The intrinsic topology of Martin-Löf universes

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Abstract

A construction by Hofmann and Streicher gives an interpretation of a type-theoretic universe U in any Grothendieck topos, assuming a Grothendieck universe in set theory. Voevodsky asked what space U is interpreted as in Johnstone's topological topos. We show that its topological reflection is indiscrete. We also offer a model-independent, *intrinsic* or *synthetic*, description of the topology of the universe: It is a theorem of type theory that the universe is sequentially indiscrete, in the sense that any sequence of types converges to any desired type, up to equivalence. As a corollary we derive *Rice's Theorem for the universe*: it cannot have any non-trivial, decidable, extensional property, unless WLPO, the weak limited principle of omniscience, holds.

1 Introduction

A construction by Hofmann and Streicher [15] gives an interpretation of a type-theoretic universe U in any Grothendieck topos, assuming a Grothendieck universe in set theory. Voevodsky asked us what space U is interpreted as in Johnstone's topological topos [9]. We show that its topological reflection is indiscrete.

This answer is perhaps shocking at first sight: one would maybe expect a rather elaborate and interesting topology for the universe, but it turns out to be trivial in this model. Perhaps the topological topos, lacking univalence [16], falls short of giving an informative interpretation of the universe in type theory, or perhaps the Hofmann–Streicher interpretation of the universe is at fault.

None of these is the case. It is a theorem *of* type theory that the universe is *intrinsically indiscrete*: Any sequence X_n of types in the universe converges to any desired type X_∞, up to equivalence. We work with the notion of equivalence from homotopy type theory [16], denoted by $\simeq$. Type equivalence is logically equivalent to the existence of back and forth maps that pointwise compose to the identities, but is defined in a subtler way.

A convergent sequence is defined as a map from the *one-point compactification* $\mathbb{N}_\infty$ of the discrete type $\mathbb{N}$, constructed as the type of decreasing binary sequences. The idea is that the points $\underline{n} = 1^n 0^\omega$ are thought to converge to the point $\infty = 1^\omega$. This is in the spirit of *synthetic topology*, but we don't need to postulate any axiom, as opposed to the usual situations in synthetic topology [2] or other synthetic theories, to prove this.

The crucial observation is that the point at infinity is not detachable, without excluded middle or similar constructive taboo, from the finite points. In particular, we cannot define a function $\mathbb{N}_\infty \to X$ by case analysis on whether the argument is ∞ or not, as this amounts to the *weak limited principle of omniscience* WLPO [7].

We say that a sequence $x\colon \mathbb{N} \to X$ converges to a point $x_\infty : X$ if it extends to a function $\mathbb{N}_\infty \to X$ that maps ∞ to x_∞. All functions automatically preserve limits of convergent sequences, or are sequentially continuous, by composition. If WLPO holds, then any sequence converges to any point, rendering all types indiscrete. From a topological point of view, WLPO violates traditional continuity axioms, and, from a computational point of view, it gives an oracle for solving the Halting Problem, which shows that WLPO fails to be constructive [4].

The above type-theoretic theorem then is that for any sequence $X : \mathbb{N} \to U$ of types and any type $X_\infty : U$, there is $A : \mathbb{N}_\infty \to U$ such that $A_{\underline{n}} \simeq X_n$ and $A_\infty \simeq X_\infty$, which can be formulated as saying that any sequence X_n converges to any desired type X_∞, up to equivalence. If the univalence axiom holds, then of course X_n converges to X_∞ on the nose, as the univalence axiom implies that equivalent types are equal. We do not assume univalence, but we do need to assume the axiom of functional extensionality (any two pointwise equal functions are equal, which is a consequence of univalence) to prove some basic properties of the type $\mathbb{N}_\infty$.

In order to relate the results proved inside type theory to the counterexample as given by the topological topos, let us first recall the definition of the latter and some its properties as relevant for this work. To build the topological topos, one starts with the monoid of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of the discrete natural numbers, and then takes sheaves for the canonical topology of this monoid considered as a category. The category of sequential topological spaces is fully embedded into the topological topos. Their subobjects are precisely the Kuratowski limit spaces (sets equipped with convergent sequences subject to suitable axioms), which form a locally cartesian closed subcategory of the topological topos, and an exponential ideal. The image of the Yoneda embedding is $\mathbb{N}_\infty$. It is well-known, and easy to check, that in any topological space, and also in any limit space X, the convergent sequences are precisely the continuous maps $\mathbb{N}_\infty \to X$. Although the topological topos hosts the sequential topological spaces and the Kuratowki limit spaces, many of its objects are not (limit or topological) spaces, including the subobject classifier and the interpretation of the type-theoretic universe following [15]. However, the limit spaces and the sequential topological spaces form two different full reflective subcategories of the topological topos.

When the above theorem in type theory is interpreted in the topological topos, it gives that the quotient of U by $\simeq$ is indiscrete. But, as discussed above, U itself is indiscrete, and this can be formulated by saying that all maps into the Sierpiński space (open-subobject classifier) are constant. This also gives that all maps from U to the two-point discrete space are constant, which is a form of Rice's Theorem for the universe, saying that all decidable predicates on U are trivial. We also formulate and prove this internally in type theory: if a non-trivial, extensional, decidable property exists, then WLPO holds.

Notice that the simplicial sets model of univalence does validate WLPO since it actually validates classical logic. It remains as an open question whether univalence is consistent with continuity principles that entail the negation of WLPO.

In Section 2, we prove in type theory that the universe is indiscrete in the above sense, and then derive a version of Rice's Theorem for the universe from this. In Section 3, we look at this from a semantical point of view. In realizability models and in the topos of reflexive graphs, the functions from the universe to the booleans need not be extensional, but nevertheless they are all constant. The same phenomenon takes place in Johnstone's topological topos, and, moreover, even the functions from the universe to the Sierpiński space are all constant.

2 Some theorems of type theory

We work in an intensional Martin-Löf type theory with Π, Σ, Id (identity type), $\mathbb{0}$ (empty type), $\mathbb{1}$ (unit type), $\mathbb{2}$ (booleans, with elements written $0, 1$), and $\mathbb{N}$ (natural numbers) and a universe U closed under these operations [13]. In particular, we reason without assuming Streicher's eliminator K, i.e. without uniqueness of identity proofs (UIP) [8]. But we use function extensionality as an axiom (any two pointwise equal functions are equal), as explained in the introduction. The univalence axiom [16] is not needed, but we briefly emphasize some of its immediate consequences for this work.

We reason informally, but rigorously, as in the homotopy type theory book [16]. Formal counter-parts of our proofs are available at [5]. We denote the identity type by "=", and we refer to it as simply equality. A type is a proposition iff any two of its elements are equal, and it is a set iff the equality of any two of its points is a proposition. Sometimes we write function types $X \to Y$ in exponential notation Y^X. The canonical maps $x = y \to Ax \to Ay$ and $x = y \to f(x) = f(y)$ are written transport and $\mathrm{ap}\, f$.

2.1 The indiscreteness of the universe

The relation $x \geq y$ in $\mathbb{2}$ can be defined as $x = 0 \to y = 0$. Because any equality in a type with decidable equality is a proposition, by Hedberg's Theorem, function extensionality gives that $x \geq y$ is also a proposition. Hence, by a further application of function extensionality, the type

$$\mathrm{IsDecreasing}\, \alpha = \prod_{i:\mathbb{N}} (\alpha_i \geq \alpha_{i+1})$$

is a proposition too. Define, as in the introduction,

$$\mathbb{N}_\infty = \sum_{\alpha:\mathbb{2}^{\mathbb{N}}} \mathrm{IsDecreasing}\, \alpha.$$

In view of the previous remarks, this is a set. Moreover, the projection $e : \mathbb{N}_\infty \to \mathbb{2}^{\mathbb{N}}$ is an embedding in the sense of [16]: the map $\mathrm{ap}\, e : x = y \to e(x) = e(y)$ is an equivalence. The points $\underline{n}, \infty : \mathbb{N}_\infty$ mentioned in the introduction can be defined by providing evident accompanying witnesses in the types ($\mathrm{IsDecreasing}\, 1^n 0^\omega$) and ($\mathrm{IsDecreasing}\, 1^\omega$). We assume the definition of convergence up to equivalence given in the introduction.

Theorem 2.1 (Universe indiscreteness) *Any sequence $X : \mathbb{N} \to U$ of types converges to any desired type Y, up to equivalence.*

Hence univalence implies that X converges to Y on the nose.

Proof We first claim that X_n converges to $\mathbb{1}$. Define $A : \mathbb{N}_\infty \to U$ by

$$A_i = \prod_{j:\mathbb{N}} X_j^{i=\underline{j}}.$$

Then

$$A_\infty = \prod_{j:\mathbb{N}} X_j^{\infty=\underline{j}} \simeq \prod_{j:\mathbb{N}} X_j^{\mathbb{0}} \simeq \prod_{j:\mathbb{N}} \mathbb{1} \simeq \mathbb{1}.$$

To see that $A_{\underline{n}} \simeq X_n$ for any $n : \mathbb{N}$, define maps

$$\phi : A_{\underline{n}} \leftrightarrows X_n : \psi$$

by

$$\begin{array}{rcl} \phi(f) & = & f(n)(\mathrm{refl}\,\underline{n}), \\ \psi(x) & = & \lambda j : \mathbb{N}.\lambda p : \underline{n} = \underline{j}.\, \mathrm{transport}\, p' x, \end{array}$$

where $p' : n = j$ is the path derived from $p : \underline{n} = \underline{j}$ using the fact that the inclusion $n \mapsto \underline{n} : \mathbb{N} \to \mathbb{N}_\infty$ is an embedding, which in turn uses the fact that the type $(\mathrm{IsDecreasing}\,\alpha)$ is a proposition for any $\alpha : \mathbb{2}^{\mathbb{N}}$. Now

$$\phi(\psi(x)) = \mathrm{transport}(\mathrm{refl}\,\underline{n})' x = \mathrm{transport}(\mathrm{refl}\,n) = x,$$

where the second last step uses the fact that $\mathbb{N}$ is an set to get $(\mathrm{refl}\,\underline{n})' = \mathrm{refl}\,n$. For the other direction we have that

$$\psi(\phi(f))(j)(p) = \mathrm{transport}\, p'(f(n)(\mathrm{refl}\,\underline{n})) \overset{\text{wish}}{=} f(j)(p).$$

To prove the wish, we first claim that, for any $q \colon n = j$,

$$\mathrm{transport}\, q(fn(\mathrm{refl}\,\underline{n})) = fj(\mathrm{ap}(\lambda n.\underline{n})q).$$

This is immediate when $q = \mathrm{refl}\,n$, and hence the claim follows by path induction (Martin-Löf's elimination rule J for identity types). Using the fact that $\mathbb{N}_\infty$ is a set, we know that

$$\mathrm{ap}(\lambda n.\underline{n})p' = p,$$

from which the wish follows considering $q = p'$. Therefore $A_{\underline{n}} \simeq X_n$, and so X_n converges to $\mathbb{1}$ as claimed.

Next we claim that the constant sequence $\mathbb{1}$ converges to Y. Define $B : \mathbb{N}_\infty \to U$ by

$$B_i = Y^{i=\infty}.$$

Then, for any $n : \mathbb{N}$,

$$\begin{array}{rcl} B_{\underline{n}} & = & Y^{\underline{n}=\infty} \simeq Y^{\mathbb{0}} \simeq \mathbb{1}, \\ B_\infty & = & Y^{\infty=\infty} \simeq Y^{\mathbb{1}} \simeq Y, \end{array}$$

again using the fact that $\mathbb{N}_\infty$ is a set to deduce that $(\infty = \infty) \simeq \mathbb{1}$, which concludes the proof of the claim.

To complete the proof of the theorem, define $C : \mathbb{N}_\infty \to U$ by

$$C_i = Y^{i=\infty} \times \prod_{j:\mathbb{N}} X_j^{i=\underline{j}} = B_i \times A_i.$$

Then, for any $n : \mathbb{N}$,

$$\begin{array}{r} C_{\underline{n}} = B_{\underline{n}} \times A_{\underline{n}} \simeq \mathbb{1} \times X_n \simeq X_n, \\ C_\infty = B_\infty \times A_\infty \simeq Y \times \mathbb{1} \simeq Y, \end{array}$$

which shows that X_n converges to Y up to equivalence. □

2.2 Rice's Theorem for the universe

As a corollary of the universe indiscreteness theorem, we get an internal version of Rice's Theorem for the universe. We say that a function $P \colon U \to \mathbb{2}$ is *extensional* if $X \simeq Y$ implies $P(X) = P(Y)$. Of course all maps $P \colon U \to \mathbb{2}$ are extensional under univalence.

The weak limited principle of omniscience WLPO is usually formulated as

$$\forall p : \mathbb{N} \to \mathbb{2}.(\forall n : \mathbb{N}.p(n) = 1) \vee \neg\forall n : \mathbb{N}.p(n) = 1.$$

In type theory it becomes

$$\prod_{p:\mathbb{N}\to\mathbb{2}} \left(\prod_{n:\mathbb{N}} p(n) = 1\right) + \neg\left(\prod_{n:\mathbb{N}} p(n) = 1\right).$$

Under function extensionality, this is a proposition, and it is easy to check that, again using function extensionality, it is equivalent to

$$\prod_{x:\mathbb{N}_\infty} x = \infty + x \neq \infty.$$

If we deny WLPO on the grounds that it contradicts some basic continuity or computability assumptions, then our version of Rice's Theorem amounts to saying that the universe cannot have any non-trivial, extensional, decidable property. But Martin-Löf type theory does not prove or disprove WLPO, because WLPO is refuted by realizability models and validated by the classical set-theoretic model. Hence we formulate the theorem as follows, where say that P is *non-trivial* if there are $X, Y : U$ with $P(X) \neq P(Y)$.

Theorem 2.2 (Rice's Theorem for the universe) *If there is a non-trivial extensional $P\colon U \to \mathbb{2}$, then* WLPO *holds.*

Proof Let $X, Y : U$ such that $P(X) = 0$ and $P(Y) = 1$. From these data we construct a function $p\colon \mathbb{N}_\infty \to \mathbb{2}$ such that

$$p(\infty) = 1 \quad \text{and} \quad p(\underline{n}) = 0 \;\text{ for all } n : \mathbb{N} \tag{1}$$

as follows. We first apply Theorem 2.1 to the constant sequence $\lambda n.\, X : \mathbb{N} \to U$ and to the type $Y : U$, to get $C\colon \mathbb{N}_\infty \to U$ such that

$$C_\infty \simeq Y \quad \text{and} \quad C_{\underline{n}} \simeq X \;\text{ for all } n : \mathbb{N}.$$

We can then let $p = P \circ C$ so that (1) holds by the extensionality of P. Now, using p, we can decide $x = \infty$ by case analysis on $p(x)$, which amounts to WLPO. □

Corollary 2.3 *If there is $P\colon U \to \mathbb{2}$ such that $P(X) = 0$ if and only if X is inhabited, then* WLPO *holds.*

Proof Any such predicate is extensional, and so Rice's Theorem applies to it. □

This amounts to an internal version of the well known meta-theoretical fact that there is no algorithm that decides whether any given closed type (proposition) is inhabited by a closed term (has a proof).

The above version of Rice's Theorem for the universe remains valid when type theory is extended with any kind of postulated axiom, e.g. Univalence, Church's Thesis, Brouwerian continuity axioms, Markov principle, to name a few of the contentious axioms that one may wish to consider in constructive mathematics [3, 4], simply because when we add more axioms, the old theorems remain valid.

One possible reaction to Rice's Theorem for the universe is that this is to be expected: after all, there are no elimination rules for the universe, as it is not inductively defined. But our arguments show that, even if there were, Rice's Theorem for the universe would still hold, which justifies the lack of elimination rules, at least if one wishes to retain the compatibility of type theory with extensionality.

In the model of simplicial sets, which validates univalence, there are non-trivial, extensional functions $U \to \mathbb{2}$ as e.g. the inhabitedness predicate. In the topological topos, which validates continuity axioms, such a map $U \to \mathbb{2}$ doesn't exist, because continuity contradicts WLPO. Hence, in any model satisfying both univalence and continuity axioms, if such a model exists, the existence of a non-trivial predicate $U \to \mathbb{2}$ is impossible, again because WLPO fails.

2.3 Failure of total separatedness

Thorsten Altenkirch, Thomas Anberrée and Nuo Li asked whether for every definable $A : U$ and for any two distinct elements of A, there is a function $A \to \mathbb{2}$ that separates them. If one thinks of $\mathbb{2}$-valued maps as characteristic functions of clopen sets in a topological view of types, which is compatible with the topological topos interpretation, then their question amounts to asking whether the definable types are *totally separated* (see e.g. [10]), that is, whether the clopens separate the points. This logical version of the topological notion is investigated in [6]. The above technical ingredients (chiefly using identity types as exponents in the construction of types with certain desired properties) can be used to give a negative answer to their question.

We begin with a construction that explodes a given point a of a type X into two copies a_0 and a_1. Let

$$E_a(X) = \sum_{x:X} \mathbb{2}^{x=a}$$

and define $s : \mathbb{2} \to X \to E_a(X)$ by

$$s(n)(x) = (x, \lambda q.n).$$

Then the maps $s(0), s(1) : X \to E_a(X)$ are both sections of the first projection. For any $x : X$ and $n : \mathbb{2}$ write

$$x_n = s(n)(x),$$

so that, in particular, a_n is defined from a, which gives the desired two new copies a_0 and a_1 of $a : X$ in $E_a(X)$, as we show below.

Lemma 2.4 *$x_0 = x_1$ for all $x \neq a$.*

Proof We have $(\lambda q : x = a.0) = (\lambda q : x = a.1)$ by function extensionality, as the type $x = a$ is empty by assumption. Applying $\lambda f : \mathbb{2}^{x=a}.(x, f)$ to this, we get $(x, \lambda q.0) = (x, \lambda q.1)$, which amounts to $x_0 = x_1$. □

Lemma 2.5 *The points a_0 and a_1 are distinct.*

Proof Define $p : E_a(X) \to U$ by

$$p(x, f) = \sum_{q:x=a} f(q) = 1.$$

Then

$$\begin{aligned} p(a_0) &= p(a, \lambda q.0) = \sum_{q:a=a} (0 = 1) = (a = a) \times (0 = 1) \simeq \mathbb{0}, \\ p(a_1) &= p(a, \lambda q.1) = \sum_{q:a=a} (1 = 1) = (a = a) \times (1 = 1) \simeq (a = a). \end{aligned}$$

Hence if a_0 and a_1 were equal then the empty type $\mathbb{0}$ would be equivalent to the inhabited type $(a = a)$. □

Lemma 2.6 *If there is $p : E_a(X) \to \mathbb{2}$ with $p(a_0) \neq p(a_1)$, then $x \neq a$ is decidable for every $x : X$.*

Proof Define $f : X \to \mathbb{2}$ by $f(x) = p(x_0) \oplus p(x_1)$ where $\oplus$ is addition modulo two. By the assumption on p and Lemma 2.4, we have

$$x = a \to f(x) = 1, \qquad x \neq a \to f(x) = 0,$$

whose contra-positives are respectively

$$f(x) = 0 \to x \neq a, \qquad f(x) = 1 \to \neg(x \neq a).$$

Hence we can decide $x \neq a$ by case analysis on the value of $f(x)$. □

We can then answer the question by Altenkirch et al. as follows:

Theorem 2.7 *There is a type A with two distinct elements such that if any function $A \to \mathbb{2}$ separates them, then WLPO holds.*

Proof Take $X = \mathbb{2}^{\mathbb{N}}$ and $a = \lambda i.1$, and let $A = E_a(X)$ with a_0, a_1 constructed from a as above. Then $\neg(x \neq a) \to x = a$ for all $x : X$. To see this, assume $\neg(x \neq a)$. If $x(i) \neq a(i)$ then $x \neq a$, and hence we must have $x(i) = a(i)$ because the type $\mathbb{2}$ has decidable equality. Hence $x = a$ by function extensionality. From this it follows that if there is $p : E_a(X) \to \mathbb{2}$ with $p(a_0) \neq p(a_1)$, then, by Lemma 2.6, it is decidable for all $x : X$ whether $x = a$, which amounts to WLPO. □

Altenkirch et al. originally formulated their question at the meta-level, as follows: Is it the case that for every definable type X and for any two provably distinct closed terms of type X there is a closed term of type $X \to \mathbb{2}$ separating them? The above gives a negative answer to that, because an inhabitant of WLPO is not definable. Moreover, we can remove the assumption of function extensionality for the meta-level question: such a function is not definable with function extensionality, but without function extensionality there are fewer terms, and hence such a function remains not definable in the absence of function extensionality.

2.4 A characterization of Rice's Condition for the universe

As observed above, if excluded middle, which is consistent with type theory, and valid in the model of simplicial sets, is postulated, one *can* define an extensional predicate $P : U \to \mathbb{2}$, with two different values, even in the absence of elimination rules, for instance $P(X) = 0$ iff X is inhabited. What our version of Rice's Theorem for the universe says is that, conversely, the assumption of such a P gives a non-provable instance of excluded middle. This can be strengthened. After we publicly presented the above version of Rice's Theorem for the universe in the conference *Mathematical Foundations of Programming Semantics* in Bristol, UK, 2012, Alex Simpson proved the following, included with his permission:

Theorem 2.8 (Alex Simpson (2012), personal communication) *The following are logically equivalent:*

(1) (Rice's Condition) *There is a non-trivial, extensional $P : U \to \mathbb{2}$.*

(2) (Weak excluded middle) *$\neg A + \neg\neg A$ for every $A : U$.*

Proof ($\Uparrow$): Trivial: define $P(X) = 0$ if $\neg X$ and $P(X) = 1$ if $\neg\neg X$. This is clearly extensional, and non-trivial because $P(0) = 0$ and $P(1) = 1$.

($\Downarrow$): Let $X, Y : U$ with $P(X) = 0$ and $P(Y) = 1$, and define

$$Q(A) = \neg A \times X + \neg\neg A \times Y.$$

Then, by the extensionality of P, and by function extensionality,

(a) if $\neg A$ then $Q(A) \simeq X$ and thus $P(Q(A)) = 0$,

(b) if A then $Q(A) \simeq Y$ and thus $P(Q(A)) = 1$.

The contra-positives of these two implications are respectively

$$\begin{array}{lcl} P(Q(A)) = 1 & \to & \neg\neg A \\ P(Q(A)) = 0 & \to & \neg A. \end{array}$$

Hence we can decide $\neg A$ by case analysis on the value of $P(Q(A))$. $\square$

3 The unverse in some categorical models

We have shown above in Theorem 2.2 that WLPO follows from the existence of a predicate $p : U \to 2$ which is extensional in the sense that $p(a) = p(b)$ whenever $E(a) \cong E(b)$. Since for most universes U isomorphic elements are not equal it is interesting to show that **all** definable 2-valued predicates on a universe U are constant.

This holds already for the well-known realizability models of type theory (see e.g. [14]) where types are interpreted as assemblies over a partial combinatory algebra $\mathcal{A}$ and the universe U is interpreted as $\nabla(\mathsf{PER}(\mathcal{A}))$, i.e. the assembly whose underlying set is the collection $\mathsf{PER}(\mathcal{A})$ of *partial equivalence relations* on $\mathcal{A}$ and every element is realized by all elements of $\mathcal{A}$. For this reason all realizable maps from U to 2 are constant (since 2 is *modest*, i.e. realizers determine realized objects uniquely).

But there are also universes U in Grothendieck toposes $\mathcal{E}$ (over **Set**) such that U is *connected* in the sense that every morphism $p : U \to 2$ in $\mathcal{E}$ is constant. It has been shown in [15] how to construct universes U in Grothendieck toposes $\mathcal{E}$ from a Grothendieck universe $\mathcal{U}$. For sake of convenience we recall the construction from [15].

Let $\mathcal{C}$ be a small category then in $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$ one can define a universe U as follows

$$U(I) = \mathcal{U}^{(\mathcal{C}/I)^{\mathsf{op}}} \qquad U(J \xrightarrow{u} I)(A) = A \circ (\Sigma_u)^{\mathsf{op}}$$

where $\mathcal{U}$ is a Grothendieck universe[1]. In order to endow U with the structure of a universe we have to consider the object E in $\widehat{\mathcal{C}}$ which is defined as

$$E(I) = \{\langle A, a\rangle \mid A \in U(I), a \in A(\mathrm{id}_I)\} \quad E(u)(\langle A, a\rangle) = \langle U(u)(A), A(u \xrightarrow{u} \mathrm{id}_I)(a)\rangle$$

for $I \in \mathcal{C}$ and $u : J \to I$ in $\mathcal{C}$. We write $p_{\mathcal{U}}$ or simply p for the natural transformation from E to U which sends $\langle A, a\rangle$ to A.[2] The point of this construction now is that all maps in $\widehat{\mathcal{C}}$ with fibres in $\mathcal{U}$ can be obtained as pullback of $p_{\mathcal{U}}$ along some (typically non-unique) map to U.

[1]Actually, much weaker closure properties are sufficient for our purposes. It suffices that $\mathcal{U}$ is closed under dependent sums and products and contains $\mathbb{N}$. In most cases $\mathcal{C}$ will be a category internal to $\mathcal{U}$ but this is only needed for having all representable objects in the universe U (defined from $\mathcal{U}$).

[2]The second clause in the definition of E is motivated by the "fibration of pointed objects" for a category $\mathcal{B}$ with finite limits. Its objects over $I \in \mathcal{B}$ are pairs (s, a) where $a : A \to I$ and $s : I \to A$ with $a \circ s = \mathrm{id}_I$. A morphism from (t, b) over J to (s, a) over I is a pair (u, f) of

As observed in [15] this construction can be adapted[3] to a sheaf topos $\mathcal{E} = \mathsf{Sh}(\mathcal{C}, \mathcal{J})$ in a fairly simple way. Let $\mathsf{i} : \mathcal{E} \hookrightarrow \widehat{\mathcal{C}}$ be the inclusion of sheaves into presheaves and a its left adjoint which is known to preserves finite limits.[4] From this it follows that $\mathsf{a}(p_{\mathcal{U}})$ is a universe in $\mathcal{E}$.

3.1 A connected universe in the topos of graphs

As a first application of the universe construction considered above we apply it to the topos of (irreflexive) graphs. Let $\mathbb{G}$ be the category with two objects V and E whose only nontrivial morphisms are $s, t : V \to E$. Obviously $\widehat{\mathbb{G}}$ is the topos of irreflexive graphs. Let $\mathcal{U}$ be some Grothendieck universe. This gives rise to a universe U in $\widehat{\mathbb{G}}$ where $U(V) = \mathcal{U}$, $U(E)$ is the set of spans $s_X : X \to X_0$, $t_X : X \to X_1$ in $\mathcal{U}$ and $U(s)(s_X, t_X) = X_0$ and $U(t)(s_X, t_X) = X_1$, respectively.

In this universe isomorphic elements are typically not equal. But, nevertheless, the object U is connected since for $X_0, X_1 \in U(V)$ we have $U(s)(\pi_0^{X_0,X_1}, \pi_1^{X_0,X_1}) = X_0$ and $U(t)(\pi_0^{X_0,X_1}, \pi_1^{X_0,X_1}) = X_1$. Thus, all maps from U to $2 = \Delta(2)$ are constant.

3.2 A connected universe in Johnstone's topological topos

In [9] P. T. Johnstone's introduced the so-called *topological topos* $\mathcal{T}$. We will show that $\mathcal{T}$ hosts a universe U for which all morphism $U \to 2 = \Delta(2)$ are constant (where $\Delta \dashv \Gamma : \mathcal{T} \to \mathbf{Set}$ is the unique geometric morphism from $\mathcal{T}$ to $\mathbf{Set}$).

We first recall the definition of $\mathcal{T}$ (see also [11] and in particular [9] for more detailed information). Let $\mathbb{M}$ be the monoid of continuous endomaps of $\mathbb{N}_\infty$, the one-point-compactification of $\mathbb{N}$. A map u in $\mathbb{M}$ is called *singular* iff its image is finite. One easily shows that non-singular maps send ∞ to ∞. On $\mathbb{M}$ we consider the Grothendieck topology $\mathcal{J}$ consisting of sieves (i.e. right ideals in $\mathbb{M}$) S such that

(1) all constant maps are in S and

(2) for every infinite subset A of $\mathbb{N}$ there is a non-singular $f \in \mathbb{M}$ whose image is a subset of $A \cup \{\infty\}$.

In [9] it is shown that the $\neg\neg$-separated object of $\mathcal{T}$ are those objects X in $\mathcal{T}$ where $x = x'$ whenever $X(c)(x) = X(c)(x')$ for all constant $c \in \mathbb{M}$. The latter objects correspond to limit spaces (called "subsequential spaces" in [9]), i.e. sets X endowed with a notion of convergence as given by a relation $\to_X$ between sequence in A and elements of X satisfying the requirements

morphisms in $\mathcal{B}$ such that

$$\begin{tikzcd}
J \arrow[r, "u"] \arrow[d, "t"'] & I \arrow[d, "s"] \\
B \arrow[r, "f"] \arrow[d, "b"'] & A \arrow[d, "a"] \\
J \arrow[r, "u"'] & I
\end{tikzcd}$$

commutes. Such a morphism is over u. The fibration of pointed objects is given by first projection to $\mathcal{B}$. We get the above E when instantiating $\mathcal{B}$ by $\widehat{\mathcal{C}}$ and then restricting along Yoneda.

[3] As pointed out in *loc.cit.* a requirement for this to work is that small maps are stable under sheafification which, however, is satisfied in case of sheaf toposes since if all fibres of a map f are isomorphic to a cardinal in $\mathcal{U}$ then this also holds for the sheafification $\mathsf{a}(f)$ of f.

[4] Such an adjunction $\mathsf{a} \dashv \mathsf{i}$ is known as a *localization* of $\widehat{\mathcal{C}}$. It is known that (see e.g. [11]) that Grothendieck toposes over $\mathbf{Set}$ are precisely the localizations of presheaf toposes.

(1) for all $x \in X$ the constant sequence (x) converges to x

(2) if (x_n) converges to y then all subsequence of (x_n) also converge to y

(2) if (x_n) is a sequence such that all its subsequences contain a subsequence converging to y then (x_n) converges to y.

Notice that the subobject classifier $\Omega_{\mathcal{T}}$ will not correspond to a limit space because equality on $\Omega_{\mathcal{T}}$ is not $\neg\neg$-closed (since otherwise $\mathcal{T}$ would be boolean, which it isn't!). Since $\Omega_{\mathcal{T}}$ is a subobject of the universe U (induced by a Grothendieck universe $\mathcal{U}$) it follows that U also doesn't correspond to a limit space.

In $\widehat{\mathbb{M}}$ the universe U is given by the set of contravariant functors from $\mathbb{M}/*$ to $\mathcal{U}$. For $u \in \mathbb{M}$ the action $U(u)$ is given by precomposition with Σ_u^{op}. The presheaf E over $\mathbb{M}$ has underlying set $\{\langle A, a\rangle \mid A \in U(*), a \in A(\mathrm{id}_{\mathbb{N}_\infty})\}$ and the action of $\mathbb{M}$ on E is given by $E(u)(\langle A, a\rangle) = \langle U(u)(A), A(u \xrightarrow{u} \mathrm{id}_{\mathbb{N}_\infty})(a)\rangle$. The universe in $\widehat{\mathbb{M}}$ is given by $p_{\mathcal{U}} : E \to U$ sending $\langle A, a\rangle$ to A. The universe in the topological topos $\mathcal{T}$ is obtained by sheafifying $p_{\mathcal{U}}$, i.e. $\mathsf{a}(p_{\mathcal{U}})$ where a is the left adjoint to the inclusion of $\mathcal{T}$ into $\widehat{\mathbb{M}}$.

Theorem 3.1 *In the topological topos the universe* $\mathsf{a}(U)$ *is connected, i.e. every* $f : \mathsf{a}(U) \to \Delta(2)$ *is constant.*

Proof Since $\Delta(2)$ is a sheaf it suffices to show that in $\widehat{\mathbb{M}}$ every map $f : U \to \Delta(2)$ is constant where U is the universe in $\widehat{\mathbb{M}}$.

Recall that $\Delta(2)$ consists of all continuous maps from $\mathbb{N}_\infty$ to the discrete space 2 on which $\mathbb{M}$ acts by precomposition.

For every $A \in \mathcal{U}$ let A_c be the constant presheaf on $\mathbb{M}/*$ with value A. Obviously, for every $u \in \mathbb{M}$ we have $U(u)A_c = A_c$. Thus, by naturality of f we have that $f(A_c) = f(U(u)A_c) = f(A_c) \circ u$ for all $u \in \mathbb{M}$ from which it follows that $f(A_c)$ is constant.

Let $A \in \mathcal{U}$ and $\alpha \in \mathbb{N}_\infty$. We write c_α for the constant map with value α. We define A_α as the presheaf over $\mathbb{M}/*$ with $A_\alpha(c_\alpha) = A$, $A_\alpha(u) = \emptyset$ for $u \in \mathbb{M} \setminus \{c_\alpha\}$ and $A(u : c_\alpha \to c_\alpha) = \mathrm{id}_A$ for all $u \in \mathbb{M}$. Obviously, we have $U(c_\alpha)(A_\alpha) = A_c$ and $U(c_\beta)(A_\alpha) = \emptyset_c$ for $\beta \neq \alpha$.

Thus, for $\alpha, \beta \in \mathbb{N}_\infty$ with $\alpha \neq \beta$ we have

$$f(A_c) = f(U(c_\alpha)(A_\alpha)) = f(A_\alpha) \circ c_\alpha \quad \text{and} \quad f(\emptyset_c) = f(U(c_\beta)(A_\alpha)) = f(A_\alpha) \circ c_\beta$$

from which it follows that $f(A_c) = f(\emptyset_c)$ since otherwise $f(A_\alpha)$ were not continuous (at α).

Suppose $F \in U$ and $\alpha \in \mathbb{N}_\infty$. Then $U(c_\alpha)(F) = A_c$ for a unique $A \in \mathcal{U}$. Thus, we have $f(F) \circ c_\alpha = f(U(c_\alpha)(F)) = f(A_c) = f(\emptyset_c)$. Since this holds for all $\alpha \in \mathbb{N}_\infty$ it follows that $f(F) = f(\emptyset_c)$ for all $F \in U$.

Thus, we have shown that f is constant as desired. $\square$

Recall from [9] that every limit space X may be identified with the object $\mathsf{y}(X)$ of $\mathcal{T}$ consisting of continuous maps from $\mathbb{N}_\infty$ to X on which $\mathbb{M}$ acts by precomposition. Notice that $\Delta(2)$ is nothing but $\mathsf{y}(2)$ where 2 is the discrete space with two elements. Let Σ be the Sierpiński space with two elements $\perp$ and $\top$ whose only non-trivial open subset is $\{\top\}$. Maps $X \to \mathsf{y}(\Sigma)$ in $\mathcal{T}$ correspond to sequentially open subsets of the reflection of X to limit spaces which motivates calling an object X of $\mathcal{T}$ *indiscrete* iff all morphisms from X to $\mathsf{y}(\Sigma)$ are constant. Next we show that

Theorem 3.2 *In the topological topos the universe* $\mathsf{a}(U)$ *is* indiscrete*.*

Proof Since $\mathsf{y}(\Sigma)$ is a sheaf it suffices to show that in $\widehat{\mathbb{M}}$ every map $f : U \to \mathsf{y}(\Sigma)$ is constant where U is the universe in $\widehat{\mathbb{M}}$.

For every $A \in \mathcal{U}$ let A_c be the constant presheaf on $\mathbb{M}/*$ with value A. Obviously, for every $u \in \mathbb{M}$ we have $U(u)A_c = A_c$. Thus, by naturality of f we have $f(A_c) = f(U(u)A_c) = f(A_c) \circ u$ for all $u \in \mathbb{M}$ from which it follows that $f(A_c)$ is constant.

Let $A \in \mathcal{U}$ and $\alpha \in \mathbb{N}_\infty$. We write c_α for the constant map with value α. We define A_α as the presheaf over $\mathbb{M}/*$ with $A_\alpha(c_\alpha) = A$, $A_\alpha(u) = \emptyset$ for $u \in \mathbb{M} \setminus \{c_\alpha\}$ and $A(u : c_\alpha \to c_\alpha) = \mathrm{id}_A$ for all $u \in \mathbb{M}$. Obviously, we have $U(c_\alpha)(A_\alpha) = A_c$ and $U(c_\beta)(A_\alpha) = \emptyset_c$ for $\beta \neq \alpha$.

Thus, for $\alpha, \beta \in \mathbb{N}_\infty$ with $\alpha \neq \beta$ we have

$$f(A_c) = f(U(c_\alpha)(A_\alpha)) = f(A_\alpha) \circ c_\alpha \quad \text{and} \quad f(\emptyset_c) = f(U(c_\beta)(A_\alpha)) = f(A_\alpha) \circ c_\beta$$

from which it follows that $f(A_c) = f(\emptyset_c)$ since otherwise $f(A_\alpha)$ were not continuous (at α).

Suppose $F \in U$ and $\alpha \in \mathbb{N}_\infty$. Then $U(c_\alpha)(F) = A_c$ for a unique $A \in \mathcal{U}$. Thus, we have $f(F) \circ c_\alpha = f(U(c_\alpha)(F)) = f(A_c) = f(\emptyset_c)$. Since this holds for all $\alpha \in \mathbb{N}_\infty$ it follows that $f(F) = f(\emptyset_c)$ for all $F \in U$.

Thus, we have shown that f is constant as desired. $\square$

We conclude by observing that analogous results hold in some realizability models of type theory since they host a type Σ which behaves very much like the Sierpiński space (see e.g. [12]). Since this Σ is always modest and the universe U is of the form $\nabla(\mathsf{PER}(\mathcal{A}))$ all maps from U to Σ will be constant. In case of function realizability (see [17]) the modest sets of the ensuing topos host a full reflective subcategory $\mathbf{QCB}_0$ which is equivalent to a full subcategory of the category $\mathbf{Sp}$ of topological spaces and continuous maps. Up to equivalence the objects of $\mathbf{QCB}_0$ are T_0 spaces arising as subquotients of Baire space. Obviously, the space Σ is in $\mathbf{QCB}_0$. See [1] for a more detailed treatment of this.

Notice also that the above universe U in the function realizability topos is also *path connected* since any map from the unit interval $[0, 1]$ to U is realizable (because U is $\nabla(\mathsf{PER}(\mathcal{A}))$ where $\mathcal{A}$ is Baire space endowed with the pca structure employed in function realizability).

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