The intrinsic topology of a Martin-Löf universe

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Abstract

Assuming the propositional axiom of extensionality, we show that a Martin-Löf universe à la Russell is indiscrete in its *intrinsic topology*. This doesn't invoke Brouwerian continuity principles. As a corollary, we derive Rice's Theorem for the universe: the existence of a non-trivial, decidable, extensional property of the universe implies the weak limited principle of omniscience. This is a theorem in type theory. Without assuming extensionality, we deduce the following metatheorem: in intensional Martin-Löf type theory with a universe, there is no closed term defining a non-trivial, decidable, extensional property of the universe.

1 Introduction

We show that any universe $\mathbb{U}$ of types closed under the usual constructions of Martin-Löf type theory is indiscrete, in the sense that every sequence of types converges to any type, up to isomorphism. This is a theorem of type theory, rather than a metatheorem or a theorem about models. Convergence is defined using $\mathbb{N}_\infty$, the *generic convergent sequence* $\mathbb{N}_\infty$, also known as the one-point compactification of the discrete set of natural numbers $\mathbb{N}$. The type $\mathbb{N}_\infty$ adds a new point ∞ to $\mathbb{N}$ as the limit of the sequence of points $n\colon \mathbb{N}$. For more information see Section 2 below.

We say that a sequence $x\colon \mathbb{N} \to X$ in a type X converges to a limit $x_\infty\colon X$ if it extends to a *limiting sequence* $\mathbb{N}_\infty \to X$ that maps ∞ to x_∞. We refer to the collection of limiting sequences as the *intrinsic (sequential) topology* of the type X. This is simply the function type $(\mathbb{N}_\infty \to X)$. Every function of any two types is trivially sequentially continuous with respect to the intrinsic topologies, without considering any model or any Brouwerian continuity axiom.

One possible motivation for the formulation of this notion is that if X is given the structure of a metric space, then the *continuous* maps $\mathbb{N}_\infty \to X$ *are* the convergent sequences (see [8, Lemma 5.5] for a precise formulation). We are not assuming Brouwerian continuity axioms, and most types don't have a metrizable intrinsic topology, but all functions are in some sense "secretly" continuous, which brings us to the second, model-theoretic, motivation of this definition of convergence.

There is an interpretation of type theory, Johnstone's topological topos [14], in which types are spaces and all functions are continuous. In this interpretation, $\mathbb{N}$ is the discrete space of natural numbers and the space $\mathbb{N}_\infty$ is the one-point compactification of $\mathbb{N}$ in the usual (classical) topological sense. Moreover, in this interpretation, convergence defined in the above sense coincides with topological convergence.

Using a general construction by Streicher [20], assuming a Grothendieck universe in set theory, one can build an object in the topological topos that is the interpretation of the universe. Vladimir Voevodsky asked what the topology of this interpretation of the Martin-Löf universe is. We don't know the answer, but it follows from what we prove here that the quotient by type isomorphism is the indiscrete topology. Moreover, we conjecture that the Grothendieck universe with the indiscrete topology can be given the structure needed to interpret a Martin-Löf universe, but this may be a bit too audacious. A space is indiscrete if the only open sets are the empty set and the whole space. This is equivalent to saying that every sequence converges to any point, which motivates the terminology we have adopted above.

The appropriate notion of equality for elements of the universe $\mathbb{U}$ of types is isomorphism. Hence we reformulate the above definition of limit for sequences of types as follows. We say that a sequence of types $X\colon \mathbb{N} \to \mathbb{U}$ converges to a limit $X_\infty\colon \mathbb{U}$ if one can find a *limiting sequence* $X'\colon \mathbb{N}_\infty \to \mathbb{U}$ such that

$$\begin{aligned} X_n &\cong X'_n \\ X_\infty &\cong X'_\infty. \end{aligned}$$

If one assumes the univalence axiom of Homotopy Type Theory [22], one can replace the isomorphisms by equalities to get an equivalent notion. But notice that in the topological topos interpretation, isomorphism is not the same thing as equality.

We show that the universe is indiscrete in the sense that every sequence of types converges to any type, and use this to deduce a version of Rice's Theorem [18] for the universe. This universe indiscreteness theorem may be surprising, because types like the Cantor space of infinite binary sequences are far from indiscrete in the sense considered here, as they have plenty of decidable properties. The Cantor space also fails to be discrete, because it doesn't have decidable equality, and this fact shows up in the proof of Rice's Theorem.

Computer checked proofs written down in Agda notation [23, 4] are available at [10], but the technical development below is largely self-contained.

2 Preliminaries

The generic convergent sequence. The type $\mathbb{N}_\infty$ can be constructed as

$$\mathbb{N}_\infty = \left\{\alpha\colon 2^{\mathbb{N}} \mid \forall i\colon \mathbb{N}(\alpha_i \geq \alpha_{i+1})\right\} = \sum_{\alpha\colon 2^{\mathbb{N}}} \prod_{i\colon \mathbb{N}} (\alpha_i = 0 \to \alpha_{i+1} = 0)\,.$$

Here $2^{\mathbb{N}} = (\mathbb{N} \to 2)$ is the Cantor space of binary sequences, $2 = \{0, 1\} \cong 1 + 1$ is the type of binary numbers, and $(=)$ denotes the identity type so that, by an abuse of notation, $\mathrm{Id}\, X\, x\, y$ is written $x =_X y$ or simply $x = y$ when X can be inferred from the context. As usual, we shall write $\forall$ and $\exists$ as alternative notations for $\prod$ and $\sum$. The type $\mathbb{N}_\infty$ has points

$$\underline{n} = 1^n 0^\omega,$$

the sequence of n ones followed by infinitely many zeros, usually written simply n by an abuse of notation, and

$$\infty = 1^\omega,$$

the constantly one sequence. The sequence n converges to ∞ in the usual metric of the Cantor space $2^{\mathbb{N}}$, and $\mathbb{N}_\infty$ is a closed subspace of $2^{\mathbb{N}}$. In fact, it is the closure of $\underline{\mathbb{N}} \cup \{\infty\}$ where $\underline{\mathbb{N}} = \{\underline{n} \mid n\colon \mathbb{N}\}$. The set $\underline{\mathbb{N}} \cup \{\infty\}$ has empty complement in $\mathbb{N}_\infty$, but is equal to $\mathbb{N}_\infty$ if and only if LPO holds [8, Section 3], and hence not equal if WLPO fails, where

$$\mathrm{WLPO} \iff \forall u\colon \mathbb{N}_\infty(u = \infty \vee u \neq \infty).$$

However, $\mathbb{N}_\infty \setminus \underline{\mathbb{N}} = \{\infty\}$ always holds [8, Lemma 3.3]:

$$\forall u\colon \mathbb{N}_\infty(\forall n\colon \mathbb{N}(u \neq n) \implies u = \infty). \tag{1}$$

For more information see [8, 9].

The axiom of extensionality. Extensional Martin-Löf type theory is intensional Martin-Löf type theory with the *rule* that says that propositionally equal terms are judgementally equal, which is known to render type checking (proof checking) undecidable. An alternative approach to extensionality is to postulate the *propositional axiom* of extensionality

$$\forall f, g \colon X \to Y((\forall x \colon X.\, f(x) = g(x)) \to f = g)$$

in intensional Martin-Löf type theory. Unless otherwise stated, we assume this axiom (whose computational content is actively under research). We don't rely on the UIP principle (uniqueness of identity proofs) or the K-rule [19], except for some types for which it can be proved from the propositional axiom of extensionality [12], which include $\mathbb{N}$, $2^{\mathbb{N}}$ and $\mathbb{N}_\infty$.

It is well known that for any type X and any $A \colon X \to \mathbb{U}$ if $\forall x \colon X \forall p, q \colon Ax(p = q)$ then the projection $\sum_{x \colon X} Ax \to X$ is a monomorphism, so that $\sum_{x \colon X} Ax$ can be regarded as a subset of the type X. Our main use of the propositional axiom of extensionality is to establish the condition $\forall x \colon X \forall p, q \colon Ax(p = q)$ for suitable A, for example in the definition of $\mathbb{N}_\infty$ so that it is a subset of $2^{\mathbb{N}}$. The need for this becomes clear in the formalization [10] of the proofs sketched here.

3 The Universe Indiscreteness Theorem

The crucial construction is this:

Lemma 3.1 *Any sequence of types converges to the terminal type* 1*.*

Given a sequence of types

$$X \colon \mathbb{N} \to \mathbb{U},$$

we extend it to a limiting sequence

$$X' \colon \mathbb{N}_\infty \to \mathbb{U}$$

with

$$X'_\infty \cong 1$$

by defining

$$X'_u = \prod_{n \colon \mathbb{N}} (u = n \to X_n).$$

The idea is that if $u = n$ then a function of type $u = n \to X_n$ simply picks an element of X_n, so that $(u = n \to X_n) \cong X_n$, and otherwise such a function has empty graph so that $(u = n \to X_n) \cong 1$. In particular, if $u = \infty$, then the latter holds for every n, so that $X'_\infty \cong \prod_{n \colon \mathbb{N}} 1 \cong 1$. At the intuitive level, it should be clear that X' has the required property, but the argument that it does work, given in [10], requires subtle reasoning with identity types, and relies on the propositional axiom of extensionality.

The above can be regarded as the topological step of the argument, so to speak. The remaining steps of the argument are of an algebraic nature, using type addition $X + Y$, multiplication $X \times Y$ and exponentiation $Y^X = (X \to Y)$, and their basic arithmetic laws expressed in terms of isomorphisms. We start from this:

Corollary 3.2 *The constant sequence* 0 *converges to the type* 1*.*

Using $0 \times X \cong 0$ and $1 \times X \cong X$, and multiplying the limiting sequence of Corollary 3.2 pointwise with any given type X, we conclude that:

Lemma 3.3 *The constant sequence* 0 *converges to any type.*

Using $0^0 \cong 1$ and $0^1 \cong 0$, and applying pointwise the function $X \mapsto 0^X$ to the constant sequence 0 converging to 1, we get:

Lemma 3.4 *The constant sequence* 1 *converges to* 0*.*

Using $1 \times X_n \cong X_n$ and $0 \times 1 \cong 0$, and multiplying the limiting sequence of Lemma 3.4 pointwise with a sequence X_n that converges to 1 constructed by Lemma 3.1, we get that:

Lemma 3.5 *Any sequence converges to* 0*.*

Finally, using $X_n + 0 \cong X_n$ and $0 + X_\infty \cong X_\infty$, and adding pointwise the limiting sequence X_n converging to 0 constructed by Lemma 3.5 with the constant sequence 0 converging to X_∞ constructed by Lemma 3.3, we get that the universe is indiscrete:

Theorem 3.6 *Every sequence of types converges to any type.*

4 Rice's Theorem for the universe

As a corollary of the universe indiscreteness theorem, we get a version of Rice's Theorem for the universe. We say that a decidable predicate $p \colon \mathbb{U} \to 2$ is extensional if $X \cong Y$ implies $p(X) = p(Y)$. Recall that WLPO is the weak limited principle of omniscience [3], a non-provable instance of the principle of excluded middle, which says that every binary sequence is constantly one or not (Section 2). Notice that although the proof uses topological techniques, the formulation of the theorem doesn't mention topology:

Theorem 4.1 *If there are an extensional $P \colon \mathbb{U} \to 2$ and types X and Y with $P(X) \neq P(Y)$, then* WLPO *holds.*

Proof Assume w.l.o.g. that $P(X) = 0$ and $P(Y) = 1$, swapping the roles of X and Y if necessary, using the decidability of equality of the type 2. From this assumption we construct a function $p \colon \mathbb{N}_\infty \to 2$ such that, for every $n \colon \mathbb{N}$,

$$p(n) = 0, \qquad p(\infty) = 1. \tag{2}$$

This is a discontinuous function, and the existence of such a function implies WLPO — see e.g. [8, 9]. To construct p, apply Theorem 3.6 to the constant sequence $\lambda n.\, X : \mathbb{N} \to \mathbb{U}$ and to the type $Y : \mathbb{U}$, to get $Q \colon \mathbb{N}_\infty \to \mathbb{U}$ such that

$$Q_n \cong X, \qquad Q_\infty \cong Y.$$

Now let $p = P \circ Q$, and (2) holds by the extensionality of P, as promised. □

Corollary 4.2 *If there is $P \colon \mathbb{U} \to 2$ such that $P(X) = 0 \iff X$ is inhabited, then* WLPO *holds.*

Proof Any such predicate is extensional, and so Rice's Theorem applies to it. □

This amounts to the well known fact that there is no algorithm that decides whether any given proposition has a proof.

Metatheorem 4.3 (Without assuming extensionality) *For all closed terms $P \colon \mathbb{U} \to 2$ and $X, Y : \mathbb{U}$ with a given proof term of the extensionality of P, there is no closed term of type $P(X) \neq P(Y)$.*

Proof Assuming the axiom of extensionality, there can't be such closed terms, as there is a realizability interpretation of intensional Martin-Löf type theory with the axiom of extensionality, e.g. Hyland's effective topos [13], where WLPO solves the Halting Problem. Without postulating the axiom, fewer terms are definable in the language, and hence the omission of extensionality gives the same conclusion. □

The above version of Rice's Theorem for the universe remains true when type theory is extended with any kind of postulated propositional axiom, e.g. univalence, Church's thesis, Brouwerian continuity axioms, Markov principle, to name a few of the contentious axioms that one may wish to consider in constructive mathematics [2, 5]. One possible reaction to our result is that this is to be expected: after all, there are no elimination rules for the universe. But our arguments show that, even if there were, Rice's Theorem for the universe would still hold, which justifies the lack of elimination rules, at least if one wishes to retain extensionality. Notice that, if excluded middle is postulated, one *can* define an extensional predicate with two different values, even in the absence of elimination rules, e.g. $p(X) = 0 \iff X$ is inhabited. What our theorem says is that, conversely, the assumption of such a p gives a non-provable instance of excluded middle.

A universe à la Tarski, on the other hand, consists of codes of types, and the type of codes does have non-trivial decidable properties. These decidable properties will be extensional in the sense that equal codes have the same value, but not extensional in the sense that two codes of isomorphic types have the same value.

5 Failure of total separateness

Thorsten Altenkirch, Thomas Anberrée and Nuo Li asked whether for every definable type X and for any two distinct elements of X, there is a function $X \to 2$ that separates them. Of course the universe is a counter-example. We construct a small counter-example, again with a continuity argument, with the same idea as the construction of Lemma 3.1 and as the proof of Theorem 4.1:

$$X = \sum_{u\colon \mathbb{N}_\infty} (u = \infty \to 2).$$

The idea is that X is $\mathbb{N}_\infty$ with ∞ split into two copies. Because $(n = \infty) \cong 0$ for any $n\colon \mathbb{N}$, and because any two functions defined on 0 are equal by extensionality, there is a unique function $e\colon n = \infty \to 2$, with empty graph, and so X has points

$$(n, e).$$

But it has two copies of ∞:

$$\infty_0 = (\infty, \lambda r.\, 0), \qquad \infty_1 = (\infty, \lambda r.\, 1).$$

These elements are indeed different [10], but this is not trivial to prove, in view of the following:

Proposition 5.1 *For every $p\colon X \to 2$, if $p(\infty_0) \neq p(\infty_1)$ then* WLPO *holds.*

Proof Define $p_i\colon \mathbb{N}_\infty \to 2$ for $i\colon 2$ by

$$p_i(u) = p(u, \lambda r.\, i).$$

Then $p_i(\infty) = p(\infty_i)$ and so $p_0(\infty) \neq p_1(\infty)$ by the assumption. For every $n\colon \mathbb{N}$ we have that $p_i(n) = p(n, e)$, and hence $p_0(n) = p_1(n)$. Thus, if we now define $f\colon \mathbb{N}_\infty \to 2$ by $f(u) = p_0(u) \oplus p_1(u)$, using addition modulo 2, then $f(n) = 0$ and $f(\infty) = 1$, and so f is discontinuous, from which we conclude WLPO. $\square$

In the topological topos, the interpretation of X is a non-Hausdorff, but T_1, compact space, and removing any of the two points at infinity one gets a space homeomorphic to $\mathbb{N}_\infty$. If one thinks of 2-valued maps as characteristic functions of clopen sets in a topological view of types, which is compatible with the topological topos interpretation, then their question amounts to asking whether the definable types are *totally separated* (see e.g. [15]), that is, whether the clopens separate the points. This logical version of the topological notion is investigated in [11].

6 Subspaces of the intrinsic topology

In categories of spaces, subobjects (monomorphisms) don't need to be subspaces. For example, the map $\mathbb{R} \to \nabla\mathbb{R}$ that is the identity on points, where $\mathbb{R}$ has the usual topology and $\nabla\mathbb{R}$ is $\mathbb{R}$ retopologized with the indiscrete topology, is a monomorphism but clearly not a subspace embedding. The same phenomenon takes place in type theory, and Voevodsky, in the Agda mailing list when the results of this paper where advertised there, provided the example of the subtype

$$\sum_{X\colon \mathbb{U}} (X \cong 0 \vee X \cong 1)$$

of the universe, which does have a non-trivial decidable property, and hence cannot be indiscrete. Of course, here it is the disjunction that gives the information necessary to perform the decision, and not the isomorphism conditions.

We say that a function $f\colon X \to Y$ is a *pre-embedding* if x_n converges to x_∞ whenever $f(x_n)$ converges to $f(x_\infty)$, and that it is a *subspace embedding* if additionally it is a monomorphism [17]. For example, any retract is a subspace with the section as the embedding. For any $A\colon X \to \mathbb{U}$, the first projection

$$\left(\sum_{x\colon X} A(x)\right) \to X$$

is a monomorphism if $\forall x\colon X\, \forall p, q\colon A(x)(p = q)$. It is easy to see that it is a pre-embedding if and only if, for every $x\colon \mathbb{N}_\infty \to X$,

$$(\forall n\colon \mathbb{N}(Ax_n) \wedge A(x_\infty) \to \forall u\colon \mathbb{N}_\infty(Ax_u).$$

By [8, Lemmas 3.3-3.4], this holds for any decidable $A\colon X \to \mathbb{U}$. More generally:

Lemma 6.1 *For any $B\colon X \to \mathbb{U}$, the type $\sum_{x\colon X} \neg B(x)$ is a subspace of X.*

Proof Assume $\forall n\colon \mathbb{N}(\neg B(x_n))$ and $\neg B(x_\infty)$. Let $u\colon \mathbb{N}_\infty$, and assume $B(x_u)$ for the sake of contradiction. Then $u \neq n$ by the assumption, for any $n\colon \mathbb{N}$, and hence $u = \infty$ by [8, Lemma 3.3], which also contradicts the hypothesis, and so $\neg B(x_u)$, which shows that the first projection into X is a pre-embedding. By extensionality, any two functions with values on the type 0 are equal, and hence $\neg B(x) = (B(x) \to 0)$ has at most one proof, and so the the projection is also a monomorphism. $\square$

Considering $B(x) = \neg A(x)$, we conclude that $\sum_{x\colon X} \neg\neg A(x)$ is always a subspace of X. Of course, if A is *separated* in the sense that $\forall x\colon X(\neg\neg A(x) \to A(x))$, then $\sum_{x\colon X} A(x)$ is a subspace. This is the case, for example, if A is defined without using $\exists$, $(\vee)$ or $(=_X)$ other than with X that has separated equality. Types with separated equality include the metrizable ones, such as 2, $\mathbb{N}$, $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

Hence the following is a subspace of the universe:

$$\sum_{X\colon \mathbb{U}} \neg\neg(X \cong 0 \vee X \cong 1).$$

Being a subspace, it is indiscrete. The obvious map from Voevodsky's example to this subspace is analogous to the map $\mathbb{R} \to \nabla\mathbb{R}$ discussed above.

We conclude with some questions. Simplifying the logic and replacing isomorphism by identity, we get the following alternative indiscrete subspace of the universe:

$$\nabla 2 \stackrel{\text{def}}{=} \sum_{X\colon \mathbb{U}} ((X = 0 \to 0) \to (X = 1 \to 0) \to 0). \tag{3}$$

It has points

$$0 \stackrel{\text{def}}{=} (0, \lambda f_0 f_1 .\, f_0(\text{refl})), \qquad 1 \stackrel{\text{def}}{=} (1, \lambda f_0 f_1 .\, f_1(\text{refl})). \tag{4}$$

Because any two functions with values on the type 0 are equal by extensionality, no point of $\nabla 2$ can be distinct from both 0 and 1, although one cannot assert that it is either 0 or 1 without getting an unwanted instance of excluded middle. There is a map $2 \to \nabla 2$ which is the identity on points by the above abuse of notation. Is there a small, definable version of $\nabla 2$?

Is there a (small or large) definable type $\mathbb{S}$ whose interpretation in the topological topos is the Sierpinski space, so that one can work with the intrinsic topology $\mathbb{S}^X$ on any type X, as in [6, 21, 1, 16], rather than the intrinsic sequential topology $X^{\mathbb{N}_\infty}$ considered here. Is the interpretation of $\mathbb{S}$ in the topological topos the Sierpinski space?

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