Kreisel's counter-example to full abstraction of the set-theoretical model of Gödel's system T

Martín Escardó

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Introduction. I have written this note because it doesn't seem to be well known that the set-theoretical model of Gödel's system T fails to be fully abstract. By the context lemma, full abstraction is equivalent to the statement that the substructure of definable elements is extensional; that is, that any two definable functionals that agree on definable arguments must agree on all arguments. Kreisel gave a counter-example to this, reported in [1, page 581, Exercise 1] by Barendregt. A ground-valued definable functional that is constant on definable arguments but non-constant on arbitrary arguments is exhibited.

The surprise in the construction is that exhaustive search over the one-point compactification of the discrete natural numbers is definable in system T. Once one knows this, it is straightforward to produce the counter-example, using the well-known fact that definable functionals $2^{\mathbb{N}} \to 2$ are continuous with respect to the discrete topology on 2 and the product topology on $2^{\mathbb{N}}$. The connection with exhaustive search is in fact made explicit in Exercise 2 of [1, page 581]. We propose a slight improvement of the suggested solution, and use this to formulate Kreisel's counter-example.

System T. We take system T to be the simply typed lambda-calculus with base types for natural numbers (including zero, successor and higher-type primitive recursion) and booleans (including if-then-else).

The set-theoretical model. The set theoretical model interprets the type of natural numbers as the set $\mathbb{N}$ of natural numbers and the type of booleans as the set $2 = \{0, 1\}$, with the convention that 0 is false and 1 is true. Function types are interpreted as exponentials in the category of sets (=sets of all functions).

The one-point compactification of $\mathbb{N}$. Let

$$\mathbb{N}_\infty = \{x \in 2^{\mathbb{N}} \mid \forall i \leq j . x_i \leq x_j\} = \{\bar{n} \mid n \in \mathbb{N}\} \cup \{\infty\},$$

where we write $\bar{n} = 0^n 1^\omega$ and $\infty = 0^\omega$. The last equation requires classical logic but we don't worry about this (see below). The argument given below uses the fact that definable functions $2^{\mathbb{N}} \to 2$ are continuous with respect to the discrete topology on 2 and the product topology on $2^{\mathbb{N}}$ (Cantor space), and the fact that $\infty = \lim_n \bar{n}$ in this topology. Notice that the pointwise order on $\mathbb{N}_\infty$ restricts to the natural order on $\mathbb{N}$ and has ∞ as a top element. For $\alpha \in 2^{\mathbb{N}}$, denote by $\alpha + 1$ the sequence 0α. Then $\bar{n} + 1 = \overline{n+1}$ and $\infty + 1 = \infty$.

Exhaustive searchability of $\mathbb{N}_\infty$. Define $\varepsilon_{\mathbb{N}_\infty} : 2^{2^{\mathbb{N}}} \to 2^{\mathbb{N}}$ by

$$\varepsilon_{\mathbb{N}_\infty}(p)(i) = 1 \text{ iff } p(\bar{n}) \text{ for some } n \leq i.$$

Because the bounded existential quantification can be reduced to primitive recursion, this is T-definable. By construction,

$$\varepsilon_{\mathbb{N}_\infty}(p) = \overline{\mu n . p(\bar{n})} \text{ if } p(\bar{n}) \text{ for some } n \in \mathbb{N},$$

and

$$\varepsilon_{\mathbb{N}_\infty}(p) = \infty \text{ if there isn't } n \in \mathbb{N} \text{ such that } p(\bar{n}).$$

Or, combining these two statements,

$$\varepsilon_{\mathbb{N}_\infty}(p) = \inf\{x \in \mathbb{N}_\infty \mid p(x)\}$$

because the infimum of the empty set is always the top element. Hence the image of $\varepsilon_{\mathbb{N}_\infty}$ is $\mathbb{N}_\infty$, and

$$p(\varepsilon_{\mathbb{N}_\infty}(p)) \text{ iff } p(x) \text{ for some some } x \in \mathbb{N}_\infty.$$

In the terminology of [2], the set $\mathbb{N}_\infty$ is searchable. Notice that $\varepsilon_{\mathbb{N}_\infty}$ implements search even for *discontinuous* p.

Kreisel's counter-example. Define $\mathrm{test}\colon 2^{2^{\mathbb{N}}} \to 2$ by

$$\mathrm{test}(f) = f\,[\inf x \in \mathbb{N}_\infty . f(x+1) = f(\infty)] = f(\infty) \implies f(\bar{0}) = f(\infty).$$

By the above, this is T-definable. If $f\colon 2^{\mathbb{N}} \to 2$ is continuous, then $\mathrm{test}(f)$ holds, because there is $n \in \mathbb{N}$ such that $f(\bar{n}) = f(\infty)$ as $\lim_n \bar{n} = \infty$. Hence the functional test is constant on T-definable arguments. However, $\mathrm{test}(f)$ fails for f defined by

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha = \infty, \\ 0 & \text{otherwise,} \end{cases}$$

because in this case $[\inf x \in \mathbb{N}_\infty . f(x+1) = f(\infty)] = \infty$ and hence

$$f\,[\inf x \in \mathbb{N}_\infty . f(x+1) = f(\infty)] = f(\infty) \text{ but } f(\bar{0}) \neq f(\infty).$$

A fully abstract model. Kleene–Kreisel functionals (exponentials in k-spaces starting from discrete natural numbers and booleans) are an example, using the fact that the Kleene–Kreisel density theorem gives dense sequences which are actually definable in system T.

Classical logic. The above argument relies on classical logic (in a weak form: a sequence is either ∞ or not). This is necessarily the case. From a fully abstract model, taking presheaves one gets a topos (model of bounded set theory based on intuitionistic logic) that gives an equivalent fully abstract model!

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References

[1] H.P. Barendregt. *The Lambda-Calculus: its Syntax and Semantics.* North-Holland, 1984.

[2] M.H. Escardó. Infinite sets that admit fast exhaustive search. In L. Ong, editor, *22nd Annual IEEE Symposium on Logic in Computer Science*, pages 443–452. IEEE Computer Society, 2007.