The regular-locally-compact coreflection of a stably locally compact locale

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Abstract

The Scott continuous nuclei form a subframe of the frame of all nuclei. We refer to this subframe as the *patch frame*. We show that the patch construction exhibits (i) the category of regular locally compact locales and perfect maps as a coreflective subcategory of the category of stably locally compact locales and perfect maps, (ii) the category of compact regular locales and continuous maps as a coreflective subcategory of the category of stably compact locales and perfect maps, and (iii) the category of Stone locales and continuous maps as a coreflective subcategory of the category of spectral locales and perfect maps. (Here a *stably locally compact* locale is not necessarily compact, and a *stably compact* locale is a compact and stably locally compact locale.) We relate our patch construction to Banaschewski and Brümmer's construction of the dual equivalence of the category of stably compact locales and perfect maps with the category of compact regular biframes and biframe homomorphisms.

Keywords: Frame of nuclei, Scott continuous nucleus, patch topology, stably locally compact locale, perfect map, compact regular locale, regular locally compact locale.

Mathematics Subject Classification: 06B35, 06E15, 54A10, 54H10.

1 Introduction

A *nucleus* on a frame is a finite-meet-preserving closure operator [22,17]. The nuclei on a frame form themselves a frame [8], with the Scott continuous nuclei as a subframe [14,3] (this is elaborated in Section 2). Karazeris [14] showed that the frame of Scott continuous nuclei on the frame of opens of a spectral space is isomorphic to the patch topology of the space. It is a corollary of our results that, more generally, this construction produces a frame isomorphic to the patch topology of a stably compact space.

We show that, via the frame of Scott continuous nuclei, (i) the category of regular locally compact locales and perfect maps appears as a coreflective subcategory of the category of stably locally compact locales and perfect maps, (ii) the category of compact regular locales and continuous maps appears as a coreflective subcategory of the category of stably compact locales and perfect maps and (iii) the category of Stone locales and continuous maps appears as a coreflective subcategory of the category of spectral locales and perfect maps.

The proofs of these localic facts are intuitionistic in the sense of topos logic. Notice, however, that their connections with topology discussed below unavoidably rely on excluded-middle and choice principles. In particular, there are toposes for which locally compact locales such as the real line fail to be spatial, and for which the spatialization of compact sublocales fail to enjoy the Heine-Borel property. Moreover, there are toposes for which not all points of a stably compact locale are perfect and hence the locales and their patches fail to have the same points (see Vickers [27] for a discussion of constructive points).

Here a locale is *stably locally compact* if it is locally compact and its way-below relation is multiplicative [9], and it is *stably compact* if it is compact and stably locally compact. A continuous map of locales is *perfect* if the right adjoint of its defining frame homomorphism preserves directed joins. This is elaborated in Section 4.

We refer to a sober space whose topology defines a stably compact locale as a *stably compact space*, and we recall the following facts [4,5]. A sober space is stably compact iff it is locally compact and its compact saturated sets are closed under finite intersections, where a set is *saturated* if it is an upper set in the specialization order iff it is the intersection of its neighbourhoods. The *cocompact topology* of a stably compact space consists of the complements of the compact saturated sets. The *patch topology* is the join of the given topology and the cocompact topology. The patch topology of a stably compact space is compact Hausdorff. A continuous map of stably compact spaces is perfect (in the localic sense defined above) iff it reflects compact saturated sets. Hence a perfect map is continuous with respect to the cocompact topologies of its domain and codomain. Since in a T_1 space all sets are saturated, and since in a compact Hausdorff space the compact sets are the closed sets, the patch of a compact Hausdorff space is itself, and a perfect map of a compact Hausdorff space into a stably compact space remains continuous if the topology of its codomain is refined to the patch topology. Moreover, every continuous map of compact Hausdorff spaces is perfect. Therefore the patch construction exhibits the category of compact Hausdorff spaces and continuous maps as a coreflective subcategory of the category of stably compact spaces and perfect maps—although the author has never seen an explicit formulation of this fact. Since universal constructions are unique up to isomorphism, we immediately conclude from the localic coreflection stated above that the frame of Scott continuous nuclei on the frame of open sets of a stably compact space is isomorphic to the patch topology of the space.

The main application of the patch topology (in this generality) is to show that the category of stably compact spaces and perfect maps is equivalent to Nachbin's category of compact Hausdorff ordered spaces and monotone continuous maps [18]. The earliest explicit reference to this fact seems to be [4, Section VII-1]—see also [5,15]. This extends the earlier result by Priestley [19,20] that the category of spectral spaces and perfect maps (a full subcategory of the category of stably compact spaces and perfect maps) is equivalent to the category of ordered Stone spaces and monotone continuous maps (see Townsend [24] for an intuitionistic localic version).

Salbany [21] keeps the given topology and the cocompact topology separated, obtaining an alternative formulation of the equivalence via bitopological spaces. A localic version of this approach is obtained by combining the work of Banaschewski and Brümmer [2] with the work of Townsend [24]. Banaschewski and Brümmer showed that the category of stably compact locales and perfect maps is dually equivalent to the category of compact regular biframes and biframe homomorphisms. Townsend defined ordered locales and proved that the category of compact regular ordered locales and monotone continuous maps is dually equivalent to the category of compact regular biframes and biframe homomorphisms, concluding that the category of compact regular ordered locales and monotone continuous maps is equivalent to the category of stably compact locales and perfect maps.

Given the coreflections stated above, it is natural to ask whether there is an equivalence between the category of regular locally compact ordered locales and monotone perfect maps with the category of stably locally compact locales and perfect maps, via (suitably defined) regular locally compact biframes. The anonymous referee pointed out that there are some difficulties in trying to achieve such an equivalence, considering the real line $\mathbb{R}$ as an example. The corresponding stably locally compact locale would need a point at infinity, the directed join of the finite points. But classically all points are perfect and so the point at infinity would also need to be in $\mathbb{R}$, which it isn't.

A biframe is a frame L equipped with two subframes L_1 and L_2 that together generate L. As part of their construction, Banaschewski and Brümmer assign a biframe L to every stably compact locale A by explicitly constructing L_1 and L_2 as subframes of the frame of all nuclei on A and then declaring that L is the subframe generated by L_1 and L_2. (To be precise, they work with the frame of congruences, which, as they emphasize, is isomorphic to the frame of nuclei.) The frame L_1 consists of the closed nuclei, and the frame L_2 is isomorphic to the frame of Scott open filters on A. We show that L turns out to be the frame of Scott continuous nuclei, and that L_2 turns out to be the frame of Scott continuous fitted nuclei. Here a nucleus is *fitted* if it induces a sublocale which is the intersection of its neighbourhoods.

The above remark is related to the Hofmann-Mislove Theorem, which says that in a sober space, the set of Scott open filters of open sets, ordered by inclusion, is dual to the set of compact saturated sets, also ordered by inclusion [7]. (See also the earlier [4, Lemma V-5.3], which is attributed to Hofmann and Lawson [6].) The Hofmann-Mislove Theorem unavoidably relies on excluded-middle and choice principles. An intuitionistic localic version, which says that in any locale there is an order-reversing bijection between compact fitted sublocales and Scott open filters of open sublocales, was proved by Johnstone [11, Lemma 3.4] (see Vickers [27] for another proof).

Banaschewski and Brümmer state that their arguments are specifically chosen to be independent of any choice principle. However, they give a contrapositive proof of their Lemma 3 and hence their argument is valid only in Boolean toposes. But, as remarked by Karazeris (personal communication), their conclusion is actually valid in any topos. In fact, by the above observations, their lemma turns out to be a particular case of the fact that the frame of Scott continuous nuclei on a compact locale defines a compact locale (Lemma 2.5 below).

Smyth [23] and Jung and Sünderhauf [13] consider finitary presentations of stably compact locales via certain proximity lattices. We leave as an open problem to present our localic patch construction in their finitary framework.

Notice that, following Isbell [8] and Johnstone [10], we adopt the geometrical point of view and regard locales as generalized (sober) spaces rather than special kinds of lattices. The category **Loc** of locales and continuous maps is thus defined as the opposite of the category of frames and frame homomorphisms. We adopt the following notation and terminology, which emphasizes this point of view. We denote locales by X, Y, Z (and also by A, B, C in order to emphasize different categories of locales when we consider adjunctions), and the corresponding frame of a locale X by $\mathcal{O}\,X$. The elements of $\mathcal{O}\,X$ are referred to as *opens* and are ranged over by the letters U, V, W. A continuous map $f : X \to Y$ is given by a frame homomorphism $f^* : \mathcal{O}\,Y \to \mathcal{O}\,X$, with right adjoint denoted by $f_* : \mathcal{O}\,X \to \mathcal{O}\,Y$.

I benefited from remarks by Panagis Karazeris and Steve Vickers on a previous version of this paper. In particular, Panagis let me know about his work [14], and Steve drew my attention to the paper [2] by Banaschewski and Brümmer. Detailed comments by the anonymous referee helped to improve the exposition. Also, the referee proposed a strengthening of the original version of Corollary 5.2, with an elegant simplification of its proof.

2 The patch frame

We begin by recalling the definitions and facts concerning nuclei needed in the development that follows (see [10,17,11]) and establishing some terminology and notation. We then consider Scott continuous nuclei.

In the category of topological spaces and continuous maps, the natural notion of subobject, namely that of (homeomorphically embedded) subspace, is not captured by the notion of monomorphism, but rather by the more restrictive notion of *regular* monomorphism. The same is true for the category of locales and continuous maps, where nuclei are used as canonical representatives of equivalence classes of regular monomorphisms.

A *nucleus* on a frame is a finite-meet-preserving inflationary idempotent. A *sublocale* of a locale X is a locale of the form X_j, where j is a nucleus on $\mathcal{O}\,X$, with frame of opens defined by $\mathcal{O}\,X_j = \{U \in \mathcal{O}\,X | j(U) = U\}$. For every sublocale X_j of X, there is a regular monomorphism $e : X_j \to X$ given by $e^*(U) = j(U)$. Conversely, any regular monomorphism $e : X' \to X$ induces the nucleus $j = e_* \circ e^*$ on $\mathcal{O}\,X$, which makes X' isomorphic to X_j. A sublocale of a sublocale is a sublocale of the original locale, and the subobject order on sublocales coincides with set-theoretic inclusion of frames.

The nuclei on a frame form themselves a frame when endowed with the pointwise order. Thus, for any locale X, there is a locale $\mathrm{N}\,X$ defined by stipulating that $\mathcal{O}\,\mathrm{N}\,X$ is the frame of nuclei on $\mathcal{O}\,X$. An arbitrary meet of nuclei is computed pointwise. A join is strictly above the pointwise join in general, and a non-empty join coincides with the pointwise join iff the latter is idempotent. The frame of nuclei is dual to the lattice of sublocales, in the sense that $j \leq k$ iff $X_j \geq X_k$.

For each open $U \in \mathcal{O}\,X$ there is a nucleus 'U' $\in \mathcal{O}\,\mathrm{N}\,X$ defined by

$$\text{`}U\text{'}(V) = U \vee V.$$

This is often referred to as a *closed* nucleus, because it induces a closed sublocale of X thought as the complement of U. But, since the frame of nuclei is dual to the lattice of sublocales, a nucleus should be thought as the formal complement of its induced sublocale. We thus think of 'U' as a copy of the open $U \in \mathcal{O}\,X$ in the frame $\mathcal{O}\,\mathrm{N}\,X$. In fact, the assignment $U \mapsto$ 'U' is a frame homomorphism. We denote by $\rho_X : \mathrm{N}\,X \to X$ the continuous map defined by $\rho_X^*(U) =$ 'U'. The right adjoint of ρ_X^* is given by $(\rho_X)_*(j) = j(0)$. The continuous map $\rho_X : \mathrm{N}\,X \to X$ is both epi and mono (but not regular mono).

Every nucleus of the form 'U' has a boolean complement, given by

$$(\neg`U\text{'})(V) = (U \Rightarrow V).$$

The nuclei of the form 'V' $\wedge \neg$'U' constitute a base of $\mathcal{O}\,\mathrm{N}\,X$. In fact, every $j \in \mathcal{O}\,\mathrm{N}\,X$ is the join of the nuclei $j_U \stackrel{\text{def}}{=}$ '$j(U)$' $\wedge \neg$'U' for $U \in \mathcal{O}\,X$. This can be proved by first showing that $j_U \leq j$ for any U and then observing that for any nucleus l with $j_U \leq l$, one has that $j(U) = j_U(U) \leq l(U)$.

For every continuous map $f : X \to Y$ there is a continuous map $\mathrm{N}\,f : \mathrm{N}\,X \to \mathrm{N}\,Y$ uniquely specified by the condition $(\mathrm{N}\,f)^*($'V'$) =$ '$f^*(V)$'. This makes N into a functor **Loc** $\to$ **Loc** and ρ into a natural transformation $\mathrm{N} \to 1$.

For every nucleus j, the set $j^{-1}(1)$ is a filter. In fact, $j^{-1}(1)$ is an upper set because j is monotone, and it is closed under finite meets because j preserves them. Moreover, this filter is the set of open neighbourhoods of the sublocale induced by j, because $j(U) = 1$ iff $\neg$'U' $\leq j$ iff (by duality) the open sublocale induced by $\neg$'U' contains the sublocale induced by j.

A sublocale is *fitted* if it is an intersection of open sublocales. Dually, a nucleus is *fitted* if it is of the form

$$\kappa_{\mathcal{U}} \stackrel{\text{def}}{=} \bigvee\{\neg`U\text{'} \mid U \in \mathcal{U}\}$$

for some set $\mathcal{U}$ of opens. For every nucleus j, the nucleus $\kappa_{j^{-1}(1)}$ induces the same filter as j and hence is the largest fitted nucleus below j.

2.1 LEMMA (Johnstone [11, Lemma 3.4])

(1) A nucleus j induces a compact sublocale iff the filter $j^{-1}(1)$ is Scott open (that is, inaccessible by directed joins).
(2) For any Scott open filter ϕ of opens of any locale, there is a (largest) nucleus j with $j^{-1}(1) = \phi$ (which has to be κ_ϕ).

2.2 COROLLARY *For any locale, there is an order-reversing bijection between compact fitted sublocales and Scott open filters of opens.*

2.3 DEFINITION A continuous map of locales is *perfect* if the right adjoint of its defining frame homomorphism preserves directed joins. A nucleus is *perfect* if it preserves directed joins.

Some authors refer to perfect maps as *proper maps*. But the latter terminology is often used for a condition stronger than perfectness [25]. Banaschewski [1] refers to perfect nuclei as *finitary nuclei*. Our terminology is due to the fact that a nucleus is perfect iff it is induced by a perfect map.

The perfect nuclei form a subframe of the frame of all nuclei [14, Theorem 4.1] [3, Lemma 3.1.8]. The proof given in [3] observes that the join of a set J of perfect nuclei is given by the pointwise join of the finite compositions of members of J (which are not necessarily nuclei). The proof given in [14], which is based on transfinite iteration, works more generally for "compact" topologies on locally finitely presentable categories. But in the calculations given below, we are only going to use the fact that *directed joins* of perfect nuclei are computed pointwise—see [14, Proposition 4.3] or the remark preceding [3, Lemma 3.1.8].

2.4 DEFINITION The *patch* of a locale X is the locale $\operatorname{Patch} X$ defined by stipulating that $\mathcal{O} \operatorname{Patch} X$ is the frame of perfect nuclei on $\mathcal{O} X$.

Since $\mathcal{O} \operatorname{Patch} X$ is a subframe of $\mathcal{O} \operatorname{N} X$, there is an epimorphism $q_X : \operatorname{N} X \to \operatorname{Patch} X$ given by $q_X^*(j) = j$. Since a nucleus of the form 'U' is perfect, the map $\rho_X : \operatorname{N} X \to X$ factors as $\operatorname{N} \xrightarrow{q_X} \operatorname{Patch} X \xrightarrow{\varepsilon_X} X$ for a unique continuous map $\varepsilon_X : \operatorname{Patch} X \to X$, given by $\varepsilon_X^*(U) =$ 'U'. Notice that $(\varepsilon_X)_*(j) = j(0)$.

2.5 LEMMA *$\varepsilon_X : \operatorname{Patch} X \to X$ is a perfect map.*

PROOF Let $J \subseteq \mathcal{O} \operatorname{Patch} X$ be directed. Since directed joins in $\mathcal{O} \operatorname{Patch} X$ are computed pointwise, we have that $(\varepsilon_X)_*(\bigvee J) = (\bigvee J)(0) = \bigvee\{j(0) \mid j \in J\} = \bigvee\{(\varepsilon_X)_*(j) \mid j \in J\}$. $\square$

2.6 LEMMA *If $f : X \to Y$ is perfect and Y is compact, so is X.*

In particular, if X is a compact locale so is $\operatorname{Patch} X$, and a perfect nucleus on a compact locale induces a compact sublocale.

PROOF The defining frame homomorphism of a perfect map always preserves the way-below relation, and in particular compact opens [4]. $\square$

3 The patch frame of a spectral locale

A *clopen* is an open with a boolean complement. A locale is *zero-dimensional* if the clopens form a base (that is, every open is a join of clopens). A *Stone locale* is a zero-dimensional compact locale. The category of Stone locales and continuous maps is denoted by **Stone**.

A *spectral locale* is a locale for which the compact opens form a base closed under finite meets. Since this includes the empty meet, a spectral locale is compact. A continuous map is *spectral* if its defining frame homomorphism preserves compact opens. The category of spectral locales and spectral maps is denoted by **Spec**.

Every clopen of a compact locale is compact. Since the clopens are closed under finite meets, Stone locales are spectral. Since frame homomorphisms preserve finite meets and joins, they also preserve clopens and hence continuous maps of Stone locales are spectral. Therefore **Stone** is a full subcategory of **Spec**. A continuous map of spectral locales is spectral iff it is perfect [4].

3.1 LEMMA *Let A be a spectral locale.*

1. *If $U \in \mathcal{O}\, A$ is compact then 'U' has a boolean complement in $\mathcal{O}\,\mathrm{Patch}\, A$.*
2. *The nuclei of the form 'V' $\wedge \neg$'U', with $U, V \in \mathcal{O}\, A$ compact, constitute a base of $\mathcal{O}\,\mathrm{Patch}\, A$.*
3. $\mathrm{Patch}\, A$ *is a Stone locale.*
4. $\varepsilon_A : \mathrm{Patch}\, A \to A$ *is a monomorphism in* **Spec**.
5. *If A is a Stone locale then $\varepsilon_A : \mathrm{Patch}\, A \to A$ is an isomorphism.*

PROOF (1): We have to show that if $U \in \mathcal{O}\, A$ is compact then the boolean complement of 'U' in $\mathcal{O}\,\mathrm{N}\, A$ is perfect. It suffices to conclude that for all opens $V, W \in \mathcal{O}\, A$ with W compact and $W \leq (\neg$'U'$)(V)$, there is a compact open $V' \leq V$ such that already $W \leq (\neg$'U'$)(V')$. Since $\neg$'U'$(V) = (U \Rightarrow V)$, we have that $W \wedge U \leq V$. By the spectral property, $W \wedge U$ is compact. Therefore we can take $V' = W \wedge U$.

(2): Since a perfect nucleus j on a spectral locale is completely determined by its effect on compact opens, one has that j is the join of the nuclei 'V' $\wedge \neg$'U' with $U, V \in \mathcal{O}\, A$ compact and $V \leq j(U)$.

(3): By Lemmas 2.5 and 2.6, $\mathrm{Patch}\, A$ is compact, and by (2), it is zero-dimensional.

(4): We know that a map of spectral locales is spectral iff it is perfect. Hence the map is spectral by Lemma 2.5. We have to show that, for every $f : B \to \mathrm{Patch}\, A$ in **Spec**, the frame homomorphism $f^* : \mathcal{O}\,\mathrm{Patch}\, A \to \mathcal{O}\, B$ is uniquely determined by its effect on nuclei of the form 'U' with $U \in \mathcal{O}\, A$ compact. If we know $f^*($'U'$)$ then we know $f^*(\neg$'U'$)$ because frame homomorphisms preserve boolean complements. Hence we also know $f^*($'U' $\wedge \neg$'V'$)$ for all $U, V \in \mathcal{O}\, A$ compact. Therefore, by (2), we know $f^*(j)$ for any $j \in \mathcal{O}\,\mathrm{Patch}\, A$.

(5): We know that ε_A^* is always one-to-one. In a Stone locale the clopens coincide with the compact opens. Since frame homomorphisms preserve boolean complements, one has that for a clopen U the identity $\neg$'U' $=$ '$\neg U$' holds. Hence every perfect nucleus is a join of nuclei of the form 'U' and hence is itself of this form. Therefore ε_A^* is an isomorphism with inverse given by $(\varepsilon_A^{-1})^*(j) = j(0)$. □

3.2 Lemma *The functor* $\mathrm{N} : \mathbf{Loc} \to \mathbf{Loc}$ *restricts to a functor* $\mathrm{Patch} : \mathbf{Spec} \to \mathbf{Stone}$ *via the (necessarily natural) transformation* $q : \mathrm{N} \to \mathrm{Patch}$. *Moreover, the restriction presents* ε *as a natural transformation* $\mathrm{Patch} \to 1$.

Proof We have to show that $(\mathrm{N}\, f)^* : \mathcal{O}\,\mathrm{N}\,B \to \mathcal{O}\,\mathrm{N}\,A$ preserves perfect nuclei if $f : A \to B$ is a spectral map of spectral locales. Let $j \in \mathcal{O}\,\mathrm{Patch}\,B \subseteq \mathcal{O}\,\mathrm{N}\,B$. By Lemma 3.1, we know that j is a join of opens of the form 'U' $\wedge \neg$'V' with U, V compact. Hence $(\mathrm{N}\, f)^*(j)$ is a join of opens of the form '$f^*(U)$' $\wedge \neg$'$f^*(V)$', because homomorphisms preserve finite meets and boolean complements. By the spectral property of f, the open $f^*(V)$ is compact. Therefore $(\mathrm{N}\, f)^*(j)$ is perfect. □

3.3 Theorem *The patch construction exhibits the category of Stone locales and continuous maps as a coreflective subcategory of the the category of spectral locales and spectral maps.*

Proof For every spectral locale A, the map $\varepsilon_A : \mathrm{Patch}\,A \to A$ is universal among spectral maps $f : X \to A$ on Stone locales X, because the composite $\mathrm{Patch}\, f \circ \varepsilon_X^{-1}$ is a map $\bar{f} : X \to \mathrm{Patch}\,A$ with $\varepsilon_A \circ \bar{f} = f$ by naturality of ε and because a map $\bar{f}$ with this property is necessarily unique as ε_A is mono. □

Joyal and Tierney [12, pages 25–26 and 31–32] showed that the frame of nuclei on a given frame is the universal solution to the problem of adding boolean complements to the opens of the given frame. The above adjunction can be interpreted as saying that, for a spectral locale, the patch frame is the universal solution to the problem of adding boolean complements to the compact opens. (See also Vickers [26, pages 129–130].)

4 Boundedly regular locales

In this section we develop a characterization of the category of regular locally compact locales and perfect maps, which we use in the next section to prove our main theorem.

By definition, an open U is closed iff there is a (necessarily unique) open W (its boolean complement) with $U \wedge W = 0$ and $U \vee W = 1$. The well-inside relation defined below gives a relative notion of closedness, and regularity generalizes zero-dimensionality.

One says that an open U is *well inside* an open V (or that U is *closed in* V), written $U \leqslant V$, if there is an open W with $U \wedge W = 0$ and $V \vee W = 1$. Equivalently, $U \leqslant V$ iff $V \vee \neg U = 1$, where $\neg U$ is the Heyting complement of U. A locale is *regular* if every open V is a join of opens $U \leqslant V$. The category of compact regular locales and continuous maps is denoted by **KReg**. Since frame homomorphisms preserve finite meets and joins, they preserve the well-inside relation.

At this point we assume familiarity with the theory of continuous lattices [4] [10, Chapter VII]. The way-below relation gives a relative notion of compactness. One says that U is *way below* V (or that U is *compact in* V), written $U \ll V$, if every open cover of V has a finite subcover of U, or, equivalently, if every directed cover of V has a member that covers U. A locale is *locally compact* if every open V is a join of opens $U \ll V$. In other words, a locale is locally compact iff its frame is a continuous lattice. We have already mentioned that the defining frame homomorphism of a perfect map always preserves the way-below relation. For a continuous map of locally compact locales, the converse holds [4].

A locally compact locale is *stably locally compact* if for every open $U \ll 1$, the set $\twoheaduparrow U \stackrel{\text{def}}{=} \{V | U \ll V\}$ is a filter. That is, a locally compact locale is stably locally compact iff its way-below relation is *multiplicative*, in the sense that $U \ll V$ and $U \ll W$ together imply $U \ll V \wedge W$. A locale is *stably compact* if it is compact and stably locally compact. The category of stably locally compact locales and perfect maps is denoted by **SLK**, and the full subcategory on the stably compact locales is denoted by **SK**.

The well-inside relation is always multiplicative [10, Lemma III-1.1]. In a compact locale, the well-inside relation entails the way-below relation, and in a regular locale the converse holds [10, Lemma VII-3.5]. Therefore **KReg** is a full subcategory of **SK**.

4.1 Definition An open U is *bounded* if $U \ll 1$. A locale is *boundedly regular* if every open V is a join of bounded opens $U \lessdot V$. A continuous map is *cobounded* if its defining frame homomorphism preserves bounded opens. The category of boundedly regular locales and cobounded maps is denoted by **BReg**.

Clearly, every boundedly regular locale is regular, and every compact regular locale is boundedly regular. The following is a slight generalization of the fact that in a compact locale the well-inside relation entails the way-below relation [10, Lemma VII-3.5(i)].

4.2 Lemma *In any locale, $U' \lessdot U \ll 1$ implies $U' \ll U$.*

Proof In order to show that $U' \ll U$, let $\mathcal{V}$ be a directed set with $U \leq \bigvee \mathcal{V}$. Since $U' \lessdot U$, we have that $U \vee \neg U' = 1$. Then $1 = U \vee \neg U' \leq (\bigvee \mathcal{V}) \vee \neg U' = \bigvee \{V \vee \neg U' \mid V \in \mathcal{V}\}$. Since $U \ll 1$, there is some $V \in \mathcal{V}$ with $U \leq V \vee \neg U'$.

From this we deduce that $V \vee \neg U' = 1$ and hence that $U' \leq V$, which shows that $U' \ll U$. □

4.3 Lemma *The following conditions are equivalent for any locale X.*

(1) X is boundedly regular.
(2) X is regular and locally compact.
(3) X is regular and stably locally compact.

Proof (1 $\implies$ 2): By bounded regularity, any open V is a join of opens $U \lessdot V$ with $U \ll 1$. But also any such U is a join of opens $U' \lessdot U$ (necessarily with $U' \ll 1$). Hence V is a join of opens U' for which there exists an open $U \lessdot V$ with $U' \lessdot U \ll 1$. By Lemma 4.2, the latter relation entails $U' \ll U$, which in turn entails $U' \ll V$. Therefore V is a join of opens $U' \ll V$, which shows that X is locally compact.

(2 $\implies$ 1): By local compactness, every open V is a join of opens $U \ll V$. By regularity, $U \ll V$ implies $U \lessdot V$. Therefore V is a join of opens $U \lessdot V$ with $U \ll 1$, which shows that X is boundedly regular.

(2 $\implies$ 3): Assume that $U \ll V$ and $U \ll W$. By local compactness and the interpolation property, there are V' and W' with $U \ll V' \ll V$ and $U \ll W' \ll W$. By regularity, $U \lessdot V'$ and $U \lessdot W'$, and hence $U \lessdot V' \wedge W'$. Since $V' \ll 1$ and $W' \ll 1$, we have that $V' \wedge W' \ll 1$. It follows from Lemma 4.2 that $U \ll V' \wedge W'$. Since $V' \wedge W' \leq V \wedge W$, we finally conclude that $U \ll V \wedge W$, which shows that X is stably locally compact. □

4.4 Lemma *A continuous map of boundedly regular locales is cobounded iff it is perfect.*

Proof ($\implies$): Let $f : X \to Y$ be a cobounded map of boundedly regular locales, and assume that $U \ll V$ in $\mathcal{O}Y$. By Lemma 4.3, the locale Y is locally compact. Hence there is an open V' with $U \ll V' \ll V$ by the interpolation property. Then $V' \ll 1$ and, by regularity, $U \lessdot V'$. Since frame homomorphisms preserve the well-inside relation and f is cobounded, $f^*(U) \lessdot f^*(V') \ll 1$. It follows by Lemma 4.2 that $f^*(U) \ll f^*(V')$. Since $f^*(V') \leq f^*(V)$, we conclude that $f^*(U) \ll f^*(V)$. This shows that f^* preserves the way-below relation and hence that f is perfect.

($\impliedby$): The defining frame homomorphism of a perfect map preserves the way-below relation, and any frame homomorphism preserves the open 1. □

Lemmas 4.3 and 4.4 show that **BReg** is a full subcategory of **SLK**. The terminal locale is trivially boundedly regular. And, clearly, a locale is compact iff the unique continuous map to the terminal locale is cobounded. Since there are boundedly regular, non-compact locales, e.g. the localic real line, we conclude that there are continuous maps of boundedly regular locales that are not cobounded.

5 The patch frame of a stably locally compact locale

The following lemma is our main tool.

5.1 LEMMA *If ϕ is a Scott open filter of opens of a stably locally compact locale, then the nucleus κ_ϕ is perfect and its defining join can be computed pointwise.*

PROOF It is enough to show that the function $k(V) = \bigvee\{U \Rightarrow V \mid U \in \phi\}$ is Scott continuous and has $k \circ k \leq k$. By definition of Heyting implication,

(i) If $U \in \phi$ and $W \wedge U \leq V$ then $W \leq k(V)$.

Conversely, the set $\{U \Rightarrow V \mid U \in \phi\}$ is directed because ϕ is a filter and because the map $U \mapsto (U \Rightarrow V)$ transforms joins into meets. Hence if $W \ll k(V)$ there is $U' \in \phi$ with $W \ll (U' \Rightarrow V)$. Since ϕ is Scott open, there is $U \in \phi$ with $U \ll U'$. By stability, $W \wedge U \ll (U' \Rightarrow V) \wedge U' \leq V$. This shows that

(ii) If $W \ll k(V)$ then $W \wedge U \ll V$ for some $U \in \phi$.

In order to show that k is Scott continuous, it suffices to show that whenever $W \ll k(V)$, there exists $V' \ll V$ with $W \leq k(V')$. But now this is immediate because we can find $U \in \phi$ with $W \wedge U \ll V$ by (ii) and then take $V' = W \wedge U$ by (i). In order to show that $k \circ k \leq k$, assume that $W \ll k(k(V))$. By two successive applications of (ii), we first have that $W \wedge U \ll k(V)$ for some $U \in \phi$ and then that $W \wedge U \wedge U' \ll V$ for some $U' \in \phi$. Since $U \wedge U' \in \phi$ as ϕ is a filter, we conclude by (i) that $W \leq k(V)$. Therefore $k(k(V)) \leq k(V)$. □

The following is a special case of Lemma 2.1(2).

5.2 COROLLARY *If ϕ is a Scott open filter of opens of a stably locally compact locale, then $\kappa_\phi^{-1}(1) = \phi$.*

It follows from Lemma 2.1(1) that κ_ϕ induces a compact sublocale.

PROOF For suppose $\kappa_\phi(V) = 1$. By Lemma 5.1 it is computed pointwise as a directed join $\bigvee\{U \Rightarrow V \mid U \in \phi\}$ and 1 is in ϕ, so by Scott openness we have $(U \Rightarrow V) \in \phi$ for some $U \in \phi$. Then $V \geq U \wedge (U \Rightarrow V)$ is in ϕ. □

The notation κ_U is a shorthand for $\kappa_{\Uparrow U}$. Notice that $\kappa_U \leq \neg$'U'.

5.3 Lemma *If* $U \ll V \ll 1$ *holds for opens of a stably locally compact locale, then* 'U' $\lll$ 'V' *and* $\kappa_V \lll \kappa_U$.

Proof Since $U \ll 1$, the set $\Uparrow U$ is a filter by stable local compactness. Then $\kappa_U(V) = \bigvee\{U' \Rightarrow V \mid U \ll U'\}$ by Lemma 5.1. Since $(V \Rightarrow V) = 1$ we conclude that $\kappa_U(V) = 1$ by considering $U' = V$. As it is well-known and easy to check [17], for any nucleus j, one has that the join 'V' $\vee j$ is the composite $j \circ$ 'V'. This and the fact that $\kappa_U(V) = 1$ show that 'V' $\vee \kappa_U =$ '1' because ('V'$\vee \kappa_U)(U) = (\kappa_U)(V \vee U) = 1$. Since $\kappa_U \leq \neg$'U', we conclude 'U'$\wedge \kappa_U =$ '0' and hence that $U \lll V$. And since 'V'$\wedge \kappa_V =$ '0' we conclude that $\kappa_V \lll \kappa_U$. □

5.4 Lemma *Let* A *be a stably locally compact locale.*

(1) The nuclei 'V' $\wedge \kappa_\phi$*, with* $V \in \mathcal{O}A$ *and* $\phi \subseteq \mathcal{O}A$ *a Scott open filter, constitute a base of* $\mathcal{O}\,\mathrm{Patch}\,A$.
(2) $\mathrm{Patch}\,A$ *is boundedly regular.*
(3) If A *is boundedly regular then* $\varepsilon_A : \mathrm{Patch}\,A \to A$ *is an isomorphism with inverse given by* $(\varepsilon_A^{-1})^*(j) = j(0)$.

Proof (1): We show that any perfect nucleus j is the join of the nuclei $j_{U,V} \stackrel{\text{def}}{=}$ 'V' $\wedge \kappa_U$ with $V \ll j(U)$ and $U \ll 1$. Since $\kappa_U \leq \neg$'U', we have that j is an upper bound of such nuclei. Let l be another, and let U and V be opens with $V \ll j(U)$. By Scott continuity of j, there is $U' \ll U$ with $V \ll j(U')$. Hence $V \leq (V \vee U) \wedge 1 =$ 'V'$(U) \wedge \kappa_{U'}(U) = j_{U',V}(U) \leq l(U)$ by Lemma 5.1. Therefore j is the least upper bound.

(2): Given $V \ll j(U)$ with $U \ll 1$, we first find opens U' and V' with $U' \ll U$ and $V \ll V' \ll j(U')$ by Scott continuity of j, local compactness of A, and the interpolation property. We then conclude that the relations 'V' $\lll$ 'V'' and $\kappa_U \lll \kappa_{U'}$ hold by Lemma 5.3. Since the well-inside relation is multiplicative, we have that $j_{U,V} \lll j_{U',V'}$, and since $j_{U',V'} \leq j$, we conclude that $j_{U,V} \lll j$, which establishes regularity. Since $V \ll 1$, since the defining frame homomorphism of a perfect map preserves the way-below relation, and since $\varepsilon_A : \mathrm{Patch}\,A \to A$ is perfect by Lemma 2.5, we conclude that 'V' $\ll$ '1' by definition of ε_A. Therefore $j_{U,V} \ll$ '1' because $j_{U,V} \leq$ 'V', which establishes bounded regularity.

(3): Since compact sublocales of regular locales are closed, the basic nuclei of item (1) induce closed sublocales by Lemma 2.1. □

Banaschewski and Brümmer [2] assign a (compact regular) biframe L to each stably compact locale A as follows (where we use nuclei instead of their frame congruences):

(1) $L_1 = \{`U\text{'} \mid U \in \mathcal{O}\,A\}$,
(2) $L_2 = \{\kappa_\phi \mid \phi \subseteq \mathcal{O}\,A$ is a Scott open filter$\}$,
(3) L is the subframe of $\mathcal{O}\,\mathrm{N}\,A$ generated by L_1 and L_2.

Corollary 5.2 shows that L_2 turns out to be the subframe of perfect fitted nuclei, and Lemma 5.4(1) shows that L turns out to be $\mathcal{O}\,\mathrm{Patch}\,A$.

5.5 Lemma *If $f : A \to B$ is a perfect map of stably locally compact locales, then $(\mathrm{N}\,f)^* : \mathcal{O}\,\mathrm{N}\,B \to \mathcal{O}\,\mathrm{N}\,A$ preserves perfect nuclei. Moreover, the induced continuous map $\mathrm{Patch}\,f : \mathrm{Patch}\,A \to \mathrm{Patch}\,B$ is uniquely determined by the condition that*

$$(\mathrm{Patch}\,f)^*(`V\text{'}) = `f^*(V)\text{'}, \qquad (\mathrm{Patch}\,f)^*(\kappa_\phi) = \kappa_{f^*[\phi]}$$

for every open $V \in \mathcal{O}\,B$ and every Scott open filter $\phi \subseteq \mathcal{O}\,B$.

Proof Let $j \in \mathcal{O}\,\mathrm{Patch}\,B \subseteq \mathcal{O}\,\mathrm{N}\,B$. By Lemma 5.4(1), we know that j is a join of opens of the form $`V\text{'} \wedge \kappa_\phi$. Hence $(\mathrm{N}\,f)^*(j)$ is a join of opens of the form $(\mathrm{N}\,f)^*(`V\text{'}) \wedge (\mathrm{N}\,f)^*(\kappa_\phi)$. The nucleus $(\mathrm{N}\,f)^*(`V\text{'})$ is perfect because it is $`f^*(V)\text{'}$. Also,

$$\begin{aligned}
(\mathrm{N}\,f)^*(\kappa_\phi) &= (\mathrm{N}\,f)^*(\textstyle\bigvee\{\neg `V\text{'} \mid V \in \phi\}) && \text{by definition of } \kappa_\phi, \\
&= \textstyle\bigvee\{(\mathrm{N}\,f)^*(\neg `V\text{'}) \mid V \in \phi\} && \text{as homomorphisms preserve joins,} \\
&= \textstyle\bigvee\{\neg(\mathrm{N}\,f)^*(`V\text{'}) \mid V \in \phi\} && \text{as they preserve complements,} \\
&= \textstyle\bigvee\{\neg `f^*(V)\text{'} \mid V \in \phi\} && \text{by specification of } \mathrm{N}\,f, \\
&= \kappa_{f^*[\phi]}, && \text{by definition of } \kappa_{f^*[\phi]}.
\end{aligned}$$

The set $f^*[\phi]$ is not necessarily a filter, but it is filtered as frame homomorphisms preserve finite meets. Since f^* preserves the way-below relation, the filter γ generated by $f^*[\phi]$ is Scott open. Also, it is clear that $\kappa_{f^*[\phi]} = \kappa_\gamma$. Hence $\kappa_{f^*[\phi]}$ is perfect. Therefore so is $(\mathrm{N}\,f)^*(j)$. □

5.6 Lemma *If $f : A \to B$ is a perfect map of stably locally compact locales, then the induced continuous map $\mathrm{Patch}\,f : \mathrm{Patch}\,A \to \mathrm{Patch}\,B$ is cobounded.*

Proof Assume that $j \ll `1\text{'}$ holds in $\mathcal{O}\,\mathrm{Patch}\,B$. By local compactness and the fact that directed joins of perfect nuclei are computed pointwise, the maximum nucleus $`1\text{'}$ is the directed join of the perfect nuclei $`V\text{'}$ for $V \ll 1$. Hence $j \leq `V\text{'}$ for some $V \ll 1$. Since f is a perfect map by assumption, $f^*(V) \ll f^*(1) = 1$, and since $\varepsilon_A : \mathrm{Patch}\,A \to A$ is perfect by Lemma 2.5, we have that $`f^*(V)\text{'} \ll `1\text{'}$. An application of Lemma 5.5 gives $(\mathrm{Patch}\,f)^*(j) \leq (\mathrm{Patch}\,f)^*(`V\text{'}) = `f^*(V)\text{'}$. Therefore $(\mathrm{Patch}\,f)^*(j) \ll `1\text{'}$. □

5.7 Corollary *The functor* N : **Loc** → **Loc** *restricts to a functor* Patch : **SLK** → **BReg** *via the (necessarily natural) transformation* $q : \mathrm{N} \to \mathrm{Patch}$. *Moreover, the restriction presents* ε *as a natural transformation* Patch → 1.

5.8 Theorem *The patch construction exhibits the category of boundedly regular locales and cobounded maps as coreflective subcategory of the category of stably locally compact locales and perfect maps.*

Proof This is equivalent to saying that the functor Patch : **SLK** → **BReg** is right adjoint to the inclusion functor. For every boundedly regular locale X, we can define $\eta_X : X \to \mathrm{Patch}\, X$ as ε_X^{-1} because Lemma 5.4(3) shows that ε_X is an isomorphism. Then η is a natural transformation and, by definition, the composite $X \xrightarrow{\eta_X} \mathrm{Patch}\, X \xrightarrow{\varepsilon_X} X$ is the identity. Thus, in order to establish the claim, it is enough to show that the composite

$$\mathrm{Patch}\, A \xrightarrow{\eta_{\mathrm{Patch}\, A}} \mathrm{Patch}\,\mathrm{Patch}\, A \xrightarrow{\mathrm{Patch}\, \varepsilon_A} \mathrm{Patch}\, A$$

is also the identity [16, Theorem IV-1.2(v)]. Denote the composite by g. If $V \in \mathcal{O}\, A$ then

$$\begin{aligned}
g^*(\text{`}V\text{'}) &= \eta^*_{\mathrm{Patch}\, A} \circ (\mathrm{Patch}\, \varepsilon_A)^*(\text{`}V\text{'}) && \text{by definition of } g,\\
&= \eta^*_{\mathrm{Patch}\, A}(\text{`}\varepsilon^*_A(V)\text{'}) && \text{by Lemma 5.5},\\
&= \eta^*_{\mathrm{Patch}\, A}(\text{``}V\text{''}) && \text{by definition of } \varepsilon_A,\\
&= \text{``}V\text{''}(\text{`}0\text{'}) && \text{by Lemma 5.4(3)},\\
&= \text{`}V\text{'} \vee \text{`}0\text{'} && \text{by definition of ``}V\text{''},\\
&= \text{`}V\text{'}.
\end{aligned}$$

If $\phi \subseteq \mathcal{O}\, A$ is a Scott open filter then

$$\begin{aligned}
g^*(\kappa_\phi) &= \kappa_{\varepsilon^*_A[\phi]}(\text{`}0\text{'}) && \text{by Lemmas 5.5 and 5.4(3)},\\
&= \bigvee\{\neg\text{`}\varepsilon^*_A(V)\text{'} \mid V \in \phi\}(\text{`}0\text{'}) && \text{by definition of } \kappa_{\varepsilon^*_A[\phi]},\\
&= \bigvee\{(\neg\text{`}\varepsilon^*_A(V)\text{'})(\text{`}0\text{'}) \mid V \in \phi\} && \text{by Lemma 5.1},\\
&= \bigvee\{(\neg\text{``}V\text{''})(\text{`}0\text{'}) \mid V \in \phi\} && \text{by definition of } \varepsilon_A,\\
&= \bigvee\{\text{`}V\text{'} \Rightarrow \text{`}0\text{'} \mid V \in \phi\}\\
&= \bigvee\{\neg\text{`}V\text{'} \mid V \in \phi\}\\
&= \kappa_\phi && \text{by definition of } \kappa_\phi.
\end{aligned}$$

Since g^* is the identity on the opens of a subbase by Lemma 5.4(1), it is the identity on all opens. □

Since if A is compact so is Patch A, the functor Patch : $\mathbf{SLK} \to \mathbf{BReg}$ further restricts to a functor Patch : $\mathbf{SK} \to \mathbf{KReg}$.

5.9 COROLLARY *The patch construction exhibits the category of compact regular locales and continuous maps as coreflective subcategory of the category of stably compact locales and perfect maps.*

This adjunction can be interpreted as saying that the patch frame of the frame of opens of a stably compact locale is the universal solution to the problem of transforming the way-below relation into the well-inside relation. Of course, this generalizes the fact that the patch frame of the frame of opens of a spectral locale is the universal solution to the problem of adding complements to the compact opens, because an open U is compact iff $U \ll U$, and it is complemented iff $U \leqslant U$.

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