On the compact-regular coreflection of a stably compact locale

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Abstract

A *nucleus* on a frame is a finite-meet preserving closure operator. The nuclei on a frame form themselves a frame, with the Scott continuous nuclei as a subframe. We refer to this subframe as the *patch frame*. We show that the patch construction exhibits the category of compact regular locales and continuous maps as a coreflective subcategory of the category of stably compact locales and perfect maps, and the category of Stone locales and continuous maps as a coreflective subcategory of the category of spectral locales and spectral maps. We relate our patch construction to Banaschewski and Brümmer's construction of the dual equivalence of the category of stably compact locales and perfect maps with the category of compact regular biframes and biframe homomorphisms.

Keywords: Frame of nuclei, Scott continuous nuclei, patch topology, stably locally compact locales, perfect maps, compact regular locales.

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1 Introduction

It is well-known that the open sets of a topological space can be regarded as the (abstract) "semi-decidable properties" of its points [23]—the terminologies "observable property" [1] and "affirmable property" [28] are also suggestive. Smyth regards the patch topology as a topology of "positive and negative information" [25]. We consider the patch construction in the (localic manifestation of the) category of stably compact spaces.

This category is convenient for considerations of computability in classical spaces because it contains compact Hausdorff spaces and semantic domains at the same time [2]. In this category, the patch construction is a compact-Hausdorff coreflection. This allows us to regard domains as compact Hausdorff spaces and hence to apply classical topology to domain theory. It must be

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mentioned that the patch topology of a domain coincides with its Lawson topology [7].

In the following discussion we regard domains as topological spaces, via the Scott topology, and we refer to the space that results from refining the topology of a given space to the patch topology as the *patch space*.

The patch construction transforms typical examples of domains into typical examples of compact Hausdorff spaces: (i) the patch of a finite domain is a finite discrete space, (ii) the flat natural numbers, the first limit ordinal and the lazy natural numbers are three domains whose patches are homeomorphic to the one-point compactification of the discrete natural numbers, (iii) the patch of the unit interval endowed with the Scott topology (which coincides with the topology of lower semi-continuity [7, Example I.1.21]) is the Euclidean unit interval, (iv) the patch of the domain of closed and bounded real intervals ordered by reverse inclusion, together with a bottom element, is homeomorphic to the one-point compactification of the subspace of the Euclidean plane consisting of the points below the diagonal $y = x$.

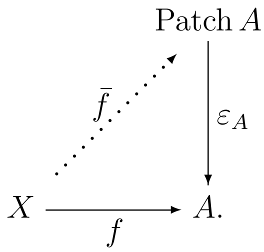
Since the patch topology is defined in terms of compact sets of points, it is not surprising that it is not amenable to constructive reasoning—one is inevitably led to appeal to the principle of excluded third and (some form of) the axiom of choice. It is the purpose of this paper to develop a simple intuitionistic account to the patch topology in a localic setting. An account based on biframes is developed in [4]—see the discussion below.

A *nucleus* on a frame is a finite-meet preserving closure operator [22,17]. The nuclei on a frame form themselves a frame [11], with the Scott continuous nuclei as a subframe [14,6] (this is elaborated in Section 2 below). Karazeris [14] showed that the frame of Scott continuous nuclei on the frame of opens of a spectral space is isomorphic to the patch topology of the space. It is a corollary of our results that, more generally, this construction produces a frame isomorphic to the patch topology of a stably compact space.

We show that, via the frame of Scott continuous nuclei, the category of compact regular locales and continuous maps appears as a coreflective subcategory of the category of stably compact locales and perfect maps, and the category of Stone locales and continuous maps appears as a coreflective subcategory of the category of spectral locales and spectral maps. The arguments are valid in any topos. Here a compact, locally compact locale is *stably compact* if its way-below relation is multiplicative, and a continuous map of locales is *perfect* if the right adjoint of its defining frame homomorphism preserves directed joins.

Explicitly, the coreflection amounts to the fact that for each stably compact locale A, there is a compact regular locale $\mathrm{Patch}\, A$ and a perfect map $\varepsilon_A : \mathrm{Patch}\, A \to A$ such that for every compact regular locale X and every perfect map $f : X \to A$, there is a unique continuous map $\bar{f} : X \to \mathrm{Patch}\, A$

with



We refer to a sober space whose topology defines a stably compact locale as a *stably compact space*, and we recall the following facts [7,8,24]. A sober space is stably compact iff it is locally compact and its compact saturated sets are closed under finite intersections, where a set is *saturated* if it is an upper set in the specialization order. The *cocompact topology* of a topological space is generated by the complements of the compact saturated sets. The *patch topology* is the join of the given topology and the cocompact topology. A continuous map of stably compact spaces is perfect (in the localic sense defined above) iff it reflects compact saturated sets. Hence a perfect map is continuous with respect to the cocompact topologies of its domain and codomain, and a perfect map remains continuous if the topology of its codomain is refined to the patch topology. The patch topology of a stably compact space is compact Hausdorff. Since in a T_1 space all sets are saturated, and since in a compact Hausdorff space the compact sets are the closed sets, every continuous map of compact Hausdorff spaces is perfect. Therefore the patch construction exhibits the category of compact Hausdorff spaces and continuous maps as a coreflective subcategory of the category of stably compact spaces and perfect maps—although the author has never seen an explicit formulation of this fact. Since universal constructions are unique up to isomorphism, we immediately conclude from the localic coreflection stated above that the frame of Scott continuous nuclei on the frame of opens of a stably compact space is isomorphic to the patch topology of the space.

The main application of the patch topology (in this generality) is to show that the category of stably compact spaces and perfect maps is equivalent to Nachbin's category of compact Hausdorff ordered spaces and monotone continuous maps [18]. The earliest explicit reference to this fact seems to be [7, Section VII-1]—see also [8,15]. This extends the earlier result by Priestley [19,20] that the category of spectral spaces and spectral maps (a full subcategory of the category of stably compact spaces and perfect maps) is equivalent to the category of ordered Stone spaces and monotone continuous maps (see Townsend [26] for a localic version).

Salbany [21] keeps the given topology and the cocompact topology separated, obtaining an alternative formulation of the equivalence via bitopological spaces. A localic version of this approach is obtained by combining the work of Banaschewski and Brümmer [4] with the work of Townsend [26]. Banaschewski and Brümmer showed that the category of stably compact locales and perfect maps is dually equivalent to the category of compact regular biframes and biframe homomorphisms. Townsend defined ordered locales and proved that the category of compact regular ordered locales and monotone continuous maps is dually equivalent to the category of compact regular biframes and biframe homomorphisms, concluding that the category of compact regular ordered locales and monotone continuous maps is equivalent to the category of stably compact locales and perfect maps.

A biframe is a frame L equipped with two subframes L_1 and L_2 that together generate L. As part of their construction, Banaschewski and Brümmer assign a biframe L to every stably compact locale A by explicitly constructing L_1 and L_2 as subframes of the frame of all nuclei on A and then declaring that L is the subframe generated by L_1 and L_2. (To be precise, they work with the frame of congruences, which, as they emphasize, is isomorphic to the frame of nuclei.) The frame L_1 consists of the closed nuclei, and the frame L_2 is isomorphic to the frame of Scott open filters on A. We show that L turns out to be the frame of Scott continuous nuclei, and that L_2 turns out to be the frame of Scott continuous fitted nuclei.

The above remark is related to the Hofmann-Mislove Theorem, which says that in a sober space, the set of Scott open filters of open sets, ordered by inclusion, is dual to the set of compact saturated sets, also ordered by inclusion [10]. (See also the earlier [7, Lemma V-5.3], which is attributed to Hofmann and Lawson [9].) It is a corollary of our main lemma that, in a stably compact locale, the frame of Scott open filters of opens is isomorphic to the frame of Scott continuous fitted nuclei. This can be seen as a reformulation of a special case of a general localic version of the Hofmann-Mislove Theorem established by Johnstone [13, Lemma 3.4]—see [29] and Section 5 below.

Banaschewski and Brümmer state that their arguments are specifically chosen to be independent of any choice principle. However, they give a contrapositive proof of their Lemma 3 and hence their argument is valid only in Boolean toposes. But, as remarked by Karazeris (personal communication), their conclusion is actually valid in any topos. In fact, by the above observations, their lemma turns out to be a particular case of the fact that the frame of Scott continuous nucleus on a compact locale defines a compact locale (Lemma 2.3 below).

In an extended version of this paper [5], we show that the category of regular locally compact locales and perfect maps appears as a coreflective subcategory of the category of stably locally compact locales and perfect maps.

Notice that, following Isbell [11] and Johnstone [12], we adopt the geometrical point of view and regard locales as generalized (sober) spaces rather than special kinds of lattices. The category **Loc** of locales and continuous maps is thus defined as the opposite of the category of frames and frame homomorphisms. We adopt the following notation and terminology, which emphasizes this point of view. We denote locales by X, Y, Z (and sometimes also by A, B, C in order to suggest different categories of locales), and the corresponding frame of a locale X by $\mathcal{O}\,X$. The elements of $\mathcal{O}\,X$ are referred to as *opens* and are ranged over by the letters U, V, W. A continuous map $f : X \to Y$ is given by a frame homomorphism $f^* : \mathcal{O}\,Y \to \mathcal{O}\,X$, with right adjoint denoted by $f_* : \mathcal{O}\,X \to \mathcal{O}\,Y$.

This paper is organized in the following sections: (2) The frame of perfect nuclei, (3) The patch frame of a spectral locale, (4) The patch frame of a stably compact locale, (5) On the Hofmann-Mislove Theorem.

I benefited from remarks by Panagis Karazeris and Steve Vickers on a previous version of this paper. In particular, Panagis let me know about his work [14], and Steve drew my attention to the paper [4] by Banaschewski and Brümmer.

2 The frame of perfect nuclei

We begin by recalling the definitions and facts concerning nuclei needed in the development that follows (see e.g. [12, Section II-2]) and establishing some terminology and notation. We then consider Scott continuous nuclei.

In the category of topological spaces and continuous maps, the natural notion of subobject, namely that of (homeomorphically embedded) subspace, is not captured by the notion of monomorphism, but rather by the more restrictive notion of *regular* monomorphism. The same is true for the category of locales and continuous maps, where nuclei are used as canonical representatives of equivalence classes of regular monomorphisms.

A *nucleus* on a frame is a finite-meet-preserving inflationary idempotent. A *sublocale* of a locale X is a locale of the form X_j, where j is a nucleus on $\mathcal{O}\,X$, with frame of opens defined by $\mathcal{O}\,X_j = \{U \in \mathcal{O}\,X | j(U) = U\}$. For every sublocale X_j of X, there is a regular monomorphism $e : X_j \to X$ given by $e^*(U) = j(U)$. Conversely, any regular monomorphism $e : X' \to X$ induces the nucleus $j = e_* \circ e^*$ on $\mathcal{O}\,X$, which makes X' isomorphic to X_j. A sublocale of a sublocale is a sublocale of the original locale, and the subobject order on sublocales coincides with set-theoretic inclusion of frames.

The nuclei on a frame form themselves a frame when endowed with the pointwise order. Thus, for any locale X, there is a locale $\mathrm{N}\,X$ defined by stipulating that $\mathcal{O}\,\mathrm{N}\,X$ is the frame of nuclei on $\mathcal{O}\,X$. An arbitrary meet of nuclei is given pointwise. A join is strictly above the pointwise join in general, and a non-empty join coincides with the pointwise join iff the latter is idempotent. The frame of nuclei is dual to the lattice of sublocales, in the sense that $j \leq k$ iff $X_j \geq X_k$.

For each open $U \in \mathcal{O}\,X$ there is a nucleus 'U' $\in \mathcal{O}\,\mathrm{N}\,X$ defined by

$$\text{`}U\text{'}(V) = U \vee V.$$

This is often referred to as a *closed* nucleus, because it induces a closed sublocale of X thought as the complement of U. But, since the frame of nuclei is dual to the lattice of sublocales, a nucleus should be thought as the formal complement of its induced sublocale. We thus think of 'U' as a copy of the open $U \in \mathcal{O}\,X$ in the frame $\mathcal{O}\,\mathrm{N}\,X$. In fact, the assignment $U \mapsto$ 'U' is a frame homomorphism. We denote by $\rho_X : \mathrm{N}\,X \to X$ the continuous map defined by $\rho_X^*(U) =$ 'U'. The right adjoint of ρ_X^* is given by $(\rho_X)_*(j) = j(0)$. The continuous map $\rho_X : \mathrm{N}\,X \to X$ is both epi and mono (but not regular mono).

Every nucleus of the form 'U' has a boolean complement, given by

$$(\neg\text{`}U\text{'})(V) = (U \Rightarrow V).$$

The nuclei of the form 'V' $\wedge\, \neg$'U' constitute a base of $\mathcal{O}\,\mathrm{N}\,X$. In fact, every $j \in \mathcal{O}\,\mathrm{N}\,X$ is the join of the nuclei $j_U \stackrel{\text{def}}{=}$ '$j(U)$' $\wedge\, \neg$'U' for $U \in \mathcal{O}\,X$. This can be proved by first showing that $j_U \leq j$ for any U and then observing that for any nucleus l with $j_U \leq l$, one has that $j(U) = j_U(U) \leq l(U)$.

For every continuous map $f : X \to Y$ there is a continuous map $\mathrm{N}\,f : \mathrm{N}\,X \to \mathrm{N}\,Y$ uniquely specified by the condition $(\mathrm{N}\,f)^*($'V'$) =$ '$f^*(V)$'. This makes N into a functor **Loc** $\to$ **Loc** and ρ into a natural transformation $\mathrm{N} \to 1$.

Definition 2.1 A continuous map of locales is *perfect* if the right adjoint of its defining frame homomorphism preserves directed joins. A nucleus is *perfect* if it preserves directed joins.

The defining frame homomorphism of a perfect map always preserves the way-below relation [7]. Some authors refer to perfect maps as *proper maps*. But the latter terminology is often used for a condition stronger than perfectness [27]. Banaschewski [3] refers to perfect nuclei as *finitary nuclei*. Our terminology is due to the fact that a nucleus is perfect iff it is induced by a perfect map.

The perfect nuclei form a subframe of the frame of all nuclei [14, Theorem 4.1] [6, Lemma 3.1.8]. The proof given in [6] observes that the join of a set J of perfect nuclei is given by the pointwise join of the finite compositions of members of J (which are not necessarily nuclei). The proof given in [14], which is based on transfinite iteration, works more generally for "compact" topologies on locally finitely presentable categories. But in the calculations given below, we are only going to use the fact that *directed joins* of perfect nuclei are computed pointwise—see [14, Proposition 4.3] or the remark preceding [6, Lemma 3.1.8].

Definition 2.2 The *patch* of a locale X is the locale $\mathrm{Patch}\,X$ defined by stipulating that $\mathcal{O}\,\mathrm{Patch}\,X$ is the frame of perfect nuclei on $\mathcal{O}\,X$.

Since $\mathcal{O}\,\mathrm{Patch}\,X$ is a subframe of $\mathcal{O}\,\mathrm{N}\,X$, there is an epimorphism $q_X : \mathrm{N}\,X \to \mathrm{Patch}\,X$ given by $q_X^*(j) = j$. Since a nucleus of the form 'U' is perfect, the map $\rho_X : \mathrm{N}\,X \to X$ factors as $\mathrm{N} \xrightarrow{q_X} \mathrm{Patch}\,X \xrightarrow{\varepsilon_X} X$ for a unique continuous map $\varepsilon_X : \mathrm{Patch}\,X \to X$, given by $\varepsilon_X^*(U) =$ 'U'. Notice that $(\varepsilon_X)_*(j) = j(0)$.

Lemma 2.3 $\varepsilon_X : \operatorname{Patch} X \to X$ *is a perfect map.*

Proof. Let $J \subseteq \mathcal{O} \operatorname{Patch} X$ be directed. Since directed joins in $\mathcal{O} \operatorname{Patch} X$ are given pointwise, we have that $(\varepsilon_X)_*(\bigvee J) = (\bigvee J)(0) = \bigvee\{j(0) \mid j \in J\} = \bigvee\{(\varepsilon_X)_*(j) \mid j \in J\}$. □

Lemma 2.4 *If $f : X \to Y$ is perfect and Y is compact, so is X.*

In particular, if X is a compact locale so is $\operatorname{Patch} X$, and a perfect nucleus on a compact locale induces a compact sublocale.

Proof. This follows from the fact that the defining frame homomorphism of a perfect map preserves the way-below relation. □

3 The patch frame of a spectral locale

Definition 3.1 A *clopen* is an open with a boolean complement. A locale is *zero-dimensional* if the clopens form a base (that is, every open is a join of clopens). A *Stone locale* is a zero-dimensional compact locale. The category of Stone locales and continuous maps is denoted by **Stone**.

Definition 3.2 A *spectral locale* is a locale for which the compact opens form a base closed under finite meets. Since this includes the empty meet, a spectral locale is compact. A continuous map is *spectral* if its defining frame homomorphism preserves compact opens. The category of spectral locales and spectral maps is denoted by **Spec**.

Every clopen of a compact locale is compact. Since the clopens are closed under finite meets, Stone locales are spectral. Since frame homomorphisms preserve finite meets and joins, they also preserve clopens and hence continuous maps of Stone locales are spectral. Therefore **Stone** is a full subcategory of **Spec**. A continuous map of spectral locales is spectral iff it is perfect [7].

Lemma 3.3 *Let A be a spectral locale.*

(i) *If $U \in \mathcal{O} A$ is compact then 'U' has a boolean complement in $\mathcal{O} \operatorname{Patch} A$.*

(ii) *The nuclei of the form 'V' $\wedge \neg$'U', with $U, V \in \mathcal{O} A$ compact, constitute a base of $\mathcal{O} \operatorname{Patch} A$.*

(iii) $\operatorname{Patch} A$ *is a Stone locale.*

(iv) $\varepsilon_A : \operatorname{Patch} A \to A$ *is a monomorphism in* **Spec**.

(v) *If A is a Stone locale then $\varepsilon_A : \operatorname{Patch} A \to A$ is an isomorphism.*

Proof. (i): We have to show that if $U \in \mathcal{O} A$ is compact then the boolean complement of 'U' in $\mathcal{O} \operatorname{N} A$ is perfect. It suffices to conclude that for all opens $V, W \in \mathcal{O} A$ with W compact and $W \leq (\neg\text{'}U\text{'})(V)$, there is a compact open $V' \leq V$ such that already $W \leq (\neg\text{'}U\text{'})(V')$. Since $\neg\text{'}U\text{'}(V) = (U \Rightarrow V)$, we have that $W \wedge U \leq V$. By coherence, $W \wedge U$ is compact. Therefore we can take $V' = W \wedge U$.

(ii): Since a perfect nucleus j on a spectral locale is completely determined by its effect on compact opens, one has that j is the join of the nuclei 'V'$\wedge\neg$'U' with $U, V \in \mathcal{O}\,A$ compact and $V \leq j(U)$.

(iii): By Lemmas 2.3 and 2.4, Patch A is compact, and by (ii), it is zero-dimensional.

(iv): We know that a map of spectral locales is spectral iff it is perfect. Hence the map is spectral by Lemma 2.3. We have to show that, for every $f : B \to \mathrm{Patch}\,A$ in **Spec**, the frame homomorphism $f^* : \mathcal{O}\,\mathrm{Patch}\,A \to \mathcal{O}\,B$ is uniquely determined by its effect on nuclei of the form 'U' with $U \in \mathcal{O}\,A$ compact. If we know f^*('U') then we know $f^*(\neg$'U') because frame homomorphisms preserve boolean complements. Hence we also know f^*('U'$\wedge$ $\neg$'V') for all $U, V \in \mathcal{O}\,A$ compact. Therefore, by (ii), we know $f^*(j)$ for any $j \in \mathcal{O}\,\mathrm{Patch}\,A$.

(v): We know that ε_A^* is always one-to-one. In a Stone locale the clopens coincide with the compact opens. Since frame homomorphisms preserve boolean complements, one has that for a clopen U the identity $\neg$'U' = '$\neg U$' holds. Hence every perfect nucleus is a join of nuclei of the form 'U' and hence is itself of this form. Therefore ε_A^* is onto. □

Lemma 3.4 *The functor* N : **Loc** $\to$ **Loc** *restricts to a functor* Patch : **Spec** $\to$ **Stone** *via the (necessarily natural) transformation* $q : \mathrm{N} \to \mathrm{Patch}$. *Moreover, the restriction presents* ε *as a natural transformation* Patch $\to 1$.

Proof. We have to show that $(\mathrm{N}\,f)^* : \mathcal{O}\,\mathrm{N}\,B \to \mathcal{O}\,\mathrm{N}\,A$ preserves perfect nuclei if $f : A \to B$ is a spectral map of spectral locales. Let $j \in \mathcal{O}\,\mathrm{Patch}\,B \subseteq \mathcal{O}\,\mathrm{N}\,B$. By Lemma 3.3, we know that j is a join of opens of the form 'U' $\wedge\neg$'V' with U, V compact. Hence $(\mathrm{N}\,f)^*(j)$ is a join of opens of the form '$f^*(U)$'$\wedge\neg$'$f^*(V)$', because homomorphisms preserve finite meets and boolean complements. By coherence of f, the open $f^*(V)$ is compact. Therefore $(\mathrm{N}\,f)^*(j)$ is perfect. □

Theorem 3.5 *The patch construction exhibits the category of Stone locales and continuous maps as a coreflective subcategory of the the category of spectral locales and spectral maps.*

Proof. For every spectral locale A, the map $\varepsilon_A : \mathrm{Patch}\,A \to A$ is universal among spectral maps $f : X \to A$ on Stone locales X, because the composite $\mathrm{Patch}\,f \circ \varepsilon_X^{-1}$ is a map $\bar{f} : X \to \mathrm{Patch}\,A$ with $\varepsilon_A \circ \bar{f} = f$ by naturality of ε and because a map $\bar{f}$ with this property is necessarily unique as ε_A is mono.□

4 The patch frame of a stably compact locale

It is plausible that the results of this section could be derived from the results of the previous, using the fact that the stably compact locales are the retracts of the spectral locales [12], but we give direct arguments.

By definition, an open U is closed iff there is a (necessarily unique) open W (its boolean complement) with $U \wedge W = 0$ and $U \vee W = 1$. The well-inside relation gives a relative notion of closedness, and regularity generalizes zero-dimensionality.

Definition 4.1 One says that an open U is *well inside* an open V (or that U is *closed in* V), written $U \lessdot V$, if there is an open W with $U \wedge W = 0$ and $V \vee W = 1$. Equivalently, $U \lessdot V$ iff $V \vee \neg U = 1$, where $\neg U$ is the Heyting complement of U. A locale is *regular* if every open V is a join of opens $U \lessdot V$. The category of compact regular locales and continuous maps is denoted by **CReg**.

Since frame homomorphisms preserve finite meets and joins, they preserve the well-inside relation.

At this point we assume familiarity with the theory of continuous lattices [7] [12, Chapter VII]. The way-below relation gives a relative notion of compactness. One says that U is *way below* V (or that U is *compact in* V), written $U \ll V$, if every open cover of V has a finite subcover of U, or, equivalently, if every directed cover of V has a member that covers U. A locale is *locally compact* if every open V is a join of opens $U \ll V$. In other words,

Definition 4.2 A locale is *locally compact* iff its frame is a continuous lattice.

We have already mentioned that the the defining frame homomorphism of a perfect map always preserves the way-below relation. For a continuous map of locally compact locales, the converse holds.

Definition 4.3 A locally compact locale is *stably compact* if for every open U, the set $\Uparrow U \stackrel{\text{def}}{=} \{V | U \ll V\}$ is a filter. The category of stably compact locales and perfect maps is denoted by **SC**.

That is, a locally compact locale is stably compact iff $1 \ll 1$ and its way-below relation is *multiplicative*, in the sense that $U \ll V$ and $U \ll W$ together imply $U \ll V \wedge W$. In a compact locale, the well-inside relation entails the way-below relation, and in a regular locale the converse holds; also, the well-inside relation is always multiplicative (see [12, Sections VII-3.5 and III.1]). Therefore **CReg** is a full subcategory of **SC**.

The (full) inclusions of categories considered in this paper are illustrated in the following diagram:

$$\begin{array}{ccc} \mathbf{Stone} & \hookrightarrow & \mathbf{CReg} \\ \downarrow & & \downarrow \\ \mathbf{Spec} & \hookrightarrow & \mathbf{SC}\,. \end{array}$$

Definition 4.4 We write $\neg j$ to denote the Heyting complement of a nucleus j in the frame of all nuclei, and $\underline{\neg} j$ to denote the Heyting complement of a perfect nucleus j in the subframe of perfect nuclei.

Lemma 4.5 *If $U \ll V$ holds for opens of a stably compact locale, then*

(i) $(\underline{\neg}$‘U’$)(V) = 1$,

(ii) ‘V’ $\vee \underline{\neg}$‘U’ $=$ ‘1’,

(iii) ‘U’ $\leqslant$ ‘V’ *and* $\underline{\neg}$‘V’ $\leqslant \underline{\neg}$‘$U$’.

Proof. For every nucleus j there is a largest perfect nucleus $\underline{j}$ below j, given by the join of the perfect nuclei below j. It is clear that $\underline{\neg} j = \underline{\neg j}$. By [6, Theorem 3.1.15], in a stably compact locale, $\underline{j}$ is explicitly given by the formula $\underline{j}(V) = \bigvee\{j(V') \mid V' \ll V\}$. In particular, one has that $(\underline{\neg}$‘U’$)(V) = \bigvee\{U \Rightarrow V' \mid V' \ll V\}$. This and the fact that $(U \Rightarrow U) = 1$ establish (i). As it is easy to check [17], for any nucleus j, one has that the join ‘V’ $\vee j$ is the composite $j \circ$ ‘V’. This and (i) establish (ii) because (‘V’ $\vee \underline{\neg}$‘U’$)(0) = (\underline{\neg}$‘U’$)(V) = 1$. Finally, the first relation of (iii) holds because ‘U’$\wedge\underline{\neg}$‘U’ $=$ ‘0’ and ‘V’ $\vee \underline{\neg}$‘U’ $=$ ‘1’ by (ii), and the second holds because ‘V’ $\wedge \underline{\neg}$‘V’ $=$ ‘0’ and $\underline{\neg}$‘U’ $\vee$ ‘V’ $=$ ‘1’. □

Lemma 4.6 *Let A be a stably compact locale.*

(i) *The nuclei* ‘V’ $\wedge \underline{\neg}$‘U’, *with $V, U \in \mathcal{O}\,A$, constitute a base of $\mathcal{O}\,\mathrm{Patch}\,A$.*

(ii) $\mathrm{Patch}\,A$ *is compact regular.*

(iii) *If A is compact regular then $\varepsilon_A : \mathrm{Patch}\,A \to A$ is an isomorphism with inverse given by $(\varepsilon_A^{-1})^*(j) = j(0)$.*

Proof. (i): We show that any perfect nucleus j is the join of the nuclei $j_{U,V} \stackrel{\text{def}}{=}$ ‘V’ $\wedge \underline{\neg}$‘U’ with $V \ll j(U)$. Since $\underline{\neg}$‘U’ $\leq \neg$‘U’, we have that j is an upper bound of such nuclei. Let l be another, and, in order to show that $j \leq l$, let U and V be opens with $V \ll j(U)$. By Scott continuity of j and local compactness of A, there is an open $U' \ll U$ with $V \ll j(U')$. It follows from Lemma 4.5(i) that $V \leq (V \vee U) \wedge 1 =$ ‘V’$(U) \wedge (\underline{\neg}$‘$U'$’$)(U) = j_{U',V}(U) \leq l(U)$. Therefore j is the least upper bound.

(ii): Compactness follows from Lemma 2.4. Given $V \ll j(U)$, we first find opens U' and V' with $U' \ll U$ and $V \ll V' \ll j(U')$ by Scott continuity of j, local compactness of A, and the interpolation property. We then conclude that the relations ‘V’ $\leqslant$ ‘V'’ and $\underline{\neg}$‘U’ $\leqslant \underline{\neg}$‘$U'$’ hold by Lemma 4.5(iii). Since the well-inside relation is multiplicative, we have that $j_{U,V} \leqslant j_{U',V'}$, and since $j_{U',V'} \leq j$, we conclude that $j_{U,V} \leqslant j$.

(iii): If j is perfect then it induces a compact sublocale by Lemma 2.4, and a compact sublocale of a regular locale is closed [12, Proposition III-1.2]. Therefore $j =$ ‘U’ for some U, which has to be $j(0)$. □

MacNab [17] considered nuclei of the form $\kappa_\phi \stackrel{\text{def}}{=} \bigvee\{\neg$‘$U$’ $\mid U \in \phi\}$ with ϕ a filter of opens. Johnstone [13, Lemma 3.4] and Banaschewski and Brümmer [4] considered the particular case in which the filter ϕ is Scott open (that is, inaccessible by directed joins). The following result on κ_ϕ seems to be new.

Lemma 4.7 *If A is a stably compact locale and $\phi \subseteq \mathcal{O}\,A$ is a Scott open filter, then the nucleus κ_ϕ is perfect and its defining join can be computed pointwise.*

Proof. It is enough to show that the function $k(V) = \bigvee\{U \Rightarrow V \mid U \in \phi\}$ is Scott continuous and has $k \circ k \leq k$. By definition of Heyting implication,

(*i*) If $U \in \phi$ and $W \wedge U \leq V$ then $W \leq k(V)$.

Conversely, the set $\{U \Rightarrow V \mid U \in \phi\}$ is directed because ϕ is a filter and because the map $U \mapsto (U \Rightarrow V)$ transforms joins into meets. Hence if $W \ll k(V)$ then there is $U' \in \phi$ with $W \ll (U' \Rightarrow V)$. Since ϕ is Scott open, there is $U \in \phi$ with $U \ll U'$. By stability, $W \wedge U \ll (U' \Rightarrow V) \wedge U' \leq V$. This shows that

(*ii*) If $W \ll k(V)$ then $W \wedge U \ll V$ for some $U \in \phi$.

In order to show that k is Scott continuous, it suffices to show that whenever $W \ll k(V)$, there exists $V' \ll V$ with $W \leq k(V')$. But now this is immediate because we can find $U \in \phi$ with $W \wedge U \ll V$ by (*ii*) and then take $V' = W \wedge U$ by (*i*). Finally, in order to show that $k \circ k \leq k$, assume that $W \ll k(k(V))$. By two successive applications of (*ii*), we first have that $W \wedge U \ll k(V)$ for some $U \in \phi$ and then that $W \wedge U \wedge U' \ll V$ for some $U' \in \phi$. Since $U \wedge U' \in \phi$ as ϕ is a filter, we conclude by (*i*) that $W \leq k(V)$. Therefore $k(k(V)) \leq k(V)$.□

It will be convenient to work with an alternative base of the patch frame of a stably compact locale.

Lemma 4.8 *For any stably compact locale A, the nuclei 'V' $\wedge\, \kappa_\phi$, with $V \in \mathcal{O}\,A$ and $\phi \subseteq \mathcal{O}\,A$ a Scott open filter, constitute a base of $\mathcal{O}\,\mathrm{Patch}\,A$.*

Proof. We show that any perfect nucleus j is the join of the nuclei $j_{U,V} \stackrel{\text{def}}{=}$ 'V'$\wedge\, \kappa_U$ with $V \ll j(U)$, where κ_U stands for κ_ϕ with $\phi = \Uparrow U$. Since $\kappa_U \leq \neg$'U', we have that j is an upper bound of such nuclei. Let l be another, and let U and V be opens with $V \ll j(U)$. By Scott continuity of j, there is $U' \ll U$ with $V \ll j(U')$. Hence $V \leq (V \vee U) \wedge 1 =$ 'V'$(U) \wedge \kappa_{U'}(U) = j_{U',V}(U) \leq l(U)$ by Lemma 4.7. Therefore j is the least upper bound. □

(Notice that, in view of the topological Hofmann-Mislove Theorem, this *property* of the patch frame is essentially the same as the *definition* of the patch topology—see Section 5.)

Banaschewski and Brümmer [4] assign a (compact regular) biframe L to each stably compact locale A as follows (where we use nuclei instead of their frame congruences):

(i) $L_1 = \{$'U' $\mid U \in \mathcal{O}\,A\}$,

(ii) $L_2 = \{\kappa_\phi \mid \phi \subseteq \mathcal{O}\,A$ is a Scott open filter$\}$,

(iii) L is the subframe of $\mathcal{O}\,\mathrm{N}\,A$ generated by L_1 and L_2.

It follows from Lemma 4.8 that L turns out to be $\mathcal{O}\,\mathrm{Patch}\,A$. Corollary 5.2

below shows that L_2 turns out to the subframe of perfect fitted nuclei.

Lemma 4.9 *If $f : A \to B$ is a perfect map of stably compact locales, then $(\mathrm{N}\, f)^* : \mathcal{O}\,\mathrm{N}\, B \to \mathcal{O}\,\mathrm{N}\, A$ preserves perfect nuclei. Moreover, the induced continuous map* $\mathrm{Patch}\, f : \mathrm{Patch}\, A \to \mathrm{Patch}\, B$ *is uniquely determined by the condition that*

$$(\mathrm{Patch}\, f)^*(`V\text{'}) = `f^*(V)\text{'}, \qquad (\mathrm{Patch}\, f)^*(\kappa_\phi) = \kappa_{f^*[\phi]}$$

for every open $V \in \mathcal{O}\, B$ and every Scott open filter $\phi \subseteq \mathcal{O}\, B$.

Proof. Let $j \in \mathcal{O}\,\mathrm{Patch}\, B \subseteq \mathcal{O}\,\mathrm{N}\, B$. By Lemma 4.8, we know that j is a join of opens of the form $`V\text{'} \wedge \kappa_\phi$. Hence $(\mathrm{N}\, f)^*(j)$ is a join of opens of the form $(\mathrm{N}\, f)^*(`V\text{'}) \wedge (\mathrm{N}\, f)^*(\kappa_\phi)$. The nucleus $(\mathrm{N}\, f)^*(`V\text{'})$ is perfect because it is $`f^*(V)\text{'}$. Also,

$$\begin{aligned}
(\mathrm{N}\, f)^*(\kappa_\phi) &= (\mathrm{N}\, f)^*(\textstyle\bigvee\{\neg`V\text{'} \mid V \in \phi\}) && \text{by definition of } \kappa_\phi, \\
&= \textstyle\bigvee\{(\mathrm{N}\, f)^*(\neg`V\text{'}) \mid V \in \phi\} && \text{as homomorphisms preserve joins,} \\
&= \textstyle\bigvee\{\neg(\mathrm{N}\, f)^*(`V\text{'}) \mid V \in \phi\} && \text{as they preserve complements,} \\
&= \textstyle\bigvee\{\neg`f^*(V)\text{'} \mid V \in \phi\} && \text{by specification of } \mathrm{N}\, f, \\
&= \kappa_{f^*[\phi]}, && \text{by definition of } \kappa_{f^*[\phi]}.
\end{aligned}$$

The set $f^*[\phi]$ is not necessarily a filter, but it is filtered as frame homomorphisms preserve finite meets. Since f^* preserves the way-below relation, the filter γ generated by $f^*[\phi]$ is Scott open. Also, it is clear that $\kappa_{f^*[\phi]} = \kappa_\gamma$. Hence $\kappa_{f^*[\phi]}$ is perfect. Therefore so is $(\mathrm{N}\, f)^*(j)$. □

Corollary 4.10 *The functor* $\mathrm{N} : \mathbf{Loc} \to \mathbf{Loc}$ *restricts to a functor* $\mathrm{Patch} : \mathbf{SC} \to \mathbf{CReg}$ *via the (necessarily natural) transformation* $q : \mathrm{N} \to \mathrm{Patch}$. *Moreover, the restriction presents ε as a natural transformation* $\mathrm{Patch} \to 1$.

Theorem 4.11 *The patch construction exhibits the category of compact regular locales and continuous maps as coreflective subcategory of the category of stably compact locales and perfect maps.*

Proof. This is equivalent to saying that the functor $\mathrm{Patch} : \mathbf{SC} \to \mathbf{CReg}$ is right adjoint to the inclusion functor. For every compact regular locale X, we can define $\eta_X : X \to \mathrm{Patch}\, X$ as ε_X^{-1} because Lemma 4.6(iii) shows that ε_X is an isomorphism. Then η is a natural transformation and, by definition, the composite $X \xrightarrow{\eta_X} \mathrm{Patch}\, X \xrightarrow{\varepsilon_X} X$ is the identity. Thus, in order to establish the claim, it is enough to show that the composite

$$\mathrm{Patch}\, A \xrightarrow{\eta_{\mathrm{Patch}\, A}} \mathrm{Patch}\,\mathrm{Patch}\, A \xrightarrow{\mathrm{Patch}\,\varepsilon_A} \mathrm{Patch}\, A$$

is also the identity [16, Theorem IV-1.2(v)]. Denote the composite by g.

If $V \in \mathcal{O}\,A$ then

$$\begin{aligned} g^*(\text{‘}V\text{’}) &= \eta^*_{\mathrm{Patch}\,A} \circ (\mathrm{Patch}\,\varepsilon_A)^*(\text{‘}V\text{’}) && \text{by definition of } g, \\ &= \eta^*_{\mathrm{Patch}\,A}(\text{‘}\varepsilon^*_A(V)\text{’}) && \text{by Lemma 4.9,} \\ &= \eta^*_{\mathrm{Patch}\,A}(\text{‘‘}V\text{’’}) && \text{by definition of } \varepsilon_A, \\ &= \text{‘‘}V\text{’’}(\text{‘}0\text{’}) && \text{by Lemma 4.6(iii),} \\ &= \text{‘}V\text{’} \vee \text{‘}0\text{’} && \text{by definition,} \\ &= \text{‘}V\text{’}. \end{aligned}$$

If $\phi \subseteq \mathcal{O}\,A$ is a Scott open filter then

$$\begin{aligned} g^*(\kappa_\phi) &= \kappa_{\varepsilon^*_A[\phi]}(\text{‘}0\text{’}) && \text{by Lemmas 4.9 and 4.6(iii),} \\ &= \bigvee\{\neg\text{‘}\varepsilon^*_A(V)\text{’} \mid V \in \phi\}(\text{‘}0\text{’}) && \text{by definition of } \kappa_{\varepsilon^*_A[\phi]}, \\ &= \bigvee\{(\neg\text{‘}\varepsilon^*_A(V)\text{’})(\text{‘}0\text{’}) \mid V \in \phi\} && \text{by Lemma 4.7,} \\ &= \bigvee\{(\neg\text{‘‘}V\text{’’})(\text{‘}0\text{’}) \mid V \in \phi\} && \text{by definition of } \varepsilon_A, \\ &= \bigvee\{\text{‘}V\text{’} \Rightarrow \text{‘}0\text{’} \mid V \in \phi\} \\ &= \bigvee\{\neg\text{‘}V\text{’} \mid V \in \phi\} \\ &= \kappa_\phi && \text{by definition of } \kappa_\phi. \end{aligned}$$

Since g^* is the identity on the opens a base by Lemma 4.8, it is the identity on all opens. □

5 On the Hofmann-Mislove Theorem

For every nucleus j, the set $j^{-1}(1)$ is a filter [17]. In fact, $j^{-1}(1)$ is an upper set because j is monotone, and it is closed under finite meets because j preserves them. (Notice that $j^{-1}(1)$ is the set of open neighborhoods of the sublocale induced by j. In fact, $j(U) = 1$ iff $\neg$‘U’ $\leq j$ iff (by duality) the open sublocale induced by $\neg$‘U’ contains the sublocale induced by j.)

Proposition 5.1 *If j is a perfect nucleus on a compact locale, then the filter $j^{-1}(1)$ is Scott open.*

Proof. If $\bigvee\mathcal{U} \in j^{-1}(1)$ for a directed set of opens $\mathcal{U}$, then $\bigvee\{j(U) \mid U \in \mathcal{U}\} = j(\bigvee\mathcal{U}) = 1$ because j is perfect, and $j(U) = 1$ for some $U \in \mathcal{U}$ because 1 is compact. Therefore $U \in j^{-1}(1)$, which shows that $j^{-1}(1)$ is Scott open. □

MacNab [17] observed that for every nucleus j there is a smallest nucleus k that induces the same filter as j, namely $\kappa_{j^{-1}(1)}$. Recall that a sublocale is fitted iff it is a meet of open sublocales, and that a nucleus is fitted iff it induces a fitted sublocale iff it is a join of nuclei of the form ¬‘U’. Since κ_ϕ is a fitted nucleus, we have the following form of the Hofmann-Mislove Theorem for stably compact locales.

Corollary 5.2 *For a stably compact locale, the assignments $j \mapsto j^{-1}(1)$ and $\phi \mapsto \kappa_\phi$ constitute an order-isomorphism between perfect fitted nuclei and Scott open filters of opens ordered by inclusion.*

Proof. In view of Lemmas 2.4 and 4.7 and Proposition 5.1, it remains to show that $\kappa_\phi^{-1}(1) = \phi$. But this is immediate, because, by Lemma 4.7, $\kappa_\phi(V) = 1$ iff $\bigvee\{U \Rightarrow V \mid U \in \phi\} = 1$ iff $(U \Rightarrow V) = 1$ for some $U \in \phi$, because 1 is compact, iff $U \leq V$ for some $U \in \phi$ iff $V \in \phi$. □

The following result by Johnstone [13, Lemma 3.4] implies that, for any locale, there is an order-reversing bijection between compact fitted sublocales and Scott open filters of opens, and hence is a general localic version the Hofmann-Mislove Theorem.

(i) A sublocale X_j of a locale X is compact iff the filter $j^{-1}(1)$ is Scott open.

(ii) For any locale X and any Scott open filter $\phi \subseteq \mathcal{O}\,X$, there is a (smallest) nucleus j on $\mathcal{O}\,X$ with $j^{-1}(1) = \phi$ (which has to be κ_ϕ).

Johnstone used transfinite iteration of the pointwise join $V \mapsto \bigvee\{(\neg\text{‘}U\text{’})(V) \mid U \in \phi\}$ in order to compute the join $\bigvee\{\neg\text{‘}U\text{’} \mid U \in \phi\}$ and obtain the conclusion. It follows by (i), which is easily proved, that a fitted nucleus on a stably compact locale induces a compact sublocale iff it is perfect. Our version of the Hofmann-Mislove Theorem is thus a reformulation of a particular case of Johnstone's. But notice that this reformulation depends on the fact that κ_ϕ is perfect in the stably compact case (Lemma 4.7) and that transfinite iteration is avoided in this special case.

Finally, we observe that in a stably compact locale, the perfect nucleus κ_ϕ can be expressed as a join of perfect nuclei as follows.

Proposition 5.3 *If A is a stably compact locale and $\phi \subseteq \mathcal{O}\,A$ is a Scott open filter, then $\kappa_\phi = \bigvee\{\underline{\neg}\text{‘}U\text{’} \mid U \in \phi\}$.*

Proof. Denote the right-hand side of the identity by j. It is clear that $j \leq \kappa_\phi$. Let $W \ll \kappa_\phi(V)$. By item *(ii)* of the proof of Lemma 4.7, there is some $U \in \phi$ with $W \wedge U \ll V$. By taking $V' = W \wedge U$, we have that $W \leq (U \Rightarrow V')$. Since j is a directed join of perfect nuclei, its defining join is given pointwise. It follows that $W \leq j(V)$. Therefore $\kappa_\phi \leq j$. □

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