

A Universal Characterization of the Closed Euclidean Interval

(EXTENDED ABSTRACT)

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Abstract *We propose a notion of interval object in a category with finite products, providing a universal property for closed and bounded real line segments. The universal property gives rise to an analogue of primitive recursion for defining computable functions on the interval. We use this to define basic arithmetic operations and to verify equations between them. We test the notion in categories of interest. In the category of sets, any closed and bounded interval of real numbers is an interval object. In the category of topological spaces, the interval objects are closed and bounded intervals with the Euclidean topology. We also prove that an interval object exists in any elementary topos with natural numbers object.*

1 Introduction

In set theory, one can implement the real numbers in many ways. For example, one can use Dedekind sections or equivalence classes of Cauchy sequences of rational numbers. But *what* is it that one is implementing? Assuming classical logic, either implementation produces a complete Archimedean field and, moreover, any two such fields are isomorphic. In fact, for the purposes of classical analysis, one never uses a particular mathematical *implementation* of the reals. One relies instead on the *specification* of the real-number system as a complete Archimedean field and works axiomatically. The only purpose of particular implementations is to be reassured that there is at least one such field.

Unfortunately, when one tries to carry out such a programme in other foundational settings, difficulties arise. One obstacle is that the categoricity of this axiomatization relies on the principle of excluded middle, which is not always available, particularly in settings that are relevant to the theory of computation. Further, one may criticize the axiomatization on the grounds that, although it is aiming to characterize the real line, which is fundamentally a geometric structure, it makes essential use of abstract concepts,

such as suprema of bounded sets of points, whose geometric meaning is unclear. In addition, the field axioms involve operations, such as multiplication and reciprocation, which one might rather see as derived from more primitive constructions.

A further objection to the field axiomatization is its lack of explicit computational content. To develop a theory of computability in the sense of Turing [32], one has to start by effectively presenting a particular implementation of the field of real numbers. For example, one can implement real numbers as Cauchy sequences of rational numbers with fixed rate of convergence [3]. Then one has to argue that the basic field operations are computable and that various methods of defining new functions from old preserve computability—see e.g. Weihrauch [34]. With this approach, computability arguments involve heavy manipulation of Gödel numberings, which are detached from the usual practice of real analysis.

The above contrasts with the natural numbers, where primitive recursion, the basic computational mechanism, is not only embodied in their usual Peano axiomatization but can also be taken as their defining property. An elegant formulation of such an axiomatization was given by Lawvere in his definition of a natural numbers object [22]. This style of axiomatization has been adopted for other inductively defined data types, such as lists and trees, which admit canonical forms of recursion that reflect their characterization as initial algebras. Dually, infinite data types, such as streams, are characterized as final coalgebras, with corresponding forms of corecursion. This formulation of data types has been convincingly exploited by Bird and de Moor in their algebraic approach to programming [2].

To place the real numbers into the above framework, one requires a notion of real number data type whose defining property embodies primitive mechanisms for recursion over the reals. In this paper, we present such an axiomatization for closed and bounded line segments, or *interval objects* for short. We characterize interval objects by a universal property that captures a basic geometrical notion and si-

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multaneously provides a computational notion of recursion. Thus, remarkably, our axiomatization reconciles geometrical and computational conceptions of the line.

In brief, our axiomatization:

- (i) is based on elementary geometrical considerations,
- (ii) has direct computational content,
- (iii) applies in a wide variety of settings,
- (iv) gives what one would expect in specific examples.

Regarding (i), we take a *midpoint* operation as the basic structure of line segments, with four axioms that correspond to intuitive geometric properties. We define a *convex body* as a midpoint algebra in which the midpoint operation can be *infinitely iterated*, in a precise sense discussed in the technical development that follows. Then an *interval object* is defined to be a free convex body over two generators, its endpoints. Geometrically, the free property amounts to the fact that any two points of a convex body are connected by a unique line segment.

Regarding (ii), the free property gives rise to an analogue of primitive recursion for defining computable functions on the interval. In particular, we use this to define basic arithmetic operations and to verify equations between them.

Regarding (iii), we make as few ontological commitments as possible by formulating our definitions in the general setting of a category with finite products. Nevertheless, to make the paper accessible to readers who are uncomfortable with category theory, we use, as far as possible, standard algebraic notation, so that everything we say can be easily understood in familiar mathematical terms. Indeed, when specialized to categories such as sets and topological spaces, our definitions assume rather concrete meanings.

Regarding (iv), we have: (1) In the category of sets, any closed and bounded interval of real numbers is an interval object (Theorem 1). (2) In the category of topological spaces, any closed and bounded interval under the usual Euclidean topology is an interval object (Theorem 2). Thus, our axiomatization of line segments exhibits the Euclidean topology as intrinsic rather than imposed structure, because it is this topology that gives rise to an interval object. This is interesting in connection with the often cited fact that the computable functions on the reals are continuous. (3) In any elementary topos with natural numbers object, an interval object is given by the Cauchy completion of the interval of Cauchy reals within the Dedekind reals (Theorem 3). In many cases this coincides with the Cauchy or Dedekind intervals; but, in general, we seem to be identifying an intriguing new intuitionistic notion of real number. For details see Section 9. Some other possible settings are discussed briefly in Section 10.

For lack of space, all proofs are omitted from this extended abstract.

Related work This paper has its origins in the first author’s work on exact real number computation [10, 11]. In this approach, real numbers are represented by concrete computational structures such as streams, allowing computations to be performed to any desired degree of accuracy [35, 6, 4, 5, 33]. Of particular relevance to our work is the issue of obtaining an abstract data type of real numbers, in which the underlying computational representation is hidden [5, 8, 10, 11].

In the programming language Real PCF [10], the abstract data type is based on simple real number *constructors* and *destructors*. Mathematically, the constructors are unary midpoint operations $x \mapsto 0 \oplus x$ and $x \mapsto x \oplus 1$ on the unit interval $[0, 1]$, where $x \oplus y = (x + y)/2$ is the binary midpoint operation. These primitives are used by Escardó and Streicher [11] to characterize the interval data type by a universal property, from which structural recursion mechanisms for real numbers are obtained. Thus, this work achieves many of the aims of the present paper. However, it crucially relies on general recursion and the consequent presence of partiality. Indeed, the interval data type includes *partial* real numbers as essential ingredients of its characterization, and the characterization only works in a domain-theoretic setting.

The goal of the present work is to obtain a characterization of the real numbers that applies to a variety of computational settings, including those, such as intuitionistic type theory [25], in which only total functions are available. Although such a programme has not been undertaken previously, algebraic and coalgebraic techniques, similar to the ones used in the present paper, do occur in previous axiomatizations of the reals.

Higgs [14] defines *magnitude algebras* and proves that the interval $[0, \infty]$ endowed with the function $x \mapsto x/2$ and the summation operation $\sum : [0, \infty]^\omega \rightarrow [0, \infty]$ is the magnitude algebra freely generated by 1. His definition is purely equational and is based on binary expansions of numbers. Although our work has some connections with Higgs’, especially regarding the idea of using an infinitary operation, there are some important differences. Firstly, in the category of topological spaces, the free magnitude algebra over one generator is the interval $[0, \infty]$ with the topology of lower semicontinuity rather than the Euclidean topology. Indeed, the infinitary summation operation is not continuous with respect to the Euclidean topology. Secondly, in general, the Dedekind or Cauchy $[0, \infty]$ intervals in an elementary topos are not magnitude algebras, let alone free ones, as there are toposes, such as Johnstone’s topological topos [17], in which these objects do not support the summation operation.

Motivated by the stream implementations of real numbers, Pavlović and Pratt [29] consider *coalgebraic* definitions of the reals. However, they do not make connections

with the computational and geometrical requirements discussed above. Peter Freyd [12] considers a more geometrical coalgebraic approach. In fact, he also places emphasis on midpoint algebras, although the midpoint operation is derived rather than primitive. His approach does appear to have some computational content, but this has yet to be elaborated.

2 Convex bodies and interval objects

This section presents the main definitions of this paper, the notions of *abstract convex body* and *interval object*.

As discussed in the introduction, we define the interval as the free convex body over two generators. To do this, we require an abstract notion of convex body that makes no reference to real numbers. We achieve this by viewing convex bodies as algebraic structures.

The algebraic structure we identify is that associated with the basic ruler-and-compass construction of bisecting a line. Given two points in a convex body A , this construction finds the point midway between them. It thus corresponds to a binary *midpoint* operation $m : A \times A \rightarrow A$. We begin by axiomatizing the equational properties of such midpoint operations.

Let \mathcal{C} be a category with finite products.

Definition 2.1 (Midpoint algebra) A *midpoint algebra* in \mathcal{C} is a pair (A, m) , where $A \times A \xrightarrow{m} A$ is any morphism, satisfying:

1. $m(x, x) = x$ (*idempotency*)
2. $m(x, y) = m(y, x)$ (*commutativity*)
3. $m(m(x, y), m(z, w)) = m(m(x, z), m(y, w))$ (*transposition*)

A midpoint algebra is said to be *cancellative* if it satisfies:

4. $m(x, z) = m(y, z)$ implies $x = y$ (*cancellation*)

A *homomorphism* from (A, m) to (A', m') is a morphism $A \xrightarrow{f} A'$ such that $f(m(x, y)) = m'(f(x), f(y))$. We write $\text{MidAlg}(\mathcal{C})$ for the category of midpoints algebras and their homomorphisms.

In order to understand such ordinary algebraic notation in an arbitrary category with finite products, the variables must be interpreted as generalized elements. Thus, for example, the homomorphism equation states: for all generalized elements $x, y : Z \longrightarrow A$ (where Z is any object), $f \circ m \circ \langle x, y \rangle = m' \circ \langle f \circ x, f \circ y \rangle$. In this case, the condition simplifies to the (unquantified) equation $f \circ m = m' \circ (f \times f)$.

The equations of midpoint algebras are not new. For example, they have appeared as the axioms of *medial means* in

the work of Kermit [20]. They have also recently been popularized by Peter Freyd in his investigations of (co)algebraic properties of the interval [12].

Example 2.2 The set \mathbb{R}^n is a cancellative midpoint algebra under the function $\oplus : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\mathbf{x} \oplus \mathbf{y} = (\mathbf{x} + \mathbf{y})/2.$$

This yields a whole range of cancellative midpoint algebras given by subsets $A \subseteq \mathbb{R}^n$ closed under \oplus . We call such midpoint algebras *standard midpoint subalgebras of \mathbb{R}^n* . Examples are: the set of dyadic rational points; the set of rational points; the set of algebraic points; any convex set.

These examples show that the midpoint axioms are still far from capturing the full power of convexity, which requires one to be able to fill in an entire connected line between any two points. Intuitively, we need to express something like a notion of Cauchy completeness for midpoint algebras. However, Cauchy completeness itself cannot be the appropriate notion, as midpoint algebras do not necessarily carry a metric structure. More fundamentally, we cannot use the notion of metric space to define the interval, because axiomatizing metric spaces already begs the question of what the real numbers are. Instead, we need a method of axiomatizing the completeness of midpoint algebras in terms of their algebraic structure alone.

Consider an arbitrary sequence of points x_0, x_1, \dots in an ordinary Euclidean convex body A . Let z be any point of A and consider the derived sequence

$$m(x_0, z), m(x_0, m(x_1, z)), m(x_0, m(x_1, m(x_2, z))), \dots$$

If A is bounded then this is a Cauchy sequence whose unique limit point lies in A and is independent of z . Thus, any sequence x_0, x_1, \dots , determines a unique point $m(x_0, m(x_1, m(x_2, \dots)))$ obtained by infinitely iterating the binary operation m over the sequence. Our notion of completeness for a midpoint algebra A is to ask that such infinite iterations always exist.

In the category of sets, such a requirement can be expressed directly, albeit clumsily—see Proposition 3.1. Remarkably, there is a very concise formulation in purely categorical terms. Infinite sequences of elements of A are naturally expressed using coalgebras for the functor $(A \times (-))$, i.e. morphisms of the form $\langle h, t \rangle : X \longrightarrow A \times X$. Indeed, any such coalgebra determines an object X of sequences of elements of A , as specified by the *head* and *tail* maps $h : X \longrightarrow A$ and $t : X \longrightarrow X$ respectively. We can now state the property of being able to iterate the midpoint operation m over any sequence so specified.

Definition 2.3 (Iterative algebra) A midpoint algebra (A, m) is *iterative* if it satisfies the *iteration axiom*: for every map $X \xrightarrow{c} A \times X$, there exists a unique $X \xrightarrow{u} A$ such that the diagram below commutes.

$$\begin{array}{ccc} A \times X & \xrightarrow{\text{id} \times u} & A \times A \\ \uparrow c & & \downarrow m \\ X & \xrightarrow{u} & A \end{array}$$

In other words, (A, m) is iterative if, for any coalgebra $c = \langle h, t \rangle : X \longrightarrow A \times X$, there exists a unique u satisfying $u(x) = m(h(x), u(t(x)))$.

The above definition states that a midpoint algebra (A, m) is iterative if it is final as an $(A \times (-))$ -algebra with respect to coalgebra-to-algebra homomorphisms from $(A \times (-))$ -coalgebras. Interestingly, the dual notion of a coalgebra being initial with respect to arbitrary algebras has arisen in recent work of Taylor [31, Section 6.3] and Eppendahl [9].

We are now in a position to formulate our abstract notion of convex body.

Definition 2.4 (Abstract convex body) An *abstract convex body* is a cancellative iterative midpoint algebra.

We henceforth omit the word abstract, except when required to avoid confusion due to alternative notions of convex body being available (for example, in Euclidean space, where ordinary convex bodies are convex sets with nonempty interior). We write $\text{Conv}(\mathcal{C})$ for the full subcategory of $\text{MidAlg}(\mathcal{C})$ whose objects are convex bodies.

Example 2.5 Continuing from Example 2.2, any bounded convex subset of \mathbb{R}^n , considered as a standard midpoint subalgebra of \mathbb{R}^n , is an abstract convex body. Indeed, given functions $h : X \rightarrow A$ and $t : X \rightarrow X$, where X is any set, the unique function $u : X \rightarrow A$ determined from the coalgebra $\langle h, t \rangle : A \rightarrow A \times X$ by the iteration axiom is

$$u(x) = \sum_{i \geq 0} 2^{-(i+1)} h(t^i(x)). \quad (1)$$

An important point is that the boundedness of A is crucial for u to be well-defined. In fact, a standard midpoint subalgebra of \mathbb{R}^n is an abstract convex body if and only if it is a bounded convex subset of \mathbb{R}^n ; and, given a bounded convex subset B of \mathbb{R}^m , a function $f : A \rightarrow B$ is a homomorphism of abstract convex bodies (i.e. a homomorphism w.r.t. \oplus) if and only if it is affine. See Section 3 for details.

Example 2.6 Let A be any bounded convex subset of \mathbb{R}^n endowed with the Euclidean topology. Then \oplus also exhibits A as a convex body in the category \mathbf{Top} of topological spaces. Indeed, given any continuous $(A \times (-))$ -coalgebra $\langle h, t \rangle : X \rightarrow A \times X$ (where X is any space), the function u defined in (1) is again the unique map required by the iteration axiom. The interesting fact here is that u is continuous. This example will be expanded upon in Section 8.

As motivated in the introduction, the interval will be defined as the free abstract convex body over two generators. This amounts to being an initial object in a suitable category of bipointed convex bodies.

A *bipointed* convex body is a structure (A, m, a, b) where (A, m) is a convex body and $a, b : \mathbf{1} \rightarrow A$ are global points. *Homomorphisms* between bipointed convex bodies are required to preserve the points as well as the binary algebra structure; i.e. $f : A \rightarrow A'$ is a homomorphism from (A, m, a, b) to (A', m', a', b') if and only if it is a homomorphism from (A, m) to (A', m') and $a' = f \circ a$ and $b' = f \circ b$. We write $\text{BiConv}(\mathcal{C})$ for the category of bipointed convex bodies and their homomorphisms.

We can now give the main definition of the paper.

Definition 2.7 (Interval object) An *interval object* in \mathcal{C} is an initial object in $\text{BiConv}(\mathcal{C})$.

Example 2.8 In \mathbf{Set} , any closed interval $[a, b] \subseteq \mathbb{R}$, with $a < b$, gives an interval object $([a, b], \oplus, a, b)$. Of course the choice of a and b makes no difference. For future convenience, we take the interval $\mathbb{I} = [-1, 1]$ as our standard closed interval and $(\mathbb{I}, \oplus, -1, 1)$ as our standard interval object. This example is discussed in more detail in Section 3.

Example 2.9 In \mathbf{Top} , $(\mathbb{I}, \oplus, -1, 1)$ is again an interval object when \mathbb{I} is equipped with the Euclidean topology. This is discussed further in Section 8.

3 Interval objects in the category of sets

In this section we study abstract convex bodies in the category \mathbf{Set} of sets, and we show that the interval object in \mathbf{Set} is indeed $(\mathbb{I}, \oplus, -1, 1)$, as claimed in Example 2.8.

The least familiar aspect of the definition of convex body is the notion of iterative algebra. We begin by showing that, in \mathbf{Set} , iterative algebras are exactly algebras supporting an additional operation of countably-infinite arity that satisfies certain characterising properties relating it to the binary operation. In general, this reformulation provides the most straightforward method of showing that an algebra is iterative.

Proposition 3.1 Let (A, m) be a midpoint algebra in **Set**.

1. (A, m) is iterative if and only if there exists a function $M : A^\omega \rightarrow A$ satisfying:

- (a) $M(x_0, x_1, x_2, \dots) = m(x_0, M(x_1, x_2, x_3, \dots))$
- (b) If $y_0 = m(x_0, y_1)$, $y_1 = m(x_1, y_2)$, $y_2 = m(x_2, y_3)$, \dots then $y_0 = M(x_0, x_1, x_2, \dots)$.

Moreover if (A, m) is iterative then there is a unique M satisfying (a).

2. If (A, m) and (A', m') are iterative midpoint algebras then any homomorphism $f : A \rightarrow A'$ is also a homomorphism with respect to the associated infinitary M and M' ; i.e. for every sequence x_0, x_1, \dots ,

$$f(M(x_0, x_1, \dots)) = M'(f(x_0), f(x_1), \dots).$$

With an appropriate reformulation, the above proposition generalizes from the category of sets to any category with finite products and a parameterized natural numbers objects.

It is useful to identify additional equational properties satisfied by the the associated infinitary operations. We use $M_i(x_i)$ as a shorthand for $M(x_0, x_1, x_2, \dots)$.

Proposition 3.2 For any iterative midpoint algebra (A, m) in **Set**, with infinitary $M : A^\omega \rightarrow A$,

- 1. $x = M(x, x, x, \dots)$,
- 2. $m(x, y) = M(x, y, y, y, \dots)$,
- 3. $M_i(M_j(x_{ij})) = M_j(M_i(x_{ji}))$,
- 4. $M_i(m(x_i, y_i)) = m(M_i(x_i), M_i(y_i))$.

For an iterative midpoint algebra to be a convex body it must also be cancellative. We have yet to see any technical consequence of this property. In fact, for iterative midpoint algebras, cancellation is equivalent to an important approximation property. To formulate this, we write m_n for the $(n + 1)$ -ary operation defined by $m_0(x) = x$ and $m_n(x_0, \dots, x_n) = m(x_0, m_{n-1}(x_1, \dots, x_n))$ for $n \geq 1$. Thus m_1 is just m itself.

Proposition 3.3 For an iterative midpoint algebra (A, m) in **Set**, the following are equivalent.

- 1. (A, m) is cancellative.
- 2. The associated $M : A^\omega \rightarrow A$ satisfies the following approximation property.

If, for all $n \geq 0$, there exist $z_n, w_n \in A$ such that $m_n(x_0, \dots, x_{n-1}, z_n) = m_n(y_0, \dots, y_{n-1}, w_n)$ then

$$M(x_0, x_1, \dots) = M(y_0, y_1, \dots).$$

This is far from immediate and is used crucially in the proof of Theorem 1.

Having obtained a good understanding of what the different aspects of the definition of convex body mean in **Set**, we return to Examples 2.5 and 2.8.

Proposition 3.4 If A is a standard midpoint subalgebra of \mathbb{R}^n , then A is an abstract convex body if and only if it is a bounded convex subset of \mathbb{R}^n .

Suppose $A \subseteq \mathbb{R}^n$ and $A' \subseteq \mathbb{R}^m$ are convex sets. Recall that a function $f : A \rightarrow A'$ is said to be *affine* if it preserves so-called convex combinations, i.e., for $\lambda_1, \dots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$,

$$f\left(\sum_{i=1}^k \lambda_i \mathbf{x}_i\right) = \sum_{i=1}^k \lambda_i f(\mathbf{x}_i).$$

The next proposition demonstrates the naturalness of homomorphisms between abstract convex bodies.

Proposition 3.5 For bounded convex sets $A \subseteq \mathbb{R}^n$ and $A' \subseteq \mathbb{R}^m$, a function $f : A \rightarrow A'$ is affine if and only if it is a homomorphism with respect to \oplus .

An example due to Peter Freyd [12], which uses the axiom of choice, can be used to show that the boundedness assumption is essential for Proposition 3.5 to hold.

Theorem 1 $(\mathbb{I}, \oplus, -1, 1)$ is an interval object in **Set**.

4 Parameterized interval objects

It is well known that Lawvere's elegant definition of a natural numbers object, which works very well in cartesian closed categories, is not powerful enough in categories with weaker structure. Instead, a modified parameterized definition is needed [21, 7]. In a category with finite products, the notion of parameterized natural numbers object supports the definition of functions by primitive recursion. Moreover, in a cartesian closed category, any ordinary natural numbers objects is automatically parameterized. Much the same situation arises for interval objects.

Definition 4.1 (Parameterized interval object) A *parameterized interval object* is a bipointed convex body $(I, \oplus, -1, 1)$ such that, for any convex body (A, m) and morphisms $X \xrightarrow{f} A$ and $X \xrightarrow{g} A$ in \mathcal{C} , there exists a unique morphism $X \times I \xrightarrow{([f, g])} A$ satisfying

$$\begin{aligned} ([f, g])(x, y \oplus z) &= m([f, g](x, y), [f, g](x, z)), \\ ([f, g])(x, -1) &= f(x), \\ ([f, g])(x, 1) &= g(x), \end{aligned}$$

i.e. there is a unique *right-homomorphism* of bipointed convex bodies from $X \times I$ to A .

By instantiating X to the terminal object, it is easily seen that any parameterized interval object is indeed an interval object. The converse holds when \mathcal{C} is cartesian closed:

Proposition 4.2 *If \mathcal{C} is cartesian closed then any interval object is parameterized.*

Henceforth in this section, let \mathcal{C} be a category with finite products and parameterized interval object $(I, \oplus, -1, 1)$. The basic arithmetic operations on I can be defined by

$$\begin{aligned} \mathbf{1} &\xrightarrow{0} I = (-1) \oplus (1), \\ I &\xrightarrow{-} I = ([1, -1]), \\ I \times I &\xrightarrow{\times} I = ([-, \text{id}_I]). \end{aligned}$$

More explicitly, the above defines multiplication as the unique morphism $I \times I \xrightarrow{\times} I$ satisfying

$$\begin{aligned} x \times (y \oplus z) &= (x \times y) \oplus (x \times z), \\ x \times (-1) &= -x, \\ x \times 1 &= x. \end{aligned}$$

Importantly, the universal property of I , stated in Definition 4.1, suffices to establish the basic equations between the above operations.

Proposition 4.3 $--x = x$,

$$\begin{aligned} x \times y &= y \times x, \\ x \times (y \times z) &= (x \times y) \times z, \\ -0 &= 0, \\ x \oplus -x &= 0, \\ -(x \oplus y) &= (-x) \oplus (-y), \\ x \times 0 &= 0, \\ x \times -y &= -(x \times y). \end{aligned}$$

The most entertaining proof is that of the commutativity of multiplication.

5 Primitive interval functions

In this section we give some preliminary results on the power of the notion of interval object with respect to defining functions on the interval. As mentioned above, any parameterized natural numbers object supports definition by primitive recursion. Here we investigate the definitional mechanisms supported by parameterized interval objects.

In fact, a parameterized interval object supports two complementary styles of definition. On the one hand, the universal property of parameterized initiality gives one mechanism for defining functions, used above to define negation and multiplication. On the other, the couniversal property of the iteration axiom supports another type of definition, needed, for example, to define non dyadic rational numbers. Parameterized interval objects allow any

combination of these two styles. We investigate the power of such combinations for the purpose of defining functions on \mathbb{I} in **Set**.

Definition 5.1 (Primitive interval functions) The *primitive interval functions* on \mathbb{I} are the functions in the smallest family $\{\mathcal{F}_n \subseteq \mathbb{I}^n \rightarrow \mathbb{I}\}_{n \geq 0}$ satisfying:

- (i) $-1, 1 \in \mathcal{F}_0$.
- (ii) If $f \in \mathcal{F}_m$ and $g_1, \dots, g_m \in \mathcal{F}_n$ then the composite $f \circ \langle g_1, \dots, g_m \rangle \in \mathcal{F}_n$.
- (iii) If $f, g \in \mathcal{F}_n$ then the function h defined below is in \mathcal{F}_{n+1} :

$$h(\mathbf{x}, y) = \frac{1}{2}(1-y)f(\mathbf{x}) + \frac{1}{2}(1+y)g(\mathbf{x}).$$

- (iv) If $f_1, \dots, f_n, g \in \mathcal{F}_n$ then the unique function h satisfying the equation below is in \mathcal{F}_n :

$$h(\mathbf{x}) = \frac{1}{2}g(\mathbf{x}) + \frac{1}{2}h(f_1(\mathbf{x}), \dots, f_n(\mathbf{x})).$$

Here (iii) corresponds to the parameterized initiality of \mathbb{I} , with respect to \mathbb{I}^n as the object of parameters, and (iv) corresponds to the iteration axiom, as induced by the coalgebra $\langle g, f_1, \dots, f_n \rangle : \mathbb{I}^n \rightarrow \mathbb{I} \times \mathbb{I}^n$. Note that property (ii) means that tuples of primitive interval functions between finite powers of \mathbb{I} form a category. This category has finite products because the projections are definable, using (iii).

The function defined by (iv) is given explicitly by

$$h(\mathbf{x}) = \sum_{i \geq 0} 2^{-(i+1)} g(\langle f_1, \dots, f_n \rangle^i(\mathbf{x})).$$

A natural generalization is to replace the sequence $(g \circ \langle f_1, \dots, f_n \rangle^i)_i$ of composite functions with an arbitrary sequence of (already defined) n -ary functions.

Definition 5.2 (Countably-primitive functions) The *countably-primitive interval functions* on \mathbb{I} are the functions in the smallest family $\{\mathcal{F}_n \subseteq \mathbb{I}^n \rightarrow \mathbb{I}\}_{n \geq 0}$ satisfying (i)–(iii) of Definition 5.1 and also

- (iv)' Given $f_0, f_1, \dots \in \mathcal{F}_n$, the function h defined below is in \mathcal{F}_n :

$$h(\mathbf{x}) = \sum_{i \geq 0} 2^{-(i+1)} f_i(\mathbf{x}).$$

Clearly every primitive interval function is a countably-primitive interval function. The converse does not hold as there are continuum many countably-primitive interval functions, but only countably many primitive interval functions. Indeed, every element of \mathbb{I} gives a countably-primitive interval function of arity 0 (i.e. a constant). Although this cannot hold for the primitive interval functions, we do at least have the following.

Proposition 5.3 *Every rational in \mathbb{I} gives a primitive interval constant.*

The proof makes crucial use of property (iv).

As in Section 4, we have $\oplus, -, \times$ as primitive interval functions. Thus every n -variable \oplus -polynomial (i.e. polynomial where \oplus replaces the usual $+$) with rational coefficients is an n -ary primitive interval function.

We are not sure how much further definability can be pushed with the primitive interval functions, as we now show that even the countably-primitive interval functions are very limited.

Proposition 5.4 *If f is an n -ary countably-primitive interval function, and $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \in \mathbb{I}$ are such that $y_i = x_i$ whenever $x_i \in \{-1, 1\}$, then $f(x_0, \dots, x_{n-1}) \in \{-1, 1\}$ implies $f(y_0, \dots, y_{n-1}) = f(x_0, \dots, x_{n-1})$.*

This is proved by induction over the defining properties of the countably-primitive interval functions.

Thus if f is a unary countably-primitive interval function and $f(x) \in \{-1, 1\}$ for some x in the interior $(-1, 1)$ then f is a constant function. Clearly then, the following *truncated double* function is not a countably-primitive interval function.

$$d(x) = \begin{cases} 1 & \text{if } 1/2 \leq x, \\ 2x & \text{if } -1/2 \leq x \leq 1/2, \\ -1 & \text{if } x \leq -1/2. \end{cases}$$

Accordingly, define the *d-primitive interval functions* to be the smallest class of functions containing d and closed under (i)–(iv). Define the *countably-d-primitive interval functions* analogously. The reason for selecting d amongst the non-countably-primitive interval functions is:

Proposition 5.5 *The n -ary countably-d-primitive interval functions are exactly the continuous functions $\mathbb{I}^n \rightarrow \mathbb{I}$.*

The proof uses the Stone-Weierstrass approximation theorem [30].

Thus including d as a basic function enormously increases definability. It is our hope that this increase in definability also means that the *d-primitive interval functions* form a useful class, somewhat analogous to the primitive recursive functions on \mathbb{N} . Although we have yet to undertake any systematic investigation of this class, we do have one important result. Recall the standard notion of an *n*-ary *computable function* on \mathbb{I} [34].

Proposition 5.6 *Every n -ary d-primitive interval function is an n -ary computable function on \mathbb{I} .*

This result follows from Theorem 3 of Section 9 below, by interpreting it in a realizability topos in which the morphisms on the interval are exactly the computable functions.

However, in the next section, we outline a direct proof, by showing that the computable functions are closed under the defining properties of the *d*-primitive interval functions.

6 An interval data type

In Proposition 3.1, we have seen that, in the category of sets, the iteration axiom is captured by the existence of an infinitary version M of the midpoint operation m . Moreover, a function of convex bodies is a homomorphism with respect to m if and only if it is a homomorphism with respect to M . Additionally, Proposition 3.2 shows that m is easily defined from M . This suggests that one might consider the ω -ary operation M as the primitive algebraic operator on convex bodies, rather than m . In this section, we exploit this idea to base a data type for the interval \mathbb{I} on the term algebra of an ω -ary operation M and two constants -1 and 1 .

We outline an implementation using a functional programming notation similar to ML [28] and Haskell [1] (it is not important whether an eager or lazy language is used). Our data type \mathbb{I} is defined as follows.

```
datatype I = -1 | 1 | M of Nat -> I
```

Within the interval type \mathbb{I} , we single out the ω -branching well-founded trees as those data elements representing points of the interval. Such trees are precisely the elements of the term algebra mentioned above. To interpret a tree as representing an element of \mathbb{I} , the infinitary operator M is interpreted as the iterated midpoint operation

$$M(x_0, x_1, x_2, \dots) = \sum_{i=0}^{\infty} 2^{-(i+1)} x_i,$$

using which any ω -branching well-founded tree evaluates to a unique point in \mathbb{I} . Thus, by this interpretation, \mathbb{I} is given as a quotient of the set of all ω -branching well-founded trees.

The iteration axiom of Definition 2.3, in the concrete form given in Example 2.5, corresponds to the following *corecursion combinator*.

```
corec : (X -> I) -> (X -> X) -> (X -> I)
corec h t x = M (\i -> h(t^i(x)))
```

In this definition, $\backslash i \rightarrow t$ is typewriter notation for the lambda expression $\lambda i.t$ and we use the evident notation for function iteration.

The initiality of \mathbb{I} , as in Definition 2.7, is exhibited by the following *recursion combinator*.

```
rec : ((Nat -> A) -> A) -> A -> A -> (I -> A)
rec N a b -1 = a
rec N a b 1 = b
rec N a b (M s) = N (\i -> rec N a b (s i))
```

In this definition, the first argument N is the infinitary midpoint operation of a given bipointed convex body A , and the second and third arguments a and b are the distinguished points. We have not built any explicit type of parameters into the type of `rec`, because parameterization is induced automatically by the functional language. For example, negation and multiplication are defined as in Section 4, using the recursion combinator.

```
neg : I -> I
neg = rec M 1 -1

mul : I -> I -> I
mul x = rec M (neg x) x
```

The recursion and corecursion combinators correspond to conditions (iii) and (iv) of Definition 5.1 respectively. The truncated double function can also be implemented using the datatype `I`, but this is surprisingly tricky. However, curiously, an algorithm for doing this occurs fairly explicitly in our (omitted) proof of Theorem 3 below. It follows that the d -primitive interval functions are definable on our interval data type `I`.

Because we are using a non-standard representation of the interval, based on the infinitary midpoint operation, it is important to show that our representation is interconvertible with the standard representations used in exact real number arithmetic. One such representation, *signed binary*, uses a data type `I'` of infinite sequences of the three digits `-1`, `0` and `1`—see [35]. It is trivial to convert from signed binary sequences to our representation `I`, using the facts that $0 = M(-1, 1, 1, 1, \dots)$ and that a signed binary expansion $0.d_0d_1d_2\dots$ is the same as $M(d_0, d_1, d_2, \dots)$. To translate in the other direction, one first defines the iterated midpoint operation $M' : (\text{Nat} \rightarrow \text{I}') \rightarrow \text{I}'$ (an interesting programming exercise), and then the conversion function $\text{I} \rightarrow \text{I}'$ is simply `rec M' (\λ i -> -1) (\λ i -> 1)`.

Although we have written this section using a functional language with general recursion, we remark that our representation of the interval can be implemented even more directly using intuitionistic type theory [25]. Indeed, by formulating the recursive definition of the data type `I` as a `W`-type, one obtains precisely the well-founded ω -branching trees over `-1` and `1`, and our recursion combinator is simply the recursor for this type.

7 Basic categorical properties

In this section, we turn our attention to general properties of convex bodies and interval objects arising from their categorical definitions. This general investigation will be useful in Sections 8 and 9, in which we study examples in categories other than `Set`.

One benefit of having simple abstract definitions of convex body and interval object is that it is easy to prove that

these notions are preserved by various categorical constructions and functors. In this section, we state basic results of this nature. The proofs are all routine.

As in Section 2, let \mathcal{C} be a category with finite products.

Proposition 7.1 *The forgetful functors $\text{Conv}(\mathcal{C}) \rightarrow \mathcal{C}$ and $\text{BiConv}(\mathcal{C}) \rightarrow \mathcal{C}$ create limits.*

In particular, if (A, m) and (A', m') are convex bodies then so is $A \times A'$ endowed with

$$(A \times A') \times (A \times A') \xrightarrow{\cong} (A \times A) \times (A' \times A') \xrightarrow{m \times m'} A \times A'$$

and an analogous statement holds for bipointed convex bodies. One simple consequence of this result is that, for any interval object (I, \oplus, a, b) , the n -dimensional cube I^n has an induced convex body structure.

As well as being closed under limits, convex bodies are also closed under internal powers.

Proposition 7.2 *If (A, m) is a convex body then so is*

$$(A^B, A^B \times A^B \xrightarrow{\cong} (A \times A)^B \xrightarrow{m^B} A^B)$$

for any exponentiable object B .

Again, the analogous result holds for bipointed convex bodies.

It is also straightforward to establish conditions under which (bipointed) convex bodies are preserved by functors. Suppose \mathcal{D} is a category with finite products, and the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves finite products. Then there is a functor $\overline{F} : \text{MidAlg}(\mathcal{C}) \rightarrow \text{MidAlg}(\mathcal{D})$ whose action on objects is:

$$\overline{F}(A, m) = (FA, FA \times FA \xrightarrow{\cong} F(A \times A) \xrightarrow{Fm} FA)$$

and whose action on morphisms is inherited from F .

Proposition 7.3 *Suppose that F has a left adjoint.*

1. *The functor $\overline{F} : \text{MidAlg}(\mathcal{C}) \rightarrow \text{MidAlg}(\mathcal{D})$ cuts down to a functor $\overline{F} : \text{Conv}(\mathcal{C}) \rightarrow \text{Conv}(\mathcal{D})$. Similarly, by extending the action of \overline{F} to bipointed objects, a functor $\overline{F} : \text{BiConv}(\mathcal{C}) \rightarrow \text{BiConv}(\mathcal{D})$ is obtained.*
2. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ also has a right adjoint $G : \mathcal{D} \rightarrow \mathcal{C}$ then $\overline{G} : \text{Conv}(\mathcal{D}) \rightarrow \text{Conv}(\mathcal{C})$ is right adjoint to the functor $\overline{F} : \text{Conv}(\mathcal{C}) \rightarrow \text{Conv}(\mathcal{D})$, and $\overline{G} : \text{BiConv}(\mathcal{D}) \rightarrow \text{BiConv}(\mathcal{C})$ is right adjoint to $\overline{F} : \text{BiConv}(\mathcal{C}) \rightarrow \text{BiConv}(\mathcal{D})$. Thus, in particular, $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves interval objects.*

It follows from 1 above that if \mathcal{C} is a full reflective subcategory of \mathcal{D} and if \mathcal{D} has an interval object $(I, \oplus, -1, 1)$ where I is an object of \mathcal{C} then $(I, \oplus, -1, 1)$ is also an interval object in \mathcal{C} .

A special case of statement 2 is that interval objects are preserved by the inverse image functors of essential geometric morphisms between elementary toposes. Thus if $f : \mathcal{E} \rightarrow \mathcal{E}'$ is an essential geometric morphism and \mathcal{E}' has an interval object then its image under f^* gives an interval object in \mathcal{E} . In particular, by Theorem 1, every presheaf topos $\mathbf{Set}^{C^{op}}$ has an interval object obtained as $\Delta(\mathbb{I})$ — recall that the constant presheaf functor, $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}^{C^{op}}$, is the inverse image functor of an essential geometric morphism [24]. More generally, in Section 9, we show that any elementary topos with natural numbers object has an interval object.

8 Interval objects in the category of topological spaces

In this section we return to the claims made earlier in Examples 2.6 and 2.9, investigating abstract convex bodies and interval objects in the category \mathbf{Top} of topological spaces.

Proposition 3.1 generalizes to \mathbf{Top} with the requirement that $M : A^\omega \rightarrow A$ be continuous with respect to the product topology. It follows that, for a bounded convex $A \subseteq \mathbb{R}^n$, the midpoint algebra (A, \oplus) with the discrete topology is *not* an abstract convex body in \mathbf{Top} , because this topology does not make the iterated midpoint operation into a continuous function. Thus the notion of abstract convex body forces one to consider more reasonable topologies on (A, \oplus) .

Proposition 8.1 *For any bounded convex subset $A \subseteq \mathbb{R}^n$ endowed with the Euclidean topology, (A, \oplus) is an abstract convex body in \mathbf{Top} .*

This result is derived from Proposition 3.4, by proving that the infinitary midpoint operation is continuous. Certain other basic information about convex bodies in \mathbf{Top} can be inferred using Proposition 7.3. The forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ has both a left adjoint Δ (giving the discrete topology) and a right adjoint ∇ (giving the indiscrete topology). Thus, both U and ∇ preserve convex bodies. As U does, we see that, by Proposition 3.4, under any topology whatsoever, for a standard midpoint subalgebra A of \mathbb{R}^n to be a convex body in \mathbf{Top} , A must be a bounded convex set. Also, for any bounded convex set, (A, \oplus) with the indiscrete topology is a convex body in \mathbf{Top} .

Also, by Proposition 3.4, if an interval object exists in \mathbf{Top} then U preserves it. In fact, we have already claimed in Example 2.9 that $(\mathbb{I}, \oplus, -1, 1)$ is an interval object in \mathbf{Top} when given the Euclidean topology. As \mathbf{Top} is not cartesian closed, it is appropriate to show that this is a parameterized interval object in the sense of Section 4.

Theorem 2 $(\mathbb{I}, \oplus, -1, 1)$ with the Euclidean topology is a parameterized interval object in \mathbf{Top} .

By Proposition 7.3.1, $(\mathbb{I}, \oplus, -1, 1)$ with the Euclidean topology is a parameterized interval object in any full reflective subcategory of \mathbf{Top} that contains the closed Euclidean interval. Thus, for example, it is a parameterized interval object in the category of compact Hausdorff spaces.

9 Interval objects in an elementary topos

In this section we prove that an interval object exists in any elementary topos with natural numbers object. There are at least two reasons to be interested in such a result. Firstly, elementary toposes include all Grothendieck and realizability toposes, of which there are numerous examples with direct geometrical and/or computational significance. Indeed, we have already mentioned that the results of this section can be used to prove Proposition 5.6.

Our second motivation is to study the notion of interval object using an intuitionistic background logic. It is well known that intuitionistic logic draws sharp distinctions between different, though classically equivalent, definitions of real number. To better understand our notion of interval object, we compare it to the competing intuitionistic accounts of the interval. Somewhat surprisingly, rather than obtaining one of the established notions, interval objects give rise to an apparently new intuitionistic notion of real number, albeit one that coincides with extant notions under the mild assumption of number-number choice.

Let \mathcal{E} be an elementary topos with natural numbers object \mathbf{N} . Among the alternative notions of real number available, two are considered as being the most natural, the Dedekind reals \mathbf{R}_D and the Cauchy (or Cantor) reals \mathbf{R}_C . Both are defined using the object of rationals \mathbf{Q} and its associated ordering. The reader is referred to [16] for details.

A basic fact is that one has inclusions

$$\mathbf{Q} \subseteq \mathbf{R}_C \subseteq \mathbf{R}_D.$$

We say that a subobject $X \subseteq \mathbf{R}_D$ is *Cauchy complete* if every Cauchy sequence in $X^{\mathbf{N}}$ (with modulus) has a limit in X . It is easy to see that the Dedekind reals are Cauchy complete. Obviously, the rationals are not Cauchy complete. The Cauchy reals partially rectify the non-completeness of \mathbf{Q} by adding all limits of Cauchy sequences of rationals. Given N-N-choice, this suffices to make \mathbf{R}_C itself Cauchy complete. However, it seems that, in general, \mathbf{R}_C is *not* Cauchy complete, as, given a Cauchy sequence of Cauchy reals, there is no mechanism for selecting representative rational sequences from which the required limiting sequence of rationals can be extracted.

The possible failure of Cauchy completeness for \mathbf{R}_C makes it natural to introduce another object of reals, namely, the *Cauchy completion of \mathbf{Q} within \mathbf{R}_D* . This object, which we call the object of *Euclidean reals* \mathbf{R}_E , is defined as the

intersection of all Cauchy complete subobjects of \mathbf{R}_D containing the rational numbers.

We have identified three objects of reals

$$\mathbf{R}_C \subseteq \mathbf{R}_E \subseteq \mathbf{R}_D.$$

In the case that \mathcal{E} satisfies N-N-choice, both inclusions are equalities. The Grothendieck topos of sheaves over the Euclidean line is a simple example in which the second inclusion is strict. To our embarrassment, we do not know an example in which the first inclusion is strict. Thus we do not know if the envisaged failure of the Cauchy completeness of \mathbf{R}_C is actually possible—although we are sure that it must be.

Each notion of real number object determines a corresponding notion of interval object; for example,

$$\begin{aligned} \mathbf{I}_D &= \{x \in \mathbf{R}_D \mid -1 \leq x \leq 1\} \\ \mathbf{I}_E &= \{x \in \mathbf{R}_E \mid -1 \leq x \leq 1\} = \mathbf{R}_E \cap \mathbf{I}_D. \end{aligned}$$

The reason for introducing the Euclidean reals in the first place is the following.

Theorem 3 ($\mathbf{I}_E, \oplus, -1, 1$) is an interval object in \mathcal{E} .

Our proof is very long and makes crucial use of Pataraia’s intuitionistic fixed-point theorem for monotonic endomaps of directed complete partial orders [27].

10 Concluding remarks

We have provided an axiomatization of the interval, by means of a geometrically motivated universal property that supports the definition of computable functions. Moreover, we have investigated this axiomatization in a number of settings.

Many other settings remain to be investigated. In the category of setoids over intuitionistic type theory [15, 26], it can be shown that any of the usual constructions of a closed real interval gives an interval object. In the category of locales over any topos, we conjecture that the standard localic interval [18] is an interval object.

By definition, an interval object is a free convex body over two generators. Freely generated convex bodies over different generating objects coincide with other familiar mathematical structures. Interesting examples occur in the category of topological spaces: (1) The free convex body over Sierpinski space is the interval with the topology of lower semicontinuity. (2) The free convex body over the flat domain of booleans under the Scott topology is the interval domain studied in [11] with its pointwise midpoint structure. (3) The free convex body over a finite discrete space of cardinality n is an n -simplex. In particular, the free convex body over three and four generators are the triangle and the tetrahedron. All the above examples are applications of the

left adjoint to the forgetful functor from topological convex bodies to topological spaces, which exists by Freyd’s Adjoint Functor Theorem [23].

There are intriguing connections between midpoint algebras and the probabilistic algebras that arise in the study of probabilistic powerdomains—see the axiomatizations discussed by Heckmann [13]. It is plausible that the free convex body over a sufficiently nice domain may be nothing but the probabilistic powerdomain of normalized valuations [19].

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