha) = f(\beta)).$$

Here $\alpha =_n \beta$ means that $\forall i < n(\alpha_i = \beta_i)$. Continuing from the first sentence of the introduction, a second reason why Bishop didn't wish to accept continuity axioms is that he wanted every theorem of constructive mathematics to be a theorem of classical mathematics, but continuity violates excluded middle and many of its classically interesting consequences.

It is interesting that Brouwer's uniform continuity principle implies theorems that belong to classical mathematics but are not provable in large fragments of Bishop mathematics such as Heyting Arithmetic with finite types (HA^ω) and with extensionality (of course it is difficult to say what is not provable in Bishop mathematics, as its boundaries are deliberately left vague). One such theorem is that the Cantor space is searchable, which can be regarded as folklore:

Lemma 4.1. *The cantor space is searchable if*

1. *excluded middle holds, or*
2. *Brouwer's uniform continuity principle holds.*

(Moreover, in the second case, the selection function can be constructed so that $\varepsilon(p) = \inf\{\alpha \in 2^{\mathbb{N}} \mid p(\alpha) = 0\}$ in the lexicographic order of the Cantor space.)

Proof. (1). In fact, more generally, any non-empty set X is searchable if EM holds. By EM there is $a \in X$. Given $p\colon X \to 2$, by EM either there is $b \in X$ with $p(b) = 0$ or not. If so, let $\varepsilon(p) = b$, and otherwise, let $\varepsilon(p) = a$. Then clearly $p(\varepsilon(p)) = 1 \implies \forall x \in X(p(x) = 1)$, because either the premise is false (if $\varepsilon(p) = b$) or the conclusion holds independently of the premise (if $\varepsilon(p) = a$).

(2). For this argument we need to assume that the uniform continuity principle states the existence of a modulus functional $H\colon (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$, or choice to get H. Given $p\colon 2^{\mathbb{N}} \to 2$, we first define a finite sequence $s \in 2^m$, where $m = H(p)$, by course-of-values induction on $k < m$. If the sequence has been defined for all $i < k$, we define $s_k = 0$ if and only if there is a finite sequence $t \in 2^{m-k-1}$ such that $p(s_0 s_1 \dots s_{k-1} 0 t 0^\omega) = 0$. We now define $\varepsilon(p) = s0^\omega$, and the result holds by construction. □

The non-provability of the searchability of the Cantor space in HA^ω with extensionality is established in [9, Section 6]. Assuming that Bishop mathematics, whatever it is, is to be compatible with classical mathematics, we have:

Metatheorem 4.2. *In Bishop mathematics or in* HA^ω *with the axioms of extensionality and choice, the searchability of the Cantor space doesn't prove continuity or uniform continuity principles.*

Proof. Because all sets are searchable in classical mathematics but certainly not all functions are continuous, and such systems cannot prove non-classical conclusions from classical assumptions. □

However, we have a situation that may be puzzling at first sight. If the Cantor space is searchable and all functions $\mathbb{N}_\infty \to \mathbb{N}$ are continuous, then all functions $2^\mathbb{N} \to \mathbb{N}$ are uniformly continuous. The reason this may be surprising is that the searchability of $2^\mathbb{N}$ doesn't give continuity, as discussed above, but, together with the simplest possible continuity assumption, which amounts to sequential continuity, it gives uniform continuity, which is much stronger than continuity, which in turn is much stronger than sequential continuity. Of course, what this means is that the searchability of $2^\mathbb{N}$ is not to be taken lightly from a constructive point of view, as expected.

Define $(- \restriction -)\colon 2^\mathbb{N} \times \mathbb{N}_\infty \to 2^\mathbb{N}$ by

$$\alpha \restriction u = (i \mapsto \min(\alpha_i, u_i)).$$

Then $\alpha \restriction \infty = \alpha$ and $\alpha \restriction n = \alpha_0 \alpha_1 \dots \alpha_{n-1} 0^\omega$. Also define

$$\alpha =_u \beta \iff \alpha \restriction u = \beta \restriction u,$$

which then extends the equivalence relation $=_n$ defined earlier so that

$$\alpha =_\infty \beta \iff \alpha = \beta.$$

Theorem 4.3. *If the Cantor space is searchable and all functions $p\colon \mathbb{N}_\infty \to 2$ are continuous, then all functions $f\colon 2^\mathbb{N} \to \mathbb{N}$ are uniformly continuous.*

Proof. Given $f\colon 2^\mathbb{N} \to \mathbb{N}$ define $p\colon \mathbb{N}_\infty \to 2$, using the searchability of $2^\mathbb{N}$, by

$$p(u) \quad = \quad [\forall \gamma \in 2^\mathbb{N}(f(\gamma) = f(\gamma \restriction u))].$$

By the continuity of p, there is $n \in \mathbb{N}$ with $p(m) = p(\infty)$ for all $m \geq n$. But $p(\infty) = 0$, and hence $p(m) = p(\infty)$ is equivalent to

$$\forall \gamma \in 2^\mathbb{N}(f(\alpha) = f(\gamma \restriction m)).$$

If $\alpha =_n \beta$ then $\alpha \restriction n = \beta \restriction n$, and considering $m = n$ and the two special cases $\gamma = \alpha$ and $\gamma = \beta$, we conclude that $f(\alpha) = f(\alpha \restriction n) = f(\beta \restriction n) = f(\beta)$, which shows that f is uniformly continuous. □

But we can do better than that: we can define a modulus of uniform continuity functional, and we can address the evident classical version of uniform continuity.

Lemma 4.4. *If the Cantor space is searchable, then there is a functional*

$$H\colon (2^\mathbb{N} \to \mathbb{N}) \to \mathbb{N}_\infty$$

such that for any map $f\colon 2^\mathbb{N} \to \mathbb{N}$ the number $H(f)$ is the least $v \in \mathbb{N}_\infty$ such that $\forall \alpha, \beta \in 2^\mathbb{N}(\alpha =_v \beta \implies f(\alpha) = f(\beta))$.

Proof. Extending the proof of Theorem 4.3, define

$$H(f) = F(u \mapsto [\forall \gamma \in 2^\mathbb{N}(f(\gamma) = f(\gamma \restriction u))]),$$

where F is constructed in Lemma 3.3. □

Hence:

1. $H(f) = \infty$ if and only if f is not uniformly continuous.
2. $H(f) \sharp \infty$ if and only if f is uniformly continuous.
3. $H(f) \neq \infty$ if and only if f is classically uniformly continuous.

Because $\neg$ WLPO implies the classical continuity of all maps $\mathbb{N}_\infty \to 2$, we conclude by Lemma 3.3 that:

Corollary 4.5. *If the Cantor space is searchable and* $\neg$ WLPO *holds, then all functions* $2^{\mathbb{N}} \to \mathbb{N}$ *are classically uniformly continuous.*

And of course further assuming MP we can remove the classicality in the conclusion. Combining this observation with Lemma 4.1, we get:

Corollary 4.6. *If* MP *and* $\neg$ WLPO *hold, then the Cantor space is searchable if and only if all functions* $2^{\mathbb{N}} \to \mathbb{N}$ *are uniformly continuous*

This method of getting uniform continuity from the searchability of $2^{\mathbb{N}}$ and from the continuity of maps $\mathbb{N}_\infty \to \mathbb{N}$ is reminiscent of Bauer and Lešnik' proof of their Theorem 4.4 and Corollary 4.5 in [1], although they refer to compactness and (uniform) bars rather than searchability.

We conclude this section with some remarks and questions about continuity. Starting from the functional $F\colon (\mathbb{N}_\infty \to \mathbb{N}) \to \mathbb{N}_\infty$ of Lemma 3.3 used above, if we assume Markov's Principle, then $F(f) \in \underline{\mathbb{N}}$, and without any assumption one can map $\underline{\mathbb{N}}$ into $\mathbb{N}$. Hence, assuming also $\neg$ WLPO, we get a modulus of continuity functional, without using choice:

$$\exists F\colon (\mathbb{N}_\infty \to \mathbb{N}) \to \mathbb{N}\, \forall f\colon \mathbb{N}_\infty \to \mathbb{N}\, \forall m \geq F(f)(f(m) = f(\infty)). \tag{12}$$

(By the same token, we get a functional $H\colon (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$ under the same assumptions.) The continuity principle (12) implies $\neg$ WLPO, but we conjecture that it doesn't imply MP. Does it imply Weak Markov's Principle [10]? Apparently it is not quite equivalent to the continuity of all $f\colon \mathbb{N}_\infty \to \mathbb{N}$, because it additionally seems to incorporate some amount of choice (but not more than that coming from MP). Of course, if choice is available, this condition is equivalent to the continuity of all functions $\mathbb{N}_\infty \to \mathbb{N}$. By search bounded by $F(f)$, one can assume w.l.o.g. that $F(f)$ is minimal with the above property, in which case F satisfies:

$$\begin{array}{lcll} F(f) & = & 0 & \text{if } \forall u \in \mathbb{N}_\infty (f(u) = f(\infty)), \\ F(f) & = & F(f \circ \mathrm{succ}) + 1 & \text{otherwise,} \end{array}$$

where $\mathrm{succ}\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ is the successor function $\mathrm{succ}(u) = u + 1$. These two equations can be considered as computation rules, because the condition is decidable by the omniscience of $\mathbb{N}_\infty$, which are a special case of Kohlenbach's bar recursion [3].

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