Topology via higher-order intuitionistic logic

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These evolving notes will eventually be used to write a paper

Martín Escardó

Abstract

When excluded middle fails, one can define a non-trivial topology on the one-point set, provided one doesn't require all unions of open sets to be open. Technically, one obtains a subset of the subobject classifier, known as a dominance, which is to be thought of as a "Sierpinski set" that behaves as the Sierpinski space in classical topology. This induces topologies on all sets, rendering all functions continuous.

Because virtually all theorems of classical topology require excluded middle (and even choice), it would be useless to reduce other topological notions to the notion of open set in the usual way. So, for example, our synthetic definition of compactness for a set X says that, for any Sierpinski-valued predicate p on X, the truth value of the statement "for all $x \in X$, $p(x)$" lives in the Sierpinski set. Similarly, other topological notions are defined by logical statements.

We show that the proposed synthetic notions interact in the expected way. Moreover, we show that they coincide with the usual notions in certain classical topological models, and we look at their interpretation in some computational models.

1 Introduction

For suitable topologies, computable functions are continuous, and semidecidable properties of their inputs/outputs are open, but the converses of these two statements fail. Moreover, although semidecidable sets are closed under the formation of finite intersections and recursively enumerable unions, they fail to form the open sets of a topology in a literal sense. Some authors have attempted to resolve this mismatch by considering effective or constructive versions of topology. (For the moment, we don't include bibliographic references in this note.)

In recent work, recorded in the *Barbados notes*, we instead propose a synthetic approach, in which both classical topology and various computational flavours arise as special cases. The idea is to (i) take continuity to mean definability in a given base language, (ii) reduce other topological notions (such as open set, closed set, compact set, discrete space, Hausdorff space) to that of continuity with the aid of the Sierpinski space, and (iii) use the lambda calculus to prove theorems about them. This is developed in detail in the Barbados notes, with some interesting computational applications.

In this note we reformulate and redevelop synthetic topology in the internal language of a topos. Exploiting the fact that excluded middle fails, one can define a non-trivial topology on the one-point set. One cannot postulate all unions of open sets to be open, because otherwise we would quickly conclude that all sets would be open. Technically, one obtains a subset of the subobject classifier, known as a dominance, which is to be thought of as a "Sierpinski set". This induces topologies on all sets, rendering all functions continuous.

In certain toposes sharing a full subcategory with that of topological spaces, such as gros toposes and Johnstone's topological topos, the Sierpinski object of the topos can be taken as the Sierpinski space. Thus, the failure of closure under arbitrary unions doesn't come from effectivity considerations, as in the first paragraph, but rather from the fact that the internal and external meanings of "all" are different.

Because virtually all theorems of classical topology require excluded middle (and even choice), it would be useless to reduce other topological notions to the notion of open set in the usual way. So, for example, our synthetic definition of compactness for a set X says that, for any Sierpinski-valued predicate p on X, the truth value of the statement "for all $x \in X$, $p(x)$" lives in the Sierpinski set. Similarly, other topological notions are defined by logical statements.

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2 The setting

If pressed, we say that our set-theoretical deveopment can be formalized in the internal language of the free topos. We sometimes consider the free topos with natural-numbers object, but much of what we say doesn't depend on natural numbers. As hinted in the introduction, we'll be concerned with interpretations of formulas in toposes other than the free one.

In practice, we work within an informal set theory in which (1) the principle of excluded middle is not assumed to hold, and (2) although it is possible to quantify over the subsets of a set (because we have powersets), it is not possible to quantify over all sets. Regarding (2), universal quantifications over all sets do occur in some theorems, but are to be understood in a schematic sense. Such quantifications cannot be used in order to construct sets. Attitude (1) doesn't reflect any philosophical position. Rather it amounts to a way of obtaining a set-theoretical universe in which all sets are spaces and all functions are automatically continuous. At this point, this is articulated by a logical statement. An equivalent mathematical statement occurs below. Just as some groups are commutative and others are not, some set-theoretical universes satisfy the principle of excluded middle and others don't. Moreover, we shall make use of some universes which do in order to manufacture suitable universes which don't, using standard topos-theoretic technology. Attitude (2) is what allows us to use toposes as examples of such universes, and it is conceivable that a different mathematical machinery will allow it to be done away with.

Axiom of choice. Because the full axiom of choice implies excluded middle, we can't assume it. However, there are toposes in which excluded middle fails and yet some instances of the axiom of choice hold. These will in fact arise in some applications of the theory.

Excluded middle. Not only we don't assume that the principle of excluded middle holds, but also we postulate axioms that imply its failure. One such axiom might be the statement that all functions $f : \mathbb{R} \to \mathbb{R}$ are continuous in the ϵ–δ sense, or an statement that implies this. With excluded middle, one can define non-continuous functions, and hence excluded middle must fail. Notice that this doesn't mean that one can find a proposition (statement with no free variables) p such that $\neg(p \vee \neg p)$, i.e. such that $\neg p \wedge \neg\neg p$, which is certainly absurd.

The powerset of the one-point set. The principle of excluded middle is equivalent to saying that $\mathcal{P}\,1 \subseteq \{\{\top\}, \emptyset\}$, where 1 is the one point set with unique element denoted by $\top$. To see that this implies excluded middle, consider the set $\{x \in 1 \mid p\}$ for a given proposition p. The hypothesis $\mathcal{P}\,1 \subseteq \{\{\top\}, \emptyset\}$ then shows that this set is either $\{\top\}$ or $\emptyset$, i.e. that either p holds or $\neg p$ holds. Because the set 1 has only one element, it is convenient to write $\{\top \mid p\}$ for $\{x \in 1 \mid p\}$, and this is the notation that we shall adopt in order to avoid having to come up with a name for the variable in contexts where the name x is needed for other purposes.

This shows that the failure of excluded middle is equivalent to the mathematical statement $\mathcal{P}\,1 \not\subseteq \{\{\top\}, \emptyset\}$. As above, this doesn't mean that we can find $S \subseteq 1$ with $S \neq \{\top\}$ and $S \neq \emptyset$, which is again absurd. Thus, even though one can consistently claim that $\mathcal{P}\,1 \not\subseteq \{\{\top\}, \emptyset\}$, it is absurd to claim that subsets other than $\{\top\}$ and $\emptyset$ can be found. Yet, as we have seen, there are "fishy" sets such as $\{\top \mid p\}$, which we cannot claim to be either $\{\top\}$ or $\emptyset$ unless we know that $p \vee \neg p$. In fact, our synthetic development of topology is based on the occurence of such sets. (We learned this terminology for this kind of set from Peter Schuster, but he attributed it to someone else.)

The set of propositions. Propositions may be generally indentified with subsets of the one point set 1. Given a proposition p, we form the set $\{\top \mid p\}$, and, conversely, given a set $S \subseteq 1$, we form the proposition "$\top \in S$". These processes are easily seen to be mutually inverse up to equivalence of propositions. This is a meta-theoretical construction, but it is common practice to identify equivalent propositions and work as if they formed a set Ω, which is then in bijection with $\mathcal{P}\,1$ by the above construction.

The set Ω has two elements $\top$ (true) and $\bot$ (false), and, as above, although it is absurd to claim that there are elements other than $\top$ and $\bot$, claiming that these are the only elements is equivalent to the principle of excluded middle. The elements of Ω are conveniently thought of as "degrees of truth", ordered by implication, and hence with $\bot$ the minimal and $\top$ the maximal degree. Thus, for instance, rather than asking whether $\{\top \mid p\} = \{\top\}$ is true, it makes more sense to to ask to what extent it holds. The answer in this example is p.

We have set the notation so that $1 \subseteq \Omega$. We also consider the subset $2 = \{\bot, \top\}$ of Ω. With this notation, excluded middle is equivalent to the statement $\Omega = 2$.

Characteristic functions. The subsets of a set X are in bijection with the functions $X \to \Omega$. Given $S \subseteq X$, one has a function $\chi_S \colon X \to \Omega$ defined by $\chi_S(x) =$ "$x \in S$", and, conversely, given $\chi \colon X \to \Omega$ one has a set $\{x \in X \mid \chi(x)\}$, and these contructions are mutually inverse.

Finite sets. In the absence of excluded middle, the various classically equivalent notions of finiteness are potentially different (in the sense that they differ in some models of the theory although they may coincide in others). We work with a notion attributed to Kuratowski. Given a set X, define a *Kuratowski system* to be a subset K of $\mathcal{P}\,X$ such that $\emptyset \in K$, $\{x\} \in K$ for every $x \in X$, and $A \cup B \in K$ whenever $A, B \in K$. Because Kuratowki systems are closed under the formation of arbitrary intersections, there is a smallest one, denoted by $\mathcal{K}\,X$. A subset S of X is said to be *finite* if $S \in \mathcal{K}\,X$. Notice that $\mathcal{K}\,1 = \{\{\top\}, \emptyset\}$, because this is a Kuratowski system.

(The set $\mathcal{K}\,X$ equipped with the empty set and the binary-union operation is a semilattice. In fact, it is the semilattice freely generated by X. More precisely, the inclusion map $X \to \mathcal{K}\,X$ that sends x to $\{x\}$ is universal among maps into semilattices. In a theory without powersets, the above impredicative definition of $\mathcal{K}X$ as the intersection of all Kuratowski systems can be replaced by this universal property, but here we are not concerned with such issues.)

The above construction gives a notion of finite subset of a set. A set is called finite if it is a finite subset of itself. It is a routine exercise to prove that a set is finite if and only if it is a finite subset of any set of which it is a subset. That is, $X \in \mathcal{K}X$ if and only if $X \in \mathcal{K}Y$ for any $Y \supseteq X$.

There is no reason why the intersection of two finite subsets of a set should be again finite. This should to be compared to the fact that the intersection of two compact subsets of a topological space doesn't need to be compact.

A subset of a set X is finite if and only if it is a compact element of the powerset of X, in the usual lattice-theoretic sense. That is, define a collection of subsets of X to be directed if its inhabited and if any two members of the collection are contained in a third member of the collection. Then a subset of X is called compact if whenever it is contained in the union of a directed collection of subsets of X, it is already contained in a member of the collection. To show that a finite subset is compact, argue that the compact subsets form a Kuratowski system. Conversely, to show that a compact set is finite, observe that any set is (the union of its singleton subsets and hence) the directed union of its finite subsets.

3 Open subsets, closed subsets, and subspaces

In the next section we topologize all sets simultaneously in such a way that all functions are continuous. From the open sets provided by the topologies, we define other types of subset, in particular subspaces and closed subsets. Not all subsets of a given set will be subspaces, but open and closed subsets will always be.

These three types of subset will share the following "reflexivity" and "transitivity" properties:

1. (a) X is an open subset of X, and (b) if U is an open subset of X and U' is an open subset of U then U' is an open subset of X.
2. (a) X is a closed subset of X, and (b) if C is a closed subset of X and C' is a closed subset of C then C' is a closed subset of X.
3. (a) X is a subspace of X, and (b) if S is a subspace of X and S' is a subspace of S then S' is a subspace of X.

Moreover, there will be three distinguished subsets Ω_o, Ω_c and Ω_s of the set Ω of propositions, called *open*, *closed* and *spatial* propositions, such that

1. $U \subseteq X$ is open iff for every $x \in X$, the proposition "$x \in U$" is open.
2. $C \subseteq X$ is closed iff for every $x \in X$, the proposition "$x \in C$" is closed.
3. $S \subseteq X$ is open iff for every $x \in X$, the proposition "$x \in S$" is spatial.

Thus, the open subsets, the closed subsets, and the subspaces are uniquely determined by the knowledge of Ω_o, Ω_c and Ω_s. Later on, Ω_c and Ω_s will be uniquely determined by the knowledge of Ω_s. It follows from this that

1. inverse images of open subsets are open,
2. inverse images of closed subsets are closed,
3. inverse images of subspaces are subspaces,

where the first and the second say that all functions are continuous, in two different ways. We use the following notation:

1. $\mathcal{O}\,X$ is the set of open subsets of X.

2. $\mathcal{C}\,X$ is the set of closed subsets of X.

3. $\mathcal{S}\,X$ is the set of subspaces of X.

For the particular case $X = 1$, where 1 is the one-point set with unique element denoted by $\top$, we conclude that

1. The map $U \mapsto$ "$\top \in U$" is a bijection $\mathcal{O}\,1 \to \Omega_{\mathrm{o}}$ with inverse $u \mapsto \{\top \mid u\}$.

2. The map $C \mapsto$ "$\top \in C$" is a bijection $\mathcal{C}\,1 \to \Omega_{\mathrm{c}}$ with inverse $c \mapsto \{\top \mid c\}$.

3. The map $S \mapsto$ "$\top \in S$" is a bijection $\mathcal{S}\,1 \to \Omega_{\mathrm{c}}$ with inverse $s \mapsto \{\top \mid s\}$.

4 Topologizing all sets simultaneously

A (synthetic) *topology* on a set is a set of subsets of the set, called *open*, subject to suitable axioms. In order to topologize all sets, we first topologize the one-point set $1 = \{\top\}$, and from this topology we derive topologies for the other sets.

Of course, non-trivial topologies on the one-point set exist only if excluded middle fails. However, even in the absence of excluded middle, if we assume that $\{\top\}$ is open and that arbitrary unions of open subsets are open, we still arrive at the trivial situation in which all subsets are open. The reason is that, for any $S \subseteq 1$, we have that $S = \bigcup\{\{\top\} \mid \top \in S\}$ and hence any $S \subseteq 1$ is a union of open sets.

Because of this and other reasons, in the axiomatization of synthetic topology we keep silent regarding what unions of open sets should be postulated to be open. This decision is postponed to the applications. This is justified by the fact that, provided the basic topological notions are suitably (re)defined from the notion of open set, much of basic topology survives this generalization, as we shall see. Occasionally, however, we consider the consequences of assuming that the topology on the one-point set is closed under the formation of finite unions (which will imply that the topology on any set also has this property).

In order to formulate the axioms for a topology, it is convenient to invoke an auxiliary notion.

4.1 Definition (Prototopology.) A *prototopology* on a set X is a set of subsets of X, again called *open*, subject to no axioms.

A prototopology on the one-point set 1 induces a prototopology on every set X, defined by stipulating that

4.2 Definition (Induced prototopology.) A set $U \subseteq X$ is open if, for each $x \in X$, the set $\{\top \mid x \in U\}$ is open in 1.

(This is the largest topology making the unique surjection $!\colon X \to 1$ into a topological quotient map, in the sense that $V \subseteq 1$ is open if and only if $!^{-1}(V)$ is open in X. (?))

4.3 Notation We denote an arbitrary prototopology on 1 by $\mathcal{O}\,1$, and the induced prototopology on a set X by $\mathcal{O}\,X$.

Of course, readers will ask themselves whether, and promptly verify that it is the case that, the prototopology on 1 induced by a protopology on 1 coincides with the inducing prototopology, so that there is no ambiguity in the notation $\mathcal{O}\,1$.

4.4 Definition (Synthetic topology.) A *topology* on the one-point set is a prototopology $\mathcal{O}\,1$ such that

S1. $\{\top\} \in \mathcal{O}\,1$, and

S2. if $U \in \mathcal{O}\,1$ and $V \in \mathcal{O}\,U$, then $V \in \mathcal{O}\,1$.

A *topology* on a set is a prototopology induced by a topology on the one-point set.

When there is no danger of ambiguity, we formulate the above requirements as

S1. $\{\top\}$ is open in 1, and

S2. if U is an open subset of 1 and V is an open subset of U, then V is open in 1.

We summarize the second requirement by saying that the open-subset relation is transitive on 1. Readers can easily verify that this implies that it is transitive on any set with the induced topology, as formulated below.

4.5 *If U is an open subset of a set X and V is an open subset of U, then V is open in X.*

This will imply that open subsets are subspaces (Section 13).

4.6 (Open subsets are closed under finite intersections.) *Any set X is open in X, and if U and V are open subsets of X, then $U \cap V$ is open.*

Proof To be included in a future version. But it is easy. □

Of course, we cannot assert that a set is open on its own — here we mean that $U \cap V$ is open in X, and analogous terminological measures will be adopted for the sake of economy in similar contexts.

4.7 (All functions are continuous.) *For any $f\colon X \to Y$ and any open subset V of Y, the set $f^{-1}(V)$ is open in X.*

Proof We have to show that, for each $x \in X$, the set $\{\top \mid x \in f^{-1}(V)\}$ is open. But this is $\{\top \mid f(x) \in V\}$, which is open because V is. □

This is nice, because we never have to check continuity, but there is a price to pay for that. For instance, although all sets are spaces, not all subsets will be subspaces (Section 13).

Before considering examples of topologies, we formulate the following technical lemma, which brings us to the next section.

4.8 Lemma (Rosolini 1986) *A prototopology $\mathcal{O}\,1$ on 1 is a topology if and only if*

1. *the set $\{\top\}$ is open, and*

2. *for any set $S \subseteq 1$ and any open set $U \subseteq 1$, if inhabitedness of U implies openness of S, then $S \cap U$ is open.*

Proof Will be provided in a future version of this note. □

This technical formulation of topology on the one-point set has the advantage of avoiding mention to induced prototopologies.

We won't have a chance of applying the following observation, which follows immediately from the above lemma:

4.9 (Protopologies as subbases of topologies.) *The topologies on the one-point set* 1 *are closed under the formation of intersections in* $\mathcal{P}\,1$*. Hence any prototopology can be extended to a smallest topology.*

5 Topology via logic

Rosolini formulated the above notion of topology on the one-point set using the following concept.

5.1 Definition (Dominance.) A *dominance* is a set Σ of propositions, called *open*, such that

D1. $\top$ is open, and

D2. for every proposition p and every open proposition u, if u implies that p is open, then $u \wedge p$ is open.

5.2 *Open propositions are closed under the formation of binary conjunctions.*

The following two propositions follow directly from Rosolini's Lemma 4.8.

5.3 (Topology induced by a dominance.) *A dominance gives rise to a topology on the one-point set* 1 *with open sets* $\{\top \mid u\}$ *for* u *open.*

5.4 (Dominance induced by a topology.) *A topology on the one-point set* 1 *gives rise to a dominance consisting of the propositions "*$\top \in U$*" for* $U \subseteq 1$ *open.*

We leave the routine verification of the following two propositions to the reader.

5.5 *The above two process are mutually inverse.*

5.6 (Dominances classify open subsets.) *For the topology* $\mathcal{O}\,1$ *on* 1 *induced by a dominance* Σ*, we have that the topology* $\mathcal{O}\,X$ *on* X *induced by* $\mathcal{O}\,1$ *consists of the sets* $U \subseteq X$ *whose characteristic functions* $\chi_U \colon X \to \Omega$ *take values in* Σ*.*

Hence the map $u \mapsto u^{-1}(\top)$ is a bijection $\Sigma^X \xrightarrow{\cong} \mathcal{O}\,X$ with inverse $U \mapsto \chi_U$. Thus, a dominance behaves as a the Sierpinski space in classical topology, and we shall sometimes refer to a dominance as a *Sierpinski set* for emphasis. The following, which is a reformulation of the previous proposition, will be our working definition of the notion of open set.

5.7 (Logical formulation of openness.) *A subset* U *of a set* X *is open if and only if, for each* $x \in X$*, the proposition "*$x \in U$*" is open.*

5.8 *If* $U \subseteq X$ *and* $V \subseteq Y$ *are open, then* $U \times V$ *is open in* $X \times Y$*.*

Proof We have to show that for any $z \in X \times Y$, the proposition "$z \in U \times V$" is open. But z is of the form (x, y) for $x \in X$ and $y \in Y$, and hence the proposition is equivalent to "$x \in U \wedge y \in U$", which is open because U and V are open and because open propositions are closed under binary conjunction. $\square$

5.9 (Standard topology.) *The following are equivalent:*

1. *$\perp$ is open and if u and v are open then so is $u \vee v$.*
2. *For any set, the empty subset is open, and the union of two open subsets is open.*

A topology satisfying these conditions is called *standard.* It is *substandard* if empty subsets are open.

Of course, the requirement that the empty subset of any set be open is equivalent to $\perp \in \Sigma$. Despite the terminology, we don't generally assume that we are working with a standard topology (we can't think of a better terminology though — suggestions are welcome).

Recall that a subset D of a set X is called *decidable* if for every $x \in X$, either $x \in D$ or $x \notin D$.

5.10 *For a substandard topology, decidable subsets are open.*

Proof Because their characteristic functions take values in $2 = \{\perp, \top\} \subseteq \Sigma$. □

We record the following for future use.

5.11 Lemma (Open subproposition of arbitrary proposition) *The following are equivalent for propositions p and u with $\{\top \mid u\} \subseteq \{\top \mid p\}$, i.e. such that $u \Rightarrow p$.*

1. *$\{\top \mid u\} \in \mathcal{O}\{\top \mid p\}$.*
2. *p implies that u is open.*

In this case, we say that u is an *open subproposition* of p, and we write $u \in \mathcal{O}\,p$.

We emphasize that if u is an open subproposition of p then u is not necessarily open. A proposition is open if and only if it is an open subproposition of $\top$. Notice that if v is open then $p \wedge v$ is an open subproposition of p for any p.

6 Examples of topologies

This will be expanded/elaborated later. In particular, we have to include examples of subspace classifiers.

6.1 Arbitrary toposes

The examples of dominaces given here are defined in the internal language (of the free topos). All the other examples given in this section are externally defined at it is open (as far as I know — check with Andrej Bauer) whether they can be defined in the internal language.

1. $\Sigma = 1 = \{\top\} \subseteq \Omega$. Then any set has precisely one open set, namely the whole set.
2. $\Sigma = \Omega$. Then all subsets of any set are open.
3. $\Sigma = 2 = \{\top, \perp\} \subseteq \Omega$. Then the open subsets of a set are the decidable subsets. Here a subset U of a set X is called decidable (or complemented) if $U \cup X \setminus U = X$, where $X \setminus U$ is the Heyting complement $\{x \in X \mid x \notin U\}$ of U in X.

4. $\Sigma = \{``\exists n \in \mathbb{N}, f(n) = 0" \mid f\colon \mathbb{N} \to \mathbb{N}\}$. This example occurs in synthetic domain theory and is often used in the effective topos.

5. Say that a proposition p is *dense* if $\neg p$ is false, which is equivalent to saying that $\neg\neg p$ holds. Then the set of dense propositions form a dominance.

6. The fixed-points of any local operator form a dominance.

6.2 Gros toposes

Consider any small full subcategory of the category of topological spaces, closed under the formation of open subspaces, and consider the topos of sheaves for any subcanonical coverage (e.g. the open-covering one defined in the obvious way). See e.g. MacLane and Moerdijk's book on sheaves and toposes or Johnstone's *Elephant*. Such a topos is known as a gros topos. The Sierpinski object can be taken as the functor represented by the Sierpinski space, if this space is in the category of spaces under consideration, and for representable sheaves the synthetic notion of open subobject coincides with that of open subspace. Of course, one can consider Gros toposes built out of small categories of locales having the Sierpinski locale as an object.

6.3 Johnstone's topological topos

This is a topos with the category of sequential spaces as a full subcategory. The reader is asked to check Johnstone's paper. The same Sierpinski object can be used, and the same observations apply.

6.4 Realizability toposes

Effective topos and relatives. One works with the internally defined dominance discussed above. This time the synthetic notion of openness captures the computational concept of semidecidability.

6.5 Toposes built out of programming languages

This is related to a construction studied by Bauer and Awodey. For a typed programming language such as PCF, one can form a category whose objects are the types and whose arrows are equivalence classes of programs with respect to observational (=contextual) equivalence (or any notion of equivalence deemed appropriate for the language under consideration). This yields a cartesian-closed category. By considering partial equivalence relations, one gets a cartesian closed embedding into a larger cartesian closed category. Then one considers the topos of sheaves for a suitable subcanonical topology. The Sierpinski object can be taken as the sheaf represented by a "Sierpinski data type" (with a top element corresponding to termination and a bottom element corresponding to non-termination). This class of examples is very much related to the gros toposes discussed above, and could be called computational gros toposes.

7 Compact sets and subsets

In classical topology, two potentially different notions of compactness agree. One is absolute and says that a space is compact if it enjoys the Heine-Borel property. In this case, we say that the space is compact on its own. The other relative and says that a subset/subspace Q of a space X is compact if it satisfies the Heine-Borel property within X; that is, every cover of Q by open subsets of X has a finite subcollection that already covers Q. In this case, we say that Q is compact in X. This terminology is unnecessary in the classical situation because a space Q is compact on its own if and only if it is compact in any space X that has Q as a subspace.

We formulate absolute and relative synthetic notions, but we don't take the Heine-Borel property as the defining condition. We use the fact that $\mathcal{O}\,X$ itself, being a set like any other, comes equipped with a topology $\mathcal{O}\,\mathcal{O}\,X$.

7.1 (Compact set.) *The following are equivalent for any a set X.*

1. *The set $\{X\}$ is open in $\mathcal{O}\,X$, i.e. $\{X\} \in \mathcal{O}\,\mathcal{O}\,X$.*
2. *For any open-valued property u of elements of X, the proposition "for all $x \in X$, $u(x)$" lives in the Sierpinski set.*

When these conditions hold, we say that X is *compact* (on its own).

7.2 (Compact subset.) *The following are equivalent for any subset A of a set X.*

1. *The set $\{U \in \mathcal{O}\,X \mid A \subseteq U\}$ is open in $\mathcal{O}\,X$.*
2. *For any open-valued property u of elements of X, the proposition "for all $x \in A$, $u(x)$" lives in the Sierpinski set.*

When these conditions hold, we say that A is *compact in* X, or that A is a *compact subset* of X.

7.3 (Images of compact subsets are compact.) *If A is a compact subset of X then $f(A)$ is a compact subset of Y for any function $f\colon X \to Y$.*

Proof The proposition "$\forall y \in f(A)$, $v(y)$" is equivalent to "$\forall x \in A$, $v(f(x))$". But, by definition of compactness of A, the latter is open if v is an open-valued property, and hence, because $u(x) = v(f(x))$ is an open-valued property, so is the former, which establishes the compactness of $f(A)$. □

Considering the inclusion map $A \hookrightarrow X$, the following consequence is obtained.

7.4 Corollary *If A is compact on its own, then A is compact in X for any X having A as a subset.*

The converse potentially fails; that is, there is no reason why compactness of A in X should imply compactness of A on its own. This is discussed in Section 13, where conditions that force it to hold are introduced.

If X and Y are compact then so is $X \times Y$. More generally:

7.5 *If $A \subseteq X$ and $B \subseteq Y$ are compact subsets, then $A \times B$ is compact in $X \times Y$.*

Proof "$\forall z \in A \times B$, $p(z)$" is equivalent to "$\forall x \in A$, $\forall y \in B$, $p(x, y)$". □

7.6 *The following hold for any set.*

1. *The union of two compact subsets is compact.*

2. *The finite subsets are compact.*

Proof (1): For subsets A and B of a set X, the proposition "$\forall x \in A \cup B,\ u(x)$" is equivalent to "$(\forall x \in A,\ u(x)) \wedge (\forall x \in B,\ u(x))$", which is open if A and B are compact and u is open-valued. Hence $A \cup B$ is compact if A and B are.

(2): The empty set is compact because the proposition "$\forall x \in \emptyset,\ u(x)$" is equivalent to "$\top$", which is open by definition. Singletons are compact because the proposition "$\forall x \in \{x_0\},\ u(x)$" is equivalent to "$u(x_0)$", which is open if u is open-valued. Hence the compact subsets form a Kuratowski system, and therefore they include the finite subsets. □

In view of the previous proposition, the following generalizes the fact that open sets are closed under the formation of finite intersections. It again relies on the fact that a topology on a set comes itself endowed with a topology. But it goes one level further — we implicitly refer to the topology $\mathcal{O}\,\mathcal{O}\,\mathcal{O}\,X$ to express the fact that a subset of the topology $\mathcal{O}\,X$ is compact. The formulation of compactness via universal quantification allows us to avoid this triple nesting of powersets in the proof.

7.7 *If $\mathcal{A} \subseteq \mathcal{O}\,X$ is compact then $\bigcap \mathcal{A}$ is open, and the converse also holds.*

Proof For any $x \in X$, the proposition "$x \in \bigcap \mathcal{A}$" is equivalent to "$\forall U \in \mathcal{A},\ x \in U$". The former is open for every x iff $\bigcap \mathcal{A}$ is open, and the latter is open for every x iff $\mathcal{A}$ is compact, because the property $u(U) =$ "$x \in U$" takes open values for any open set U. □

7.8 (Compact proposition.) *The following are equivalent for any proposition p.*

1. *$\{\top \mid p\}$ is a compact subset of 1.*

2. *For every open proposition u, the proposition "$p \Rightarrow u$" is open.*

Such a proposition is called *compact*.

8 Overt sets and subsets

Joyal introduced a certain notion of openness for locales, whose definition ressembles that of an existential quantifier as an adjoint formulated by Lawvere (see e.g. Johnstone's *Elephant*). When excluded middle holds, the notion is vacuous, as then all locales are open. Taylor reformulated this notion for the spaces of his *abstract Stone duality*, renaming it to overtness. For several reasons, Joyal's terminology is inconvenient and hence we adopt Taylor's. However, rather than working with adjoints as Joyal and Taylor, we work directly with existential quantifiers. The development of this section parallels that of Section 7.

8.1 (Overt set.) *The following are equivalent for any a set X.*

1. *The set $\{U \in \mathcal{O}\,X \mid U$ is inhabited$\}$ is open in $\mathcal{O}\,X$, i.e. belongs to $\mathcal{O}\,\mathcal{O}\,X$.*

2. *For any open-valued property u of elements of X, the proposition "there is $x \in X$ with $u(x)$" is open.*

When these conditions hold, we say that X is *overt* (on its own).

8.2 (OVERT SUBSET.) *The following are equivalent for any subset E of a set X.*

1. *The set $\{U \in \mathcal{O}\,X \mid E$ meets $U\}$ is open in $\mathcal{O}\,X$.*
2. *For any open-valued property u of elements of X, the proposition "there is $x \in E$ with $u(x)$" lives in the Sierpinski set.*

When these conditions hold, we say that E is *overt in* X, or that E is an *overt subset* of X.

8.3 (IMAGES OF OVERT SUBSETS ARE OVERT.) *If E is an overt subset of X then $f(E)$ is an overt subset of Y for any function $f\colon X \to Y$.*

PROOF The proposition "$\exists y \in f(E),\, v(y)$" is equivalent to "$\exists x \in E,\, v(f(x))$". □

8.4 COROLLARY *If E is overt on its own, then E is overt in X for any X having E as a subset.*

8.5 *If the topology is standard (Definition 5.9), then the following hold for any set.*

1. *The union of two overt subsets is overt.*
2. *The finite subsets are overt.*

PROOF (1): For subsets E and E' of a set X, the proposition "$\exists x \in E \cup E',\, u(x)$" is equivalent to "$(\exists x \in E,\, u(x)) \vee (\exists x \in E',\, u(x))$", which is open if E and E' are overt and u is open-valued. Hence $E \cup E'$ is overt if E and E' are.

(2): The empty set is overt because the proposition "$\exists x \in \emptyset,\, u(x)$" is equivalent to "$\bot$", which is open by hypothesis. Singletons are overt because the proposition "$\exists x \in \{x_0\},\, u(x)$" is equivalent to "$u(x_0)$", which is open if u is open-valued. Hence the overt subsets form a Kuratowski system, and therefore they include the finite subsets. □

In view of the previous proposition, the following generalizes the fact that open sets are closed under the formation of finite unions when we have a standard topology. However, the proposition itself doesn't assume that the topology is standard.

8.6 *If $\mathcal{E} \subseteq \mathcal{O}\,X$ is overt then $\bigcup \mathcal{E}$ is open, and the converse also holds.*

PROOF For any $x \in X$, the proposition "$x \in \bigcup \mathcal{E}$" is equivalent to "$\exists U \in \mathcal{E},\, x \in U$". □

If X and Y are overt then so is $X \times Y$. More generally:

8.7 *If $A \subseteq X$ and $B \subseteq Y$ are overt subsets, then $A \times B$ is overt in $A \times B$.*

PROOF "$\exists z \in A \times B,\, w(z)$" is equivalent to "$\exists x \in A,\, \exists y \in B,\, w(x, y)$". □

9 Discrete sets

9.1 (DISCRETE SET.) *The following are equivalent for any set X.*

1. *$\{x\}$ is open for every $x \in X$.*
2. *For every $x, y \in X$, the proposition "$x = y$" belongs to the Sierpinski set.*
3. *The diagonal $\{(x, x) \mid x \in X\}$ is open in $X \times X$.*

A set satisfying the above conditions is called *discrete*.

Recall that a set X is said to have *decidable equality* if its diagonal is decidable, i.e. for all $x, y \in X$, either $x = y$ or $x \neq y$. For example, the natural numbers have decidable equality, but the reals fail to have decidable equality in some models.

9.2 *For a substandard topology, a set with decidable equality is discrete.*

Proof This follows from Proposition 5.10, because in this case singletons are decidable sets. □

9.3 *A product of two discrete spaces is discrete.*

Proof The proposition "$(x, y) = (x', y')$" in $X \times Y$ is equivalent to the conjunction of the propositions "$x = x'$" and "$y = y'$". □

10 Closed subsets

There are at least four reasonable candidates for a notion of a closed subset F of a set X:

1. *Every limit point of F belongs to F.* Here $x \in X$ is called a limit point of F if every neighbourhood of x meets F.

2. *F is the complement of an open set.* This implies the previous condition.

3. *The complement of F is open.*

4. *For any $U \in \mathcal{O}F$, the set $(F \Rightarrow U) = \{x \in X \mid x \in F \Rightarrow x \in U\}$ is open.* (Observe that if excluded middle holds, then this set is $(X \setminus F) \cup U$.) Considering $U = \emptyset$, we see that this implies the previous condition.

The first condition is familiar from analysis, and the second gives examples of this situation. The third is familiar from the theory of computation and can be read as saying that its complement can be "detected" by observations with values in the Sierpinski set. For such F, the double complement

$$X \setminus (X \setminus F) = \{x \in X \mid \neg\neg(x \in F)\} \supseteq F$$

has the same complement as F and hence also satisfies the condition. Moreover, being the complement of an open set, the double complement additionally satisfies the second condition and hence the first. The fourth condition is probably unfamiliar (at least we haven't seen it before), but we have found it to be the right choice in a number of contexts. We adopt the following terminology for the above conditions:

1. F is *limit closed*.

2. F is the complement of an open set.

3. The complement of F is open.

4. F is *topologically closed*.

10.1 Terminology (Notions of closedness) We generally refer to the above four concepts as *notions of closedness*. However, we shall frequently refer to topologically closed sets simply as *closed sets* when there is no danger of ambiguity.

10.2 *For a subset F of X with $X \setminus (X \setminus F) = F$, the implications* $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ *hold.*

The following is an immediate consequence.

10.3 *For a substandard topology, a decidable subset is closed in the four senses.*

The following will be our working definition of the notion of closed set:

10.4 (Logical formulation of closedness.) *A subset F of a set X is closed if and only if for every open $U \in \mathcal{O}\, F$ and for every every $x \in X$, the proposition "$x \in F \Rightarrow x \in U$" is open.*

The following observation is used repeatedly without explicit mention. The point is that we can consider open subsets of X rather than of F.

10.5 *If a subset F of X is closed, then for every open $U \in \mathcal{O}\, X$ and for every every $x \in X$, the proposition "$x \in F \Rightarrow x \in U$" is open.*

The converse if not true, but see Proposition **??**.

10.6 *Finite unions of closed subsets are closed.*

Is the inverse image of a closed set closed? If not, can we find reasonable sufficient conditions. In any case, if not, then this will be too bad, independently of whether one can find sufficient conditions. Notice that this is equivalent to saying that the closed propositions form a dominance. Hence we could simply postulate that, as it is done in SDT (synthetic domain theory). Of course, we have to make sure that the intended models satisfy the condition, or else suitably adapt the intended models (e.g. by considering suitable coverages for the construction of gros toposes).

10.7 (Closed subsets of compact sets are compact.) *If $F \subseteq X$ is closed and X is compact then F is a compact in X.*

Proof "$\forall x \in F,\, p(x)$" is equivalent to "$\forall x \in F,\, x \in F \implies p(x)$". □

More generally:

10.8 *If $F \subseteq X$ is closed and $A \subseteq X$ is compact $F \cap A$ is a compact subset of X.*

Proof "$\forall x \in F,\, p(x)$" is equivalent to "$\forall x \in F,\, x \in F \implies p(x)$". □

For future use, notice that every set is contained in a smallest limit closed set, called its limit closure, which is simply the set of its limit points. As we shall see, limit closed sets arise in connection with overt sets.

We now analyse the concept of closed set via closed propositions.

10.9 (Closed proposition.) *The following are equivalent for any proposition f.*

1. $\{\top \mid f\}$ *is a closed subset of* 1.
2. *For every open subproposition u of f, the proposition "$f \Rightarrow u$" is open. (Cf. Lemma 5.11).*
3. *f is compact. (Cf. Lemma 7.8.)*

By construction, the closed propositions classify closed subsets.

10.10 *A subset F of a set X is closed if and only if for every every $x \in X$, the proposition "$x \in F$" is closed.*

Closed sets in SDT. We take some axioms for SDT from Rosolini's *notes on synthetic domain theory* (available from his web page). We assume that Σ is a substandard dominance such that the following two conditions hold:

Markov principle $u \in \Sigma \Rightarrow \neg negu = u$.

Co-dominance The set $\Sigma_\neg = \{\neg u \mid u \in \Sigma\}$ is a dominance.

[[Emphasize in an appropriate place: "complements" are Heyting complements.]]

10.11 (Closed sets in SDT) *Under these assumptions, the following are equivalent for any proposition f.*

1. *f is the complement of an open proposition (we say that f is sdt-closed).*
2. *The complement of f is open.*
3. *f is topologically closed.*

If f is topologically closed, then we already know that its complement is open. If the proposition $u = \neg f$ is open, then $\neg u = \neg\neg f$.

11 Hausdorff sets

Because there are various notions of closed set, there are various corresponding notions of Hausdorff set. All of them say that the diagonal is closed in the appropriate sense.

11.1 *For a substandard topology and any of the four notions of closed set considered in the previous section, a set with decidable equality is Hausdorff.*

So, for example, the natural numbers are Hausdorff. Recall that a set X is called $\neg\neg$-separated if for all $x, y \in X$, $\neg\neg(x = y)$ implies $x = y$. For example, the set of Dedekind reals satisfies the condition. Then Proposition 10.2 shows that, in this case, the four notions of Hausdorff set are linearly ordered by implication, with the strongest saying that the diagonal is closed and the weakest that it is limit-closed.

11.2 (Hausdorff set.) *The following are equivalent for any set X.*

1. *Its diagonal is closed.*
2. *For every open set $U \subseteq X$, and all $x, y \in X$, the proposition "$x = y \Rightarrow x \in U$" is open.*

Such a set is called *Hausdorff*.

11.3 *A compact subset of a Hausdorff set is closed.*

Proof Let A be a compact subset of a Hausdorff space X, and $U \in \mathcal{O}\,A$. We have to show that, for any $x \in X$, the proposition "$x \in A \implies x \in U$" is open. But this is equivalent to "$\forall a \in A,\ a = x \Rightarrow a \in U$", which is open because A is compact and because "$a = x \Rightarrow a \in U$" is open for every a by the Hausdorff property of X. $\square$

(Can we show that (probably for a standard topology) the product of two Hausdorff space is Hausdorff?)

12 Function spaces

13 Subspaces

If X is a subset of a set Y, with inclusion denoted by $i\colon X \to Y$, then $i^{-1}(V) = X \cap V$ is an open subset of X for every open subset V of Y, because all functions are continuous. The following definition requires that this map be a surjection:

13.1 Definition (Subspace.) A subset X of a set Y is called a *subspace* if for every open set $U \subseteq X$ there is an open set $V \subseteq Y$ with $U = X \cap V$.

A stronger condition is that it be a retraction:

13.2 Definition (Well embedded subspace.) A subset X of a set Y is called a *well embedded subspace* if there is a map $e\colon \mathcal{O}\,X \to \mathcal{O}\,Y$ with $U = X \cap e(U)$ for every $U \in \mathcal{O}\,X$.

An even stronger condition is that it be a surjection with a right adjoint:

13.3 Definition (Perfectly embedded subspace.) A subset X of a set Y is called a *perfectly embedded subspace* if there is a map $e\colon \mathcal{O}\,X \to \mathcal{O}\,Y$ with $U = X \cap e(U)$ and $V \subseteq e(X \cap V)$ for all $U \in \mathcal{O}\,X$ and $V \in \mathcal{O}\,Y$.

In classical topology, because we can take the union of all open subsets V of Y with $X \cap V = U$, this right adjoint always exists. But it is not always continuous. When it is, in an appropriate sense, the inclusion map has been called perfect (and sometimes proper, and other times semiproper), and this is the reason for our terminology.

13.4 *Any closed subset is a perfectly embedded subspace.*

Proof If F is closed subset of a set X and $U \in \mathcal{O}\,F$, then $(F \Rightarrow U)$ is the largest open set V in X with $F \cap V = U$. □

An alternative strengthening of the notion of well embedded subspace is obtained by considering a left adjoint to the surjection.

13.5 Definition (Essentially embedded subspace.) A subset X of a set Y is called an *essentially embedded subspace* if there is a map $e\colon \mathcal{O}\,X \to \mathcal{O}\,Y$ with $U = X \cap e(U)$ and $e(X \cap V) \subseteq V$ for all $U \in \mathcal{O}\,X$ and $V \in \mathcal{O}\,Y$.

13.6 *Any open subset is an essentially embedded subspace.*

Proof By definition of topology, if X is an open subset of Y then any open subset of X is an open subset of Y. Hence the inclusion $e\colon \mathcal{O}\,X \hookrightarrow \mathcal{O}\,Y$ is the desired left adjoint. □

The notions of perfect and essential embedding can be generalized to any map (Section 15).

For the purposes of doing topology via logic, we reformulate one of the above definitions:

13.7 *A subspace X of a set Y is well embedded iff there is a map $(p \mapsto \bar{p}) \colon \Sigma^X \to \Sigma^Y$ with $\bar{p}(x) = p(x)$ for all $x \in X$.*

For the moment, the only motivation to consider such notions consists of the following two propositions:

13.8 *If a compact subspace A of a set X is well embedded, then A is compact on its own.*

Proof Let p be a Sierpinski-valued property of elements of A. We have to show that the proposition "$\forall x \in A,\, p(x)$" lives in the Sierpinski set. But this is equivalent to "$\forall x \in A,\, \bar{p}(x)$", which lives in the Sierpinski space by compactness of A in X. $\square$

13.9 *If an overt subspace E of a set X is well embedded, then E is overt on its own.*

(Explain the situation regarding compactness in gross toposes.)

13.10 *Singleton subsets are subspaces.*

Because every set is the join of its singleton subsets, we cannot expect spaces to be closed under the formation of unions unless all subsets are subspaces. We are not able show that they are closed under finite unions. In fact, we don't even have a proof that doubleton subsets are subspaces. The situation isn't any easier for intersections.

The following hold for the three notions of subspace discussed above:

13.11 *If Y is a subspace of Z and X is a subspace of Y, then X is a subspace of Z.*

13.12 *If X is a subspace of Y then Z^X is a subspace of Z^Y.*

14 The Heine-Borel property

15 Injective spaces

16 Axioms for synthetic topology

17 Miscelanea

17.1 (Weakly Hausdorff set) *The following are equivalent for any set X.*

1. *The complement of $\{x\}$ is open for every $x \in X$.*
2. *For every $x, y \in X$, the proposition "$x \neq y$" belongs to the Sierpinski set.*
3. *The complement of the diagonal $\{(x,x) \mid x \in X\}$ is open in $X \times X$.*

Such a set is called *weakly Hausdorff*.

Notice that although the third condition is a familiar way of defining the notion of a Hausdorff space, the first is a familiar way of defining the weaker notion of a T_1 space.

17.2 (T_1 SET.) *The following are equivalent for any space X.*

1. *The singleton subsets of X are limit-closed.*
2. *For $x, y \in X$, if every neighbourhood of x is a neighbourhood of y then $x = y$.*

17.3 (SPECIALIZATION ORDER AND T_0 SET.) *The following are equivalent for any two points x and y of a set X.*

1. *Every neighbourhood of x is a neighbourhood of y.*
2. *x is a limit point of $\{y\}$.*

In this case we write $x \leq y$. This relation is reflexive and transitive, and it is known as the *specialization order*. If it is antisymetric (i.e. any two points with the same neighbourhoods must be the same), then the set X is called T_0. The specialization order is the identity if and only if X is T_1.

18 Questions

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