Injective locales over perfect embeddings and algebras of the upper powerlocale monad

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Abstract

We show that the locales which are injective over perfect sublocale embeddings coincide with the underlying objects of the algebras of the upper powerlocale monad, and we characterize them as those whose frames of opens enjoy a property analogous to stable supercontinuity.

Key words: Injective locale, perfect embedding, powerlocale, free frame, Kock–Zöberlein monad, stably supercontinuous lattice.

MSC: 06D22, 54C20, 54D80, 06B35

1 Introduction

An object D of a category is said to be injective over a map $j\colon X \to Y$ if for every map $f\colon X \to D$ there is at least one map $\bar{f}\colon Y \to D$ with $\bar{f} \circ j = f$. The injectives over subspace embeddings in the category of T_0 topological spaces are known to be precisely the continuous lattices under the Scott topology [24]. Although the extension $\bar{f}$ of f is far from unique in general, in this case there is a canonical choice, the greatest extension in the pointwise specialization order, denoted by f/j. It is natural to ask whether, for X and Y exponentiable topological spaces and for D injective over subspace embeddings, the greatest-extension operator $f \mapsto f/j\colon D^X \to D^Y$ is continuous [24]. It was shown in [6] that, excluding the trivial situation in which D is the one-point space, this is the case if and only if the embedding $j\colon X \to Y$ is a perfect map.

Here a continuous map $g\colon X \to Y$ of topological spaces is called perfect if the right adjoint of the frame homomorphism $g^{-1}\colon \mathcal{O}Y \to \mathcal{O}X$ preserves directed joins, where $\mathcal{O}X$ and $\mathcal{O}Y$ are the frames of open sets of X and Y. Hofmann and Lawson [12] showed that, for X and Y sober, this is equivalent to saying that (1) for every closed set $C \subseteq X$, the lower set of the direct image $g[C]$ in the specialization order of Y is closed and (2) the inverse image $g^{-1}(Q)$ of every compact saturated set $Q \subseteq Y$ is compact (saturated). Hofmann [11] showed that, under the additional assumption of local compactness, the first condition follows automatically from the second. Notice that the perfect maps are the continuous maps satisfying half of the definition of proper map of locales [27], where the missing half is the Frobenius condition. Topologically, the proper maps are known to be precisely the closed continuous maps that reflect compact saturated sets.

Perfect maps, under various names and guises, arise frequently in topology and locale theory. It is a folkloric fact that the category of perfect maps of stably locally compact spaces is equivalent to Nachbin's category [20] of monotone continuous maps of compact order-Hausdorff spaces — see e.g. [9]. This is an extension of the equivalence of the category of perfect maps of spectral spaces with that of monotone continuous maps of ordered Stone spaces, previously established by Priestley [21]. Localic versions and variations of this can be found in [2, 26, 7] — see also Section 4 below.

Coming back to the subject of the first paragraph, the second natural question is what are the injective spaces over perfect embeddings. It was also shown in [6] that they coincide with the algebras of the upper powerspace monad, but an intrinsic characterization of the algebras was left as an open problem.

This paper solves this problem in the localic case and shows that the characterization of the injectives as algebras also holds in this setting. Notice that the notion of perfect map as defined for topological spaces makes sense for locales: A continuous map of locales is called *perfect* if the right adjoint of its defining frame homomorphism preserves directed joins. For a definition of the upper powerlocale monad and its basic properties, see Section 2 below. If $u, v \in \mathcal{O}\,X$ are opens of a locale X, we write $u \prec v$ to mean that every Scott closed subset C of $\mathcal{O}\,X$ with $v \leq \bigvee C$ has u as a member.

Theorem 1.1 *The following are equivalent for any locale X.*

1. *X is injective over perfect sublocale embeddings.*
2. *X is the underlying locale of an algebra of the upper powerlocale monad.*
3. *(a) Every open $v \in \mathcal{O}\,X$ is the join of the set $\{u \in \mathcal{O}\,X \mid u \prec v\}$, and*
   (b) $1 \prec 1$, and $u \prec v$ and $u \prec w$ together imply $u \prec v \wedge w$.

Schalk [22, 23] characterized the algebras of the restriction of the monad to the category of locally compact locales using a different criterion. As already pointed out by her, the monad on the whole category is of the Kock–Zöberlein type [18]. Convenient references for our purposes are [5] or [8]. By a general result established in [6, Theorem 4.2.2], the underlying objects of the algebras of this monad coincide with the locales which are injective over perfect embeddings, using the characterization of perfect maps given in Vickers [28]. Moreover, it also follows from [6, Theorem 4.2.2] that the greatest-extension property discussed above also holds for injectives over perfect embeddings. Our main tool for identifying the algebras, and hence the injectives, is Kock's criterion: An object is the underlying object of an algebra if and only if its unit has a right adjoint, which is then its unique structure map — see Section 2 below. We emphasize that the proof of Theorem 1.1, given in Section 3 below, is constructive in the sense of topos logic.

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2 Preliminaries

We take the basic notions concerning locales and frames for granted [14] — we just emphasize that the category **Loc** of continuous maps of locales is defined to be the opposite of the category **Frm** of homomorphisms of frames. The topology, or frame of opens, of a locale X is denoted by $\mathcal{O}\,X$ and is ranged over by the letters u, v, w. The defining frame homomorphism of a continuous map $f\colon X \to Y$ of locales is denoted by $f^*\colon \mathcal{O}\,Y \to \mathcal{O}\,X$. As a map of posets, this has a right adjoint, which is denoted by $f_*\colon \mathcal{O}\,X \to \mathcal{O}\,Y$.

For the sake of completeness, we recall the definition of the upper powerlocale monad and show that it is of the Kock–Zöberlein type.

A preframe is a poset with finite meets and directed joins in which the former distribute over the latter, and a preframe homomorphism is a map that preserves both operations. The forgetful functor $G\colon$ **Frm** $\to$ **PrFrm** into the category of preframes has a left adjoint $F\colon$ **PrFrm** $\to$ **Frm**. By Banaschewski [1], and independently Heckmann [10], for any preframe L, the free frame FL can be constructed as the set of Scott closed subsets of L ordered by inclusion, with insertion of generators given by principal ideals:

$$\begin{array}{rccc} \Box\colon & L & \to & FL \\ & u & \mapsto & {\downarrow} u. \end{array}$$

If A is any frame and $h\colon L \to A$ is any preframe homomorphism, the unique frame homomorphism $\bar{h}\colon FL \to A$ with $\bar{h}(\Box u) = h(u)$ is given by

$$\bar{h}(C) = \bigvee \{h(u) \mid u \in C\}.$$

This induces a monad on **PrFrm** and a comonad on **Frm**, and hence a monad $(\mathcal{U}, \eta, \mu)$ on **Loc**, with $\mathcal{O}\mathcal{U}X = FG\,\mathcal{O}\,X$, known as the upper powerlocale monad. (Notice that, by virtue of the freeness property, the global points $1 \to \mathcal{U}X$ are in bijection with the preframe homomorphisms $\mathcal{O}\,X \to \mathcal{O}\,1$ and hence with the Scott open filters of $\mathcal{O}\,X$. Thus, by the localic Hofmann–Mislove Theorem [15, 29], they are in bijection with the compact saturated sublocales of X.)

Explicitly, the upper powerlocale monad is constructed as follows. For any continuous map $f\colon X \to Y$, the continuous map $\mathcal{U}f\colon \mathcal{U}X \to \mathcal{U}Y$ is given by

$$\begin{array}{rccc} (\mathcal{U}f)^*\colon & \mathcal{O}\mathcal{U}Y & \to & \mathcal{O}\mathcal{U}X \\ & \Box v & \mapsto & \Box f^*(v). \end{array}$$

The unit $\eta_X\colon X \to \mathcal{U}X$ of a locale X is given by

$$\begin{array}{rccc} \eta_X^*\colon & \mathcal{O}\mathcal{U}X & \to & \mathcal{O}\,X \\ & \Box u & \mapsto & u. \end{array}$$

In both cases, we are using the fact that the opens of the form $\Box u$ freely generate the frame of opens of the upper powerlocale, as explained above. Explicitly, the unit is given by

$$\eta_X^*(C) = \bigvee C,$$

a fact used in the proof of Lemma 3.1 below. Multiplication is given by

$$\begin{array}{rccc} \mu^*_X\colon & \mathcal{O}\,\mathcal{U}X & \to & \mathcal{O}\,\mathcal{U}\mathcal{U}X \\ & \Box u & \mapsto & \Box\,\Box\,u, \end{array}$$

but this is not needed for our considerations.

The category of locales is poset-enriched (and, in fact, dcpo-enriched) with $f \leq g$ in $\mathbf{Loc}(X,Y)$ if and only if $f^*(v) \leq g^*(v)$ for all $v \in \mathcal{O}\,Y$. The following lemma implicitly refers to this enrichment, which is known as the specialization ordering.

A monad (T, η, μ) on a poset-enriched category $\mathcal{C}$ is said to be of the Kock–Zöberlein type if the functor T is order-preserving and, for every object X, the inequality $\eta_{TX} \leq T\eta_X$ holds (the original, official definition has the inequality in the opposite direction, but the convention that we adopt is the one which is convenient for our purposes). In the presence of order preservation, the inequality is equivalent to saying that structure maps $\alpha\colon TX \to X$ are right adjoint to units $\eta_X\colon X \to TX$ (and hence uniquely determined by X when they exist), a fact which is also used in the proof of Lemma 3.1 below. Here, by an adjunction $f \dashv g$ in the hom-poset $\mathcal{C}(X,Y)$ it is meant a pair of maps $f\colon X \to Y$ (the left adjoint) and $g\colon Y \to X$ (the right adjoint) with $f \circ g \leq \mathrm{id}_Y$ and $g \circ f \geq \mathrm{id}_X$. The adjunction $f \dashv g$ is said to be reflective if $g \circ f = \mathrm{id}_X$. Once again for use in Lemma 3.1 below, observe that, by definition of structure map [19], a right adjoint to a unit has to be reflective in order to be a structure map. Notice that, because $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$, we have that $f \dashv g$ holds in $\mathbf{Loc}(X,Y)$ with respect to the specialization order if and only if $g^* \dashv f^*$ holds in $\mathbf{Frm}(\mathcal{O}\,Y, \mathcal{O}\,X)$ with respect to the pointwise order.

Lemma 2.1 *The upper powerlocale monad is of the Kock–Zöberlein type.*

Proof The functor is monotone: Assume that $f \leq g$ in $\mathbf{Loc}(X,Y)$ and let $v \in \mathcal{O}\,Y$. Then $\Box f^*(v) \leq \Box g^*(v)$ by monotonicity of $\Box$, which amounts to $(\mathcal{U}f)^*(\Box v) \leq (\mathcal{U}g)^*(\Box v)$. This completes the proof of monotonicity, because the opens of the form $\Box v$ form a subbase of the topology of $\mathcal{U}Y$.

The inequality $\eta_{\mathcal{U}X} \leq \mathcal{U}\eta_X$ holds in $\mathbf{Loc}(\mathcal{U}X, \mathcal{U}\mathcal{U}X)$: For $C \in \mathcal{O}\,\mathcal{U}X$, we have that $\eta^*_{\mathcal{U}X}(\Box C) = C \subseteq {\downarrow} \bigvee C = \Box \eta^*_X(C) = (\mathcal{U}\eta_X)^*(\Box C)$. The result then follows again from the fact that the opens of the form $\Box C$ with $C \in \mathcal{O}\,\mathcal{U}X$ form a subbase of the topology of $\mathcal{U}\mathcal{U}X$. $\Box$

3 Proof of the theorem

Recall the definition of the relation $\prec$ on the opens of a locale formulated in the paragraph preceding Theorem 1.1.

Lemma 3.1 *A locale is the underlying object of an algebra of the upper powerlocale monad if and only if the following two conditions hold.*

1. *Every open v is the join of the opens $u \prec v$, and*

2. *$1 \prec 1$, and $u \prec v$ and $u \prec w$ together imply $u \prec v \wedge w$.*

Proof By the Kock–Zöberlein property, a locale X is the underlying object of an algebra if and only if its unit $\eta_X\colon X \to \mathcal{U}X$ has a reflective right adjoint $\alpha\colon \mathcal{U}X \to X$. This amounts to saying that $\alpha^* \circ \eta_X^* \leq \mathrm{id}$ and $\eta_X^* \circ \alpha^* = \mathrm{id}$ hold in the category of frames. Because $\eta_X^*(C) = \bigvee C$, the adjoint-functor theorem shows that the inequalities $\alpha^* \circ \eta_X^* \leq \mathrm{id}$ and $\eta_X^* \circ \alpha^* \geq \mathrm{id}$ are equivalent to the equation $\alpha^*(v) = \bigcap\{C \in \mathcal{O}\mathcal{U}X \mid v \leq \bigvee C\}$. It follows that $u \in \alpha^*(v)$ if and only if u belongs to every $C \in \mathcal{O}\mathcal{U}X$ with $v \leq \bigvee C$, which amounts to saying that $u \prec v$; that is, $\alpha^*(v) = \{u \in \mathcal{O}X \mid u \prec v\}$. Hence the equation $\eta_X^* \circ \alpha^* = \mathrm{id}$ amounts to condition 1. Being a left adjoint, α^* preserves all joins. Preservation of finite meets amounts to condition 2. $\square$

Vickers showed that a continuous map $f\colon X \to Y$ of locales is perfect if and only if the continuous map $\mathcal{U}f\colon \mathcal{U}X \to \mathcal{U}Y$ has a left adjoint [28, Proposition 5.6]. A slight modification of his proof establishes the following.

Lemma 3.2 *A continuous map $j\colon X \to Y$ of locales is a perfect sublocale embedding if and only if $\mathcal{U}j\colon \mathcal{U}X \to \mathcal{U}Y$ has a reflective left adjoint.*

Proof Assume that $j\colon X \to Y$ is a perfect map. Being a right adjoint, $j_*\colon \mathcal{O}X \to \mathcal{O}Y$ preserves meets and hence is a preframe homomorphism by perfectness of j. By freeness of the frame $\mathcal{O}\mathcal{U}X$, there is a unique continuous map $\mathcal{U}^*j\colon \mathcal{U}Y \to \mathcal{U}X$ such that $(\mathcal{U}^*j)^*(\Box u) = \Box j_*(u)$ for all $u \in \mathcal{O}X$. Then $(\mathcal{U}j)^* \circ (\mathcal{U}^*j)^*(\Box u) = (\mathcal{U}j)^*(\Box j_*(u)) = \Box j^* j_*(u) \leq \Box u$, and a similar calculation shows that $(\mathcal{U}^*j)^* \circ (\mathcal{U}j)^*(\Box v) \geq \Box v$. Since each inequality holds for all opens of a subbase, they hold for all opens, which establishes the desired adjunction $\mathcal{U}^*j \dashv \mathcal{U}j$. Reflectivity holds if and only if $(\mathcal{U}j)^* \circ (\mathcal{U}^*j)^* = \mathrm{id}$ if and only if $\Box j^* j_*(u) = \Box u$ for all $u \in \mathcal{O}X$ if and only if $j^* j_*(u) = u$ for all $u \in \mathcal{O}X$ if and only if j is an embedding.

Conversely, assume that $\mathcal{U}j\colon \mathcal{U}X \to \mathcal{U}Y$ has a (reflective) left adjoint $\mathcal{U}^*j\colon \mathcal{U}Y \to \mathcal{U}X$ and define

$$\mathcal{O}X \xrightarrow{r} \mathcal{O}Y \quad = \quad \mathcal{O}X \xrightarrow{\Box} \mathcal{O}\mathcal{U}X \xrightarrow{(\mathcal{U}^*j)^*} \mathcal{O}\mathcal{U}Y \xrightarrow{\eta_Y^*} \mathcal{O}Y.$$

Being a composition of maps that preserve directed joins, r itself preserves directed joins. We show that $j^* \dashv r$ (reflectively), so that $j_* = r$ and hence j is a perfect map (embedding):

$$\begin{array}{rcll}
j^* \circ r(u) & = & j^* \circ \eta_Y^* \circ (\mathcal{U}^*j)^*(\Box u) & \text{by definition of } r \\
& = & (\eta_Y \circ j)^* \circ (\mathcal{U}^*j)^*(\Box u) & \text{by contravariance of } (-)^* \\
& = & (\mathcal{U}j \circ \eta_X)^* \circ (\mathcal{U}^*j)^*(\Box u) & \text{by naturality of } \eta \\
& = & \eta_X^* \circ (\mathcal{U}j)^* \circ (\mathcal{U}^*j)^*(\Box u) & \text{by contravariance of } (-)^* \\
& = & \eta_X^* \circ (\mathcal{U}^*j \circ \mathcal{U}j)^*(\Box u) & \text{by contravariance of } (-)^* \\
& \leq & \eta_X^*(\Box u) & \text{because } \mathcal{U}^*j \dashv \mathcal{U}j, \\
& = & u, &
\end{array}$$

where the inequality is an equality if the adjunction $\mathcal{U}^*j \dashv \mathcal{U}j$ is reflective, and

$$\begin{array}{rcll} r \circ j^*(v) & = & \eta_Y^* \circ (\mathcal{U}^*j)^*(\Box j^*(v)) & \text{by definition of } r \\ & = & \eta_Y^* \circ (\mathcal{U}^*j)^* \circ (\mathcal{U}j)^*(\Box v) & \text{because } (\mathcal{U}j)^*(\Box v) = \Box j^*(v) \\ & = & \eta_Y^* \circ (\mathcal{U}j \circ \mathcal{U}^*j)^*(\Box v) & \text{by contravariance of } (-)^* \\ & \supseteq & \eta_Y^*(\Box v) & \text{because } \mathcal{U}^*j \dashv \mathcal{U}j \\ & = & v. & \end{array}$$

$\Box$

Corollary 3.3 *A locale is the underlying object of an algebra of the upper powerlocale monad if and only if it is injective over perfect sublocale embeddings.*

Proof By [6, Theorem 4.2.2], for any Kock–Zöberlein monad T on a poset-enriched category, the objects which are injective over maps $j\colon X \to Y$ such that $Tj\colon TX \to TY$ has a reflective left adjoint coincide with the underlying objects of T-algebras. $\Box$

This concludes the proof of Theorem 1.1. Notice that Lemma 3.2 shows that the unit $\eta_X : X \to \mathcal{U}X$ of any locale X is a perfect sublocale embedding, because the map $T\eta_X : TX \to TTX$ has $\mu_X : TTX \to TX$ as a reflective left adjoint for any Kock–Zöberlein monad (T, η, μ) on a poset-enriched category [5]. Hence, by Corollary 3.3, every locale can be perfectly embedded into a perfectly injective locale, because $\mathcal{U}X$ is a free algebra.

4 Remarks

The (independent and unpublished) work of Ho Weng Kin and Zhao Dongsheng [17] characterizes the lattices of Scott closed sets of various kinds of posets. When applied to frames, their results imply that the free algebras of the upper powerlocale monad are precisely the locales such that every open v is the join of the opens $u \leq v$ with $u \prec u$ and such that the relation $\prec$ satisfies the stability condition (3b) of Theorem 1.1. They also consider lattices in which every element v is the join of the elements $u \prec v$, with a different motivation in mind, and establish a number of interesting results for them.

As far as we know, the characterization of the injectives over perfect embeddings given here is new. However, as shown in [6, 5, 8], several known injectivity results can be established by proofs following the above pattern, provided the monads under consideration are of the Kock–Zöberlein type. An example mentioned in [8] is that of stably locally compact locales. These coincide with the injectives over flat sublocale embeddings [13, 16], and with the algebras of the monad on locales that arises from the forgetful functor from frames into distributive lattices, whose left adjoint is ideal completion. In connection with a discussion started in the introduction, we observe that the algebra homomorphisms are precisely the perfect maps, so that there is a further equivalence with Nachbin's category of compact order-Hausdorff spaces. This is related to the work of Simmons [25]. An example not mentioned in [8] is that of injectives over arbitrary sublocale embeddings; these coincide with the algebras of the monad on locales that arises from the forgetful functor from frames into meet-semilattices.

Let's develop this second example in more detail. As in the main example discussed in this paper, an explicit construction of the free frame is known: it is the set of down sets of the meet-semilattice, ordered by inclusion. Again, the insertion of generators is given by principal ideals. Thus, one is led to consider the relation $u \prec v$ redefined to mean that every cover of v which is a down set has u as a member, or, equivalently, every cover of v has a member that covers u, and a theorem analogous to the one proved in this paper holds with an analogous proof. The frames which satisfy the third condition of the theorem, with the relation redefined as above, are called stably supercontinuous and are known to coincide with the Scott topologies of continuous lattices. Part of what is described here is proved in [3].

By the freeness property, the global points of the locale whose topology is the free frame over a frame qua semilattice coincide with the meet-semilattice homomorphisms from the frame into the initial frame, which in turn coincide with the filters of the frame. Thus, this monad is the localic version of the filter monad on the category of T_0 topological spaces discussed in [5] and previously by A. Day [4] and Wyler [30].

Notice that, in the main example developed in this paper and the ones discussed in this section, it is crucial to know an explicit construction of the free frame and of the insertion of generators. For instance, we are unable to apply our technique to the lower powerlocale monad, because we lack a sufficiently explicit description of the left adjoint of the forgetful functor from frames into sup-lattices.

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