Function-space compactifications of function spaces

Martín H. Escardó

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Abstract

If X and Y are Hausdorff spaces with X locally compact, then the compact-open topology on the set $\mathrm{C}(X, Y)$ of continuous maps from X to Y is known to produce the right function-space topology. But it is also known to fail badly to be locally compact, even when Y is locally compact. We show that for any Tychonoff space Y, there is a densely injective space Z containing Y as a densely embedded subspace such that, for every locally compact space X, the set $\mathrm{C}(X, Z)$ has a compact Hausdorff topology whose relative topology on $\mathrm{C}(X, Y)$ is the compact-open topology. The following are derived as corollaries: (1) If X and Y are compact Hausdorff spaces then $\mathrm{C}(X, Y)$ under the compact-open topology is embedded into the Vietoris hyperspace $\mathrm{V}(X \times Y)$. (2) The space of real-valued continuous functions on a locally compact Hausdorff space under the compact-open topology is embedded into a compact Hausdorff space whose points are pairs of extended real-valued functions, one lower and the other upper semicontinuous. The first application is generalized in two ways.

Keywords: Compactification of function spaces, semicontinuous function, continuous lattice, Scott domain, Scott topology, densely injective space, dual topology, Lawson topology, Vietoris hyperspace, patch topology, locally compact space, compact-open topology, core-compact space, Isbell topology.

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1 Introduction

In the *Compendium of Continuous Lattices* [6], the set $\mathrm{C}(X)$ of real-valued continuous functions on a locally compact Hausdorff space X is considered as an application of the theory (page XIV). Under the pointwise operations, this is a sublattice of the complete lattice $\mathrm{LSC}(X)$ of lower semicontinuous functions with values on the extended real line, which is an example of a continuous lattice. As any continuous lattice, it admits two canonical topologies, known as the Scott and the Lawson topologies. The Scott topology is compact and locally compact, but highly non-Hausdorff. Lower semicontinuous functions can be regarded as genuinely continuous functions by considering the topology of lower semicontinuity on the line; under this view, the Scott topology of $\mathrm{LSC}(X)$ coincides with the compact-open topology. The Lawson topology is a refinement of the Scott topology, which is characterized as the unique compact Hausdorff topology making the formation of binary meets into a continuous operation. In the light of the fact that $\mathrm{C}(X)$ with the compact-open topology is not even locally compact in general, as the example $X = [0, 1]$ shows, this is somewhat surprising. It is natural to wonder whether the subspace topology on $\mathrm{C}(X)$ induced by the Lawson topology of $\mathrm{LSC}(X)$ is the compact-open topology. Unfortunately, it turns out to be strictly weaker [14]. But a related construction does produce a Hausdorff compactification of $\mathrm{C}(X)$ and of more general function spaces.

We show that for any Tychonoff space Y there is a space Z containing Y as a densely embedded subspace such that, for every locally compact space X, the compact-open topology of the set $\mathrm{C}(X, Z)$ of continuous functions from X to Z has a compact-Hausdorff refinement whose relative topology on $\mathrm{C}(X, Y)$ is still the compact-open topology. Such a space Z is necessarily non-Hausdorff. We construct it as a continuous Scott domain endowed with the Scott topology. Then the set $\mathrm{C}(X, Z)$ is also a continuous Scott domain under the pointwise ordering, with Scott topology coinciding with the compact-open topology. The compact-Hausdorff refinement is taken as the Lawson topology.

When Y is compact Hausdorff, its closed sets form a continuous lattice under the reverse-inclusion order, with the Lawson topology coinciding with the Vietoris topology. In this case, the space Z can be taken as the closed sets under the Scott topology. From this and general properties of the compact-open topology, we derive as a corollary that if X and Y are compact Hausdorff spaces then $\mathrm{C}(X, Y)$ under the compact-open topology is embedded into the Vietoris hyperspace $\mathrm{V}(X \times Y)$ by the graph map. Generalizations of this situation are considered.

When Y is the Euclidean line, the space Z can be taken as the topological product of two copies of the extended real line, one endowed with the topology of lower semicontinuity and the other with the topology of upper semicontinuity. Thus, as a corollary, we obtain a Hausdorff compactification of a space of continuous real-valued maps by a space of pairs of semicontinuous maps.

Although the theory of continuously ordered sets is our fundamental tool, this paper is specifically written in such a way that the topologist or functional analyst who is not necessarily familiar with the theory should be able to follow the formulations of the propositions and the proposed proofs. The survey Section 2 is based on three lectures that I gave in the *Informal Analysis Seminar* series of the School of Mathematics of the University of St Andrews in February and March 2000. But only the material that is needed for the development that follows has been included—for more applications of continuously ordered sets to analysis see [1]. I am grateful to the organizers and to the audience for valuable feedback. Discussions on a previous version of this paper with Roy Dyckhoff were enjoyable and profitable. This version contains some reactions to his constructive criticism. Many thanks to Reinhold Heckmann for a careful and critical reading of a previous version.

2 Continuous lattices in analysis and topology

In this survey section we present the background material on continuous lattices that is needed for the purposes of this paper. Examples are given to illustrate the defined notions and their theory. Proofs can be found in the references [6, 7, 9, 10, 11, 12, 16]. Occasionally, however, we offer different routes to well-known facts, in which case we include proofs. For detailed historical notes, see the references [6, 10, 11]. The following topics are covered in this survey: (1) Continuously ordered sets. (2) The Scott, dual and Lawson topologies of a continuously ordered set. (3) Densely injective topological spaces. (4) The dual and patch topologies of a topological space. (5) Core-compact topological spaces. (6) Topological function spaces.

2.1 Continuously ordered sets

We are interested in continuous Scott domains, and in continuous lattices in particular, but it is convenient to start from the more general notion of a continuous poset.

The way-below relation A subset D of a partially ordered set is called *directed* if it is non-empty and any two members of D have an upper bound in D. For elements x and y of a partially ordered set, one defines

$x \ll y$ iff every directed set with join above y has a member above x,

and in this case one says that x is *way below* y. The very basic properties of the way-below relation are the following.

2.1 *In any partially ordered set,*

1. $x \ll y$ *implies* $x \leq y$,
2. $x' \leq x \ll y \leq y'$ *implies* $x' \ll y'$, *and*
3. *if* $\perp$ *is a least element then* $\perp \ll x$.

Continuous posets A partially ordered set is *continuous* if for every element x, the set

$$\mathord{\downarrow\!\!\downarrow} x \stackrel{\text{def}}{=} \{u \mid u \ll x\}$$

is directed and has x as its join. Notice that we *don't* assume that the partially ordered set is closed under the formation of arbitrary directed joins—see e.g. Example 2.2 below—but this will be the case in our applications. A *basis* of a continuous partially ordered set is a subset B such that for every member x of the partially ordered set, the set $\{b \in B \mid b \ll x\}$ is directed and has x as its join. Clearly, the set of all elements of a continuous partially ordered set is a basis. Our first example plays a major rôle, both in the theory and in the applications.

2.2 Example (The line) The real line is a continuous poset under its natural order, with way-below relation given by $x \ll y$ iff $x < y$, which shows that the rational points form a basis. Notice that, because the order is linear, a subset is directed iff it is non-empty.

2.3 Example (The plane) The plane is a continuous poset under its coordinatewise order, with way-below relation given by $x \ll y$ iff $x_1 < y_1$ and $x_2 < y_2$.

2.4 Example (Finite posets) Any finite partially ordered set is continuous, with way-below relation coinciding with the partial order.

Other examples of continuous posets are the following. (1) The open sets of a locally compact Hausdorff space under the inclusion order; in this example, $U \ll V$ iff the closure of U is a compact subset of V—see Section 2.5 below. (2) The set of extended real-valued lower semicontinuous functions on a locally compact Hausdorff space under the pointwise order—see Section 2.6 below. (3) The power set of the natural numbers under the inclusion order; in this example, $X \ll Y$ iff X is a finite subset of Y. (4) The subgroups of a group under the subgroup order; in this example, $G \ll H$ iff G is a finitely generated subgroup of H. These last two examples are irrelevant for the purposes of this paper.

Naturally occurring counterexamples are not so easy to find. By virtue of Section 2.5 below, the lattice of open sets of the topological product of countably many copies of the discrete space of natural numbers is one. An artificial counterexample is obtained by adding a top element ∞ to the natural numbers under their natural order, and an element a with $0 \leq a \leq \infty$ but incomparable with other elements. This complete lattice is not a continuous poset because $\mathord{\downarrow\!\!\downarrow} a = \{0\}$.

By (2.1), in a continuous partially ordered set, the sets $\mathord{\downarrow\!\!\downarrow} x$ are ideals (directed lower sets). A non-trivial property of the way-below relation of a continuously ordered set is the following.

2.5 *If $x \ll y$ holds in a continuous partially ordered set, then every directed set with join above y has a member way above x.*

Proof Let D be a directed set with join above y, and let I be $\bigcup\{\downarrow d \mid d \in D\}$. By continuity, I has the same join as D, and, being a directed union of ideals, it is an ideal. Hence if $x \ll y$ then $x \in I$, which means that $x \ll d$ for some $d \in D$. □

Since y is the directed join of the (basis) elements $b \ll y$, we have the following order-density property, which is known as the *interpolation property* and is a fundamental tool of the theory.

2.6 Corollary *If $x \ll y$ holds for elements of a continuous partially ordered set, then there is a (basis) element b with $x \ll b \ll y$.*

Usually, this is proved first and the above is derived as a consequence.

Continuous lattices A continuous *complete* lattice is referred to as a *continuous lattice.*

2.7 Example (The extended line) The real line under its natural order fails to be a continuous lattice only by lacking bottom and top elements (the infima and suprema of the whole line). Thus, the extended line $[-\infty, +\infty]$ is a continuous lattice under its natural order. Its way-below relation is given by $x \ll y$ iff $x < y$ or $x = y = -\infty$, which shows that the set $\{-\infty\} \cup \mathbb{Q}$ is a basis.

Continuous Scott domains By a *continuous Scott domain* we mean a continuous partially ordered set with joins of directed subsets and meets of non-empty subsets. Notice that, in any poset, existence of meets of non-empty subsets is equivalent to existence of joins of upper-bounded subsets. A continuous lattice is clearly a continuous Scott domain, and if a continuous Scott domain fails to be a continuous lattice, it does so only by lacking a top element, which can be artificially added if desired. This is sometimes expressed by saying that a continuous Scott domain is a continuous lattice modulo top element. Notice, however, that if one starts from a continuous lattice, a continuous Scott domain is obtained by removing the top element $\top$ iff $\top \ll \top$. In particular, $[0, \infty)$ is *not* a continuous Scott domain under its natural order.

2.8 Example (The interval domain) The closed intervals of the extended Euclidean line form a continuous lattice under the reverse-inclusion order, with $\mathbf{x} \ll \mathbf{y}$ iff the interior of the interval $\mathbf{x}$ contains the interval $\mathbf{y}$. If the empty interval (a top element) is removed, a continuous Scott domain is obtained.

2.2 The Scott, dual and Lawson topologies

We have seen that continuously ordered sets generalize the real line under its natural order. We now discuss three topologies on continuously ordered sets that generalize the topologies of lower and upper semicontinuity and the Euclidean topology.

The Scott topology An upper subset U of a partially ordered set L is *Scott open* iff every directed subset of L with join in U intersects U. It is an easy exercise to show that the Scott open sets form a topology. Its closed sets are the lower sets that are closed under the formation of existing directed joins. The proof of the following proposition is based on the interpolation property, but it can be proved more directly from (2.5).

2.9 *In a continuous partially ordered set with a basis B, the sets*

$$\twoheaduparrow b \stackrel{\text{def}}{=} \{x \mid b \ll x\}, \qquad b \in B,$$

form a base of the Scott topology.

The Scott topology is highly non-Hausdorff. The following example is extremal, to the extent that no two distinct points can be separated by disjoint neighbourhoods. But, as we shall see in Proposition 3.6, the Scott topology has plenty of Hausdorff subspaces in other examples.

2.10 Example (The topology of lower semicontinuity) In the real line, $\twoheaduparrow a = (a, \infty)$ for any $a \in \mathbb{R}$. Thus, in this case, the Scott topology is the topology of lower semicontinuity.

In fact, when L is the real line, the following coincides with the usual notion of lower semicontinuity that occurs in real analysis.

2.11 Corollary *A function $f : X \to L$ from a topological space to a continuous partially ordered set is continuous with respect to the Scott topology of L iff whenever $v \ll f(x)$ for $v \in L$, there is a neighbourhood U of x with $v \ll f(u)$ for all $u \in U$.*

The dual topology Similarly, one considers a generalization of the notion of upper semicontinuous function in real analysis. But this is not done by considering a symmetric definition as in real analysis, because the notion of continuity for ordered sets is not symmetric in general. The *dual topology* of a partially ordered set is generated by the complements of the principal filters

$$\uparrow v \stackrel{\text{def}}{=} \{x \mid v \leq x\}.$$

For simplicity, we denote the subbasic open sets in the dual topology by

$$\not\uparrow v \stackrel{\text{def}}{=} \{x \mid v \not\leq x\}.$$

2.12 Example (The topology of upper semicontinuity) In the real line, $\not\uparrow a = (-\infty, a)$ for any $a \in \mathbb{R}$. Hence, in this case, the dual topology is the topology of upper semicontinuity.

Thus, in this example the dual topology is the Scott topology of the dual order. But, for other examples, such as the interval domain discussed in Example 2.8, this is not the case.

The following is immediate from the definition of continuity for partially ordered sets. It is used to prove that the Lawson topology, defined below, is Hausdorff.

2.13 *If $x \not\leq y$ holds for elements of a continuous partially ordered set, then there is a (basis) element $b \ll x$ such that already $b \not\leq y$.*

Here we use it to obtain a subbase of the dual topology.

2.14 *In a continuous partially ordered set with a basis B, the sets $\not\uparrow b$ with $b \in B$ form a subbase of the dual topology.*

Proof Let $x \in \not\uparrow v$. Then $v \not\leq x$, and by continuity of the partially ordered set, there is $b \ll v$ in B such that already $b \not\leq x$ and hence $x \in \not\uparrow b$. If $y \in \not\uparrow b$ then $b \not\leq y$ and hence $v \not\leq y$, because if one had $v \leq y$ then one would have $b \leq y$ by transitivity, which shows that $y \in \not\uparrow v$, and therefore that $\not\uparrow b \subseteq \not\uparrow v$. $\square$

The Lawson topology The *Lawson topology* is the join of the Scott and the dual topologies. The following is an immediate consequence of the above development.

2.15 *In a continuous partially ordered set with a basis B, the sets $\twoheaduparrow b$ and $\not\uparrow b$ with $b \in B$ form a subbase of the Lawson topology.*

2.16 Example (The Euclidean topology) For the real line, we know from the above examples that the Lawson topology is the join of the topologies of lower and upper semicontinuity. Therefore it is the Euclidean topology.

We have seen that continuously ordered sets generalize the real line, that the Scott and dual topologies generalize the topologies of lower and upper semicontinuity, and that the Lawson topology generalizes the Euclidean topology. We shall see in Example 3.4 that the Lawson topology generalizes the Vietoris topology on the closed subsets of a compact Hausdorff space, where the following is used to establish the link.

2.17 *The Lawson topology of a continuous Scott domain is the unique compact Hausdorff topology making the formation of binary meets into a continuous operation.*

2.3 Densely injective topological spaces

By an *embedding* of topological spaces we mean a homeomorphism onto a subspace. A topological space Z is called *injective over an embedding* $j : X \hookrightarrow Y$ if any continuous map $f : X \to Z$ extends to a continuous map $\hat{f} : Y \to Z$ along j. A space is *densely injective* if it is injective over dense embeddings.

2.18 *The elements of a continuous Scott domain form a densely injective topological space under the Scott topology.*

If Z is a continuous Scott domain under the Scott topology, an explicit construction of a continuous extension $f/j : Y \to Z$ of a continuous map $f : X \to Z$ along a dense embedding $j : X \to Y$ is given by

$$f/j(y) = \sup_{y \in V \in \mathcal{O}\,Y} \inf f(j^{-1}(V)),$$

where $\mathcal{O}\,Y$ denotes the lattice of open sets of Y. Among all continuous extensions, f/j is characterized as the largest in the pointwise order.

2.19 Example (The topologist's sine curve) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that maps x to $\sin(1/x)$ is continuous, but cannot be extended to a continuous function defined on the whole real line. However, if (1) the topology of $\mathbb{R}$ is weakened to a densely injective topology or (2) its topology is kept unmodified but more points are added to $\mathbb{R}$ in such a way that a densely injective space is obtained, this becomes possible:

1. Let $\underline{\mathbb{R}}$ be the extended real line with the topology of lower semicontinuity. Since the topology of $\underline{\mathbb{R}}$ is weaker than the topology of $\mathbb{R}$, we have a continuous map $f : \mathbb{R} \setminus \{0\} \to \underline{\mathbb{R}}$. Being a continuous lattice under the Scott topology, $\underline{\mathbb{R}}$ is densely injective and hence f has a (lower semi)continuous extension $f/j : \mathbb{R} \to \underline{\mathbb{R}}$ along the dense embedding $j : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. A simple calculation shows that $f/j(0) = -1$.

2. The domain of non-empty closed intervals discussed in Example 2.8 is a densely injective space $\mathrm{I}\,\mathbb{R}$ under the Scott topology. The map $x \mapsto \{x\}$ is an embedding $k : \mathbb{R} \to \mathrm{I}\,\mathbb{R}$. The function f coextends to a continuous function $g : \mathbb{R} \setminus \{0\} \to \mathrm{I}\,\mathbb{R}$ by composition with k; that is, $g(x) = \{\sin(1/x)\}$. By dense injectivity of $\mathrm{I}\,\mathbb{R}$, the function g extends to a continuous function $g/j : \mathbb{R} \to \mathrm{I}\,\mathbb{R}$ along the dense embedding $j : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. A simple calculation shows that $g/j(0) = [-1, 1]$.

The second situation is discussed in more detail and related to the first in Section 3.4 below.

The specialization order The *specialization order* on the points of a topological space is defined by

$x \leq y$ iff every neighbourhood of x is a neighbourhood of y.

Since this means that x belongs to the closure of $\{y\}$, continuous functions preserve the specialization order. The specialization order is always reflexive and transitive, and, remembering that a space is T_0 iff no two distinct points share the same system of neighbourhoods, it is immediate that the specialization order is antisymmetric iff the space is T_0. The specialization order plays no rôle in the theory of Hausdorff spaces. In fact, a topological space is T_1 iff its specialization order is the identity.

2.20 Example (The specialization order of the topology of lower semicontinuity) In the real line with the topology of lower semicontinuity, the relation $x \leq y$ holds in the specialization order iff $x \in (a, \infty)$ implies $y \in (a, \infty)$ iff $a < x$ implies $a < y$ iff $x \leq y$ holds in the natural order.

All implicit or explicit references to order in a topological space are to be interpreted with respect to the specialization order. From now on, we assume that all our topological spaces are T_0 (this assumption could be avoided by considering continuous *pre*ordered sets).

2.21 *The points of a densely injective space form a continuous Scott domain under the specialization order. Moreover, the topology of a densely injective space coincides with the Scott topology of its specialization order.*

We thus have, as corollaries, a topological characterization of the continuous Scott domains and an order-theoretic characterization of the densely injective spaces.

2.22 *The continuous Scott domains are precisely the specialization orders of the densely injective spaces.*

2.23 *The densely injective spaces are precisely continuous Scott domains under the Scott topology.*

Injective spaces and continuous lattices The injective spaces over arbitrary (not just dense) embeddings, known simply as *injective spaces*, are precisely the continuous lattices endowed with the Scott topology, via the same constructions. Of course, there are more densely injective spaces than injective spaces, because the defining extension property is harder to be met in the latter case. Since there are continuous Scott domains which are not continuous lattices, there are strictly more densely injective spaces than injective spaces.

2.4 The dual and patch topologies of a topological space

For the purposes of this paper, compactness is understood in the sense of the Heine-Borel covering property—the Hausdorff separation axiom is not considered as part of the notion. Two basic facts of general topology are that a closed set of a compact space is compact, and that a compact set of a Hausdorff space is closed; in particular, the compact sets of a compact Hausdorff space coincide with the closed sets. However, a compact set of a non-Hausdorff space is hardly ever closed.

2.24 Example (Compact, non-closed sets) In the extended real line with the topology of lower semicontinuity, singletons are compact but not closed. In fact, the closure of $\{x\}$ is $[-\infty, x]$.

The above observations suggest that a compact non-Hausdorff space could be made into a compact Hausdorff space by taking the least refinement of its topology for which the compact sets become closed. A little reflection on the above example shows that this idea doesn't work, as a set is compact iff it has a least element, so that far too many new closed sets get added. To make it work, one considers a special class of compact spaces, which includes the example, and a special type of compact set. We begin by discussing the latter. As a motivation, we remark that, from the point of view of the Heine-Borel property, what matters of a compact set are its neighbourhoods and not its points.

Saturated sets A set of points of a topological space is called *saturated* if it is the intersection of its neighbourhoods, which is equivalent to saying that it is an upper set in the specialization order. The *saturation* of a set is the intersection of its neighbourhoods, or, equivalently, its upper set in the specialization order. Any set has the same neighbourhoods as its saturation. In particular, a set is compact iff its saturation is compact.

2.25 Example (Saturation) In the extended real line with the topology of lower semicontinuity, the saturation of a singleton $\{x\}$ is $[x, \infty]$, because the non-trivial open neighbourhoods of both sets are those of the form $(a, \infty]$ with $a < x$.

In a T_1 space (and hence in a Hausdorff space), all sets are saturated, and hence the notion of saturation, as that of specialization order, plays no rôle. The special class of spaces referred above consists of the stably compact spaces (the sober locally compact spaces for which the compact saturated sets are closed under the formation of finite intersections). But stably compact spaces are not needed for the purposes of this paper. It suffices to say that they include the densely injective spaces.

The dual topology The dual topology of a continuous Scott domain can be seen as derived from a topology rather than from an order.

2.26 *The dual topology of a continuous Scott domain has as closed sets precisely the compact saturated sets in the Scott topology.*

The *dual topology* of a topological space is generated by the complements of its compact saturated sets (for a stably compact space, the dual closed sets are precisely the compact saturated sets). The dual of a densely injective space is hardly ever a densely injective space. But it is still a stably compact space, whose dual is the original densely injective space. More generally, any stably compact space coincides with its second dual. Hence the terminology.

The patch topology The *patch topology* of a topological space is the join of its topology and its dual topology. A topological space X with its topology refined to the patch topology is denoted by

$$\operatorname{Patch} X.$$

For a stably compact space, this is a compact Hausdorff topology. In particular, the patch topology of a densely injective space is the Lawson topology of its specialization order.

2.27 *The Lawson topology of a continuous Scott domain is the patch of the Scott topology.*

Hence, as the dual topology, the Lawson topology can be seen as derived from a topology rather than from an order. Under this view, it is a compact-Hausdorff coreflection [5]. Thus, the main concepts and constructions of continuous-lattice theory have purely topological formulations. But the order-theoretic formulations remain important and useful. In fact, the results of this paper, which are developed in the next section, constitute an application of continuous-order theory to topology.

The Scott topology can be recovered from the Lawson topology *and* the order as follows, which is one manifestation of the many connections of the theory of ordered topological spaces [15] with the theory of continuous lattices.

2.28 *The Scott open sets of a continuous Scott domain are precisely the Lawson open upper sets.*

More generally, the topology of a stably compact space can be recovered from its patch topology and its specialization order in the same way. Although the following example could have been presented much earlier, we have deliberately saved it to concretely illustrate some aspects of the above discussion.

2.29 Example (The Scott and Lawson topologies of the interval domain) The Lawson topology of the interval domain $\mathrm{I}\,\mathbb{R}$ discussed in Examples 2.8 and 2.19 coincides with the topology induced by the Hausdorff metric. Hence the Scott open sets of $\mathrm{I}\,\mathbb{R}$ are the open sets U of the Hausdorff metric such that $\mathbf{x} \in U$ and $\mathbf{y} \subseteq \mathbf{x}$ together imply $\mathbf{y} \in U$. Since continuous functions preserve the specialization order, Scott continuous maps $\mathrm{I}\,\mathbb{R} \to \mathrm{I}\,\mathbb{R}$ preserve the inclusion order. Moreover, Lawson continuous maps $\mathrm{I}\,\mathbb{R} \to \mathrm{I}\,\mathbb{R}$ that preserve the inclusion order are Scott continuous—but not all Scott continuous maps $\mathrm{I}\,\mathbb{R} \to \mathrm{I}\,\mathbb{R}$ arise in this way, as illustrated by the Scott continuous function $f : \mathrm{I}\,\mathbb{R} \to \mathrm{I}\,\mathbb{R}$ defined by $f(\mathbf{x}) = \{-1\}$ if $\sup \mathbf{x} < 0$, $f(\mathbf{x}) = \{1\}$ if $\inf \mathbf{x} > 0$, and $f(\mathbf{x}) = [-1, 1]$ otherwise (that is, if $0 \in \mathbf{x}$).

2.5 Core-compact topological spaces

We have seen that there are topological spaces whose *points* form continuously ordered sets. Here we consider topological spaces whose *open sets* form continuously ordered sets. Such spaces arise, for instance, in the theory of topological function spaces, which is the topic of the next subsection. The lattice of open sets of a topological space X is denoted by

$$\mathcal{O}\, X.$$

In this lattice, the way-below relation captures a relative notion of compactness for pairs of opens.

2.30 *The relation $U \ll V$ holds in the lattice of open sets of a topological space iff every open cover of V has a finite subcover of U.*

A space is called *locally compact* iff each neighbourhood of a point contains a compact (not necessarily open) neighbourhood of the point. A Hausdorff space, as it is well-known, is locally compact iff each point has a compact neighbourhood; in particular, compact Hausdorff spaces are locally compact.

2.31 Example The relation $U \ll V$ holds in the lattice of open sets of a locally compact space iff there is a compact subset Q of the space with $U \subseteq Q \subseteq V$.

A space X is called *core-compact* if $\mathcal{O}X$ is a continuous lattice. The following is a purely topological formulation of the notion.

2.32 *A topological space is core-compact iff each neighbourhood V of a point x contains a neighbourhood U of x with the property that every open cover of V has a finite subcover of U.*

2.33 Example Locally compact spaces are core-compact.

But core-compactness is a rather mild generalization of the notion of local compactness. In fact, for Hausdorff spaces (and more generally for sober spaces) core-compactness is the same as local compactness (and a space is core-compact iff its sobrification is locally compact). We finish our brief exposition of the subject of core-compactness by remarking that every distributive continuous lattice is isomorphic to the lattice of open sets of a core-compact topological space, via a Stone-type duality—but this is not exploited in this paper.

2.6 Topological function spaces

Exponential topologies A topology on the set

$$\mathrm{C}(X, Y)$$

of continuous maps from a topological space X to a topological space Y is *exponential* if for all spaces A, continuity of a function $f : A \times X \to Y$ is equivalent to that of its *transpose* $\bar{f} : A \to \mathrm{C}(X, Y)$ defined by $\bar{f}(a)(x) = f(a, x)$. Thus, a topology on $\mathrm{C}(X, Y)$ is exponential iff for all spaces A, transposition is a well-defined bijection from continuous maps $A \times X \to Y$ to continuous maps $A \to \mathrm{C}(X, Y)$.

2.34 *An exponential topology, when it exists, is unique.*

Exponential spaces A topological space X is called *exponential* if for each space Y there is an exponential topology on the set $\mathrm{C}(X, Y)$. If X is an exponential space and Y is an arbitrary space, then the set $\mathrm{C}(X, Y)$ endowed with the exponential topology is denoted by

$$Y^X.$$

Thus, according to our convention, it doesn't make sense to write Y^X if X is not exponential. To refer to the *set* of continuous functions, we use the notation $\mathrm{C}(X, Y)$, as above. As an important property of exponential topologies, one has that for any exponential space X and any space Y, the evaluation map $(f, x) \mapsto f(x) : Y^X \times X \to Y$, having the identity as its transpose, is continuous.

The compact-open topology It is well-known that locally compact spaces are exponential. Moreover, among Hausdorff spaces (and, more generally, sober spaces), the exponential spaces are precisely the locally compact spaces. If X is locally compact then the topology of the function space Y^X, for any space Y, is the *compact-open topology*, which, by definition, is generated by the subbasic open sets

$$\{f \in \mathrm{C}(X,Y) \mid Q \subseteq f^{-1}(V)\},$$

where Q ranges over compact sets of X, and V over open sets of Y.

The Isbell topology For an exponential space that is not locally compact, a refinement of the compact-open topology is needed in order to obtain the exponential topology.

2.35 *A topological space is exponential iff it is core-compact. Moreover, for any exponential space X and any space Y, the topology of the function space Y^X is generated by the subbasic open sets*

$$N(H,V) \stackrel{\text{def}}{=} \{f \in \mathrm{C}(X,Y) \mid f^{-1}(V) \in H\},$$

where H ranges over Scott open subsets of $\mathcal{O}X$, and V over $\mathcal{O}Y$.

This construction of the exponential topology is known as the *Isbell topology*. Notice that here one is considering the Scott topology of the lattice of open sets of a topological space. For example, for $Q \subseteq X$ compact, the set $\{O \in \mathcal{O}X \mid Q \subseteq O\}$ is Scott open, which shows that every subbasic open set in the compact-open topology is open in the Isbell topology, and hence that the Isbell topology is indeed finer than the compact-open topology. Also, notice that, by core-compactness of X and (2.9), for U open, the set $\Uparrow U = \{O \in \mathcal{O}X \mid U \ll O\}$ is Scott open, and any Scott open subset H of $\mathcal{O}X$ is the union of the sets $\Uparrow U$ with $U \in H$. This shows that the sets of the form

$$\{f \in \mathrm{C}(X,Y) \mid U \ll f^{-1}(V)\},$$

where U and V range over $\mathcal{O}X$ and $\mathcal{O}Y$, also generate the exponential topology. It follows from Example 2.31 that the Isbell topology coincides with the compact-open topology if X is locally compact.

Some fundamental properties of function spaces

2.36 *A topological product of two exponential spaces is exponential.*

We have seen that a space X is exponential iff for each space Y there is a topology on $\mathrm{C}(X,Y)$ such that, for all spaces A, transposition is a well-defined bijection from the set $\mathrm{C}(A \times X, Y)$ to the set $\mathrm{C}(A, \mathrm{C}(X,Y))$. If these two sets also admit exponential topologies, then the bijection becomes a homeomorphism.

2.37 *If X, Y and A are topological spaces with X and A exponential, then the function space $Y^{A\times X}$ is homeomorphic to the iterated function space $(Y^X)^A$.*

By embedding the codomain of a function space into a larger space, the function space gets embedded into a larger function space.

2.38 *If X is an exponential topological space and $k : Y \to Z$ is an embedding of topological spaces, then the functional $K(f) = k \circ f$ is an embedding of the function space Y^X into the function space Z^X.*

(It is sometimes possible to embed a function space into a larger function space by embedding its *domain* into a larger space [4, 3]. If Z is densely injective and X is densely embedded into Y via a *proper* map j, then Z^X is embedded into Z^Y via the extension process $f \mapsto f/j$ discussed in Section 2.3; properness of the embedding j is a necessary and sufficient condition for continuity of the extension process.)

Injective spaces and function spaces From the definitions, without knowing any explicit construction of exponential topologies or any explicit characterization of the densely injective spaces, one can easily prove the following.

2.39 *If X is exponential and Z is densely injective then the function space Z^X is densely injective.*

Proof (Attributed to Joyal by Johnstone.) Let $j : I \to J$ be a dense embedding of a space I into a space J, and let $f : I \to Z^X$ be a continuous map. Then f is the transpose of a continuous function $g : I \times X \to Z$. Since Z is densely injective and $j \times \mathrm{id}_X : I \times X \to J \times X$ is a dense embedding, where $\mathrm{id}_X : X \to X$ is the identity map, the map $g : I \times X \to Z$ has a continuous extension $\hat{g} : J \times X \to Z$ along $j \times \mathrm{id}_X : I \times X \to J \times X$, which has a continuous transpose $\hat{f} : J \to Z^X$. It is immediate that $\hat{f} : J \to Z^X$ is an extension of $f : I \to Z^X$ along $j : I \to J$, which shows that Z^X is densely injective. □

The patch topology of a function space over a densely injective space An explicit construction of the patch topology of a function space Z^X from the Isbell or the compact-open topologies can be quite complicated. But the topology of the function space has a simpler alternative construction when Z is a densely injective space. We begin with a lemma on the unrestricted case, using the Isbell topology in the proof.

2.40 *If X is exponential and Z is arbitrary, then the specialization order of the function space Z^X coincides with the pointwise specialization order of the function set* $\mathrm{C}(X, Z)$.

Proof Assume that $f \leq g$ holds in the pointwise specialization order of $\mathrm{C}(X, Y)$. A simple unfolding of definitions shows that this is equivalent to saying that $f^{-1}(V) \subseteq g^{-1}(V)$ for every $V \in \mathcal{O}\,Z$. Let $N(H, V)$ be a subbasic neighbourhood of f in the Isbell topology. Then $f^{-1}(V) \in H$ by definition, and $g^{-1}(V) \in H$ by the assumption, because H, being Scott open, is an upper set. This means that $g \in N(H, V)$ and shows that $f \leq g$ in the specialization order of Z^X. Conversely, if $f \leq g$ holds in the specialization order of Z^X then $f(x) \leq g(x)$ for any $x \in X$ because evaluation at x is a continuous map and continuous maps preserve the specialization order. □

Combining this with the characterization of the densely injective spaces as the continuous Scott domains endowed with the Scott topology, one gets the following.

2.41 Corollary *If X is exponential and Z is densely injective, then the topology of the densely injective function space Z^X is the Scott topology of the pointwise specialization order of the function set* $\mathrm{C}(X, Z)$.

With this, the patch topology of Z^X is the Lawson topology of its specialization order. The complication now is that one needs an explicit description of the way-below relation of the specialization order of Z^X—but we shall not give the details [2]. What we need is the following. For $U \in \mathcal{O}\,X$ and $z \in Z$, denote by

$$(U \searrow z)$$

the *single-step function* $X \to Z$ that maps $u \in U$ to z, and $x \notin U$ to the bottom point of Z. A *step function* is a join of finitely many single-step functions.

2.42 *If X is exponential and Z is densely injective, then the step functions $X \to Z$ form a basis of Z^X qua continuous Scott domain.*

This and (2.14) entail the following.

2.43 Corollary *The sets $\uparrow(U \searrow z)$ with $U \in \mathcal{O}\,X$ and $z \in Z$ form a closed subbase of the dual topology of the function space Z^X.*

A simple calculation gives the following explicit description of the subbasic closed sets.

2.44 $\uparrow(U \searrow z) = \{h \in Z^X \mid U \subseteq h^{-1}(\uparrow z)\}$.

3 Function-space compactifications of function spaces

We have seen that if X is an exponential space and Z is a densely injective space, then the function space Z^X is also densely injective, and that if a space Y is embedded into Z then Y^X is embedded into Z^X. Being densely injective, the function space Z^X is compact and locally compact, but highly non-Hausdorff. We consider the situation in which its compact-Hausdorff refinement Patch Z^X still contains Y^X as an embedded subspace. In this case, the closure of Y^X in Patch Z^X is a Hausdorff compactification of Y^X.

3.1 Strong embeddings

We begin by solving the problem for the special case in which X is the one-point space. The first of the conditions below is thus obtained.

3.1 Proposition *The following are equivalent for any embedding $k : Y \to Z$ of topological spaces.*

1. *k is also an embedding with respect to the patch topology of Z.*
2. *k is continuous with respect to the patch topology of Z.*
3. *k is continuous with respect to the dual topology of Z.*

Proof This follows from the fact that if the topology of Z is refined in such a way that k remains continuous, then k is still an embedding, and, conversely, if the topology of Z is refined in such a way that k is still an embedding, then k remains continuous, by definition of embedding, and from the fact that the patch topology of a space Z is defined to be the join of the topology of Z and of the dual topology of Z. $\square$

By a *strong embedding* we mean an embedding satisfying the above equivalent conditions. Such embeddings, albeit not under this (or any other) name, occur in the work of Lawson on subspaces of maximal points, with Z a continuous poset under the Scott topology, and with k restricted to be onto the maximal points of Z, where the last two of the conditions below are taken as equivalent defining properties [13].

3.2 PROPOSITION *The following are equivalent for any embedding $k : Y \to Z$ of a topological space Y into a densely injective space Z.*

1. *k is a strong embedding.*
2. *$k^{-1}(\uparrow z)$ is closed for every point $z \in Z$.*
3. *On the image of k, the relative Scott and Lawson topologies of the specialization order of Z coincide.*

PROOF Here $\uparrow z$ is the upper set of z in the specialization order of Z. We have seen that sets of this form constitute a closed subbase of the dual topology when Z is densely injective, and that the topology and the patch topology of a densely injective space Z coincide with the Scott and Lawson topologies of the specialization order of Z respectively. □

The above special case actually gives a full solution to the problem posed in the opening paragraph of the section.

3.3 THEOREM *The following conditions are equivalent for an embedding $k : Y \to Z$ of a topological space Y into a densely injective space Z.*

1. *The induced embedding $K : Y^X \to Z^X$ is strong for every exponential space X.*
2. *The embedding $k : Y \to Z$ is strong.*

PROOF ($\Downarrow$): As above, choose X to be the one-point space. ($\Uparrow$): We have seen in (2.38) that k induces the embedding $K(f) = k \circ f$. By (2.43), a subbase of the dual topology of Z^X is given by sets of the form $\not\uparrow(U \searrow z)$. Let $F = K^{-1}(\not\uparrow(U \searrow z))$. Then $f \in F$ iff $k \circ f \in \not\uparrow(U \searrow z)$ iff $U \not\subseteq f^{-1}(k^{-1}(\uparrow z))$ by (2.44). Hence $F = \{f \in Y^X \mid U \not\subseteq f^{-1}(C)\}$, where $C = k^{-1}(\uparrow z)$. Since $\uparrow z$ is a closed set in the dual topology of Z and k is a strong embedding, C is closed. Let $f \in F$. Then there is some $u \in U$ with $f(u) \notin C$. Hence $G = \{g \in Y^X \mid g(u) \in Y \setminus C\}$ is a neighbourhood of f. If $g \in G$ then $g(u) \notin C$, which shows that $U \not\subseteq g^{-1}(C)$ and hence that $G \subseteq F$. Therefore F is open and K is continuous with respect to the dual topology of Z^X. □

3.2 Strongly densely embedded subspaces of densely injective spaces

We begin by considering a well-known special case.

3.4 EXAMPLE (THE VIETORIS HYPERSPACE) Let Y be a compact Hausdorff space. By local compactness, its open sets form a continuous lattice under inclusion. Via complementation, the closed sets form an isomorphic continuous lattice under *reverse* inclusion. If the empty closed set is removed, a continuous Scott domain is obtained. The Lawson topology, being the unique compact Hausdorff topology making the formation of binary meets (in this case, set-theoretical unions) into a continuous operation, coincides with the Vietoris topology. Let $\mathrm{V}\,Y$ and $\mathrm{U}\,Y$ denote the collection of non-empty closed sets under the Lawson and Scott topologies respectively. It is well-known that the singleton map $y \mapsto \{y\}$ is a dense embedding of Y into $\mathrm{U}\,Y$, and that it is also a (closed and hence non-dense) embedding of Y into $\mathrm{V}\,Y$. Therefore it is a strong dense embedding of Y into the space $\mathrm{U}\,Y$, which, being a continuous Scott domain under the Scott topology, is densely injective. If the empty closed set is allowed as a point of the hyperspace construction, we still have a strong embedding, but it isn't dense anymore. Moreover, for both topologies, the empty closed set is an isolated point of the hyperspace, and this is one reason for omitting it. Occasionally, however, it is technically convenient to admit the empty closed set as a point of the hyperspace—see for example the observation after Proposition 3.6 and Section 3.3. We denote the collection of *all* closed sets under the Scott and Lawson topologies by $\mathrm{U}_0\,Y$ and $\mathrm{V}_0\,Y$ respectively.

The following observation is immediate.

3.5 LEMMA *If $Y \hookrightarrow Y'$ is an embedding and $Y' \hookrightarrow Z$ is a strong embedding then the composite $Y \hookrightarrow Y' \hookrightarrow Z$ is also a strong embedding.*

3.6 PROPOSITION *The strongly densely embedded subspaces of the densely injective spaces are precisely the Tychonoff spaces.*

PROOF If Y is strongly densely embedded into Z, then it is embedded into Patch Z by definition of strong embedding. Hence, being a subspace of a compact Hausdorff space, it is Tychonoff. Conversely, if Y is Tychonoff, then it is densely embedded into a compact Hausdorff space Y', for example its Stone-Čech compactification. Therefore the result follows by Lemma 3.5, because we know by Example 3.4 that a compact Hausdorff space Y' is strongly densely embedded into the densely injective space $\mathrm{U}\,Y'$. □

This also holds for strongly embedded subspaces of injective spaces, by considering the empty closed set as a point of the hyperspace construction in the proof, and removing all references to density.

3.7 COROLLARY *For any Tychonoff space Y there is a densely injective space Z containing Y as a densely embedded subspace such that, for every exponential space X, the function space Z^X has a compact-Hausdorff refinement still containing Y^X as an embedded subspace.*

The space Z, being densely injective, is non-Hausdorff. Let Y be a compact Hausdorff space and, for the sake of contradiction, assume that there is a Hausdorff space Z satisfying the conclusion of the corollary. As Z contains Y as a densely embedded subspace, Z is homeomorphic to Y because Z is Hausdorff and Y is compact and hence closed in Z. Then, by the assumption, for each exponential space X, the function space $Z^X \cong Y^X$ has a compact-Hausdorff refinement. But any topology coarser than a compact topology is compact. Hence Y^X must be already compact. And, because Y is Hausdorff, so is Y^X. Thus, we conclude from the assumption that for every compact Hausdorff space Y and every exponential space X, the function space Y^X is compact Hausdorff, which is certainly not the case. For example, it is well-known that if Y is a compact interval of the Euclidean line, then the function space Y^Y is not even locally compact. But trivial counterexamples also exist. Let Y be the two-point discrete space and X be a compact Hausdorff space. A simple argument shows that Y^X is discrete, with as many points as X has clopen sets. Thus, if X is, for example, the Cantor discontinuum, then Y^X, being a countably infinite discrete space, is not compact. Therefore a space Z satisfying the conclusion of the corollary is necessarily non-Hausdorff in general.

The following is known for the case in which the exponential topology is the compact-open topology.

3.8 Corollary *For every space Y and every non-empty exponential space X, the function space Y^X is Tychonoff if and only if the space Y is.*

Proof If the function space Y^X is Tychonoff then Y, being a subspace of Y^X via the constant-maps embedding, is Tychonoff. Conversely, if Y is Tychonoff then Y^X, being a subspace of a compact Hausdorff space by the above corollary, is Tychonoff. $\square$

3.3 Relation-space compactifications of function spaces

Another interesting consequence of Theorem 3.3 is that not only continuous functions of compact Hausdorff spaces have closed graphs, but also the *graph map* $f \mapsto \{(x, f(x)) \mid x \in X\}$ is an embedding into a space of closed relations.

3.9 Proposition *If X and Y are compact Hausdorff spaces, then the graph map is an embedding of the function space Y^X into the relation space $\mathrm{V}(X \times Y)$.*

This is a particular case of a more general situation. The *Sierpinski space*, denoted by $\mathbb{S}$, is the two-point lattice $\{0, 1\}$ under the Scott topology. Thus, the singleton $\{1\}$ is the only non-trivial open set and hence the continuous maps of a space Y into $\mathbb{S}$ are the characteristic functions of opens of Y. By (2.41), the function space $\mathbb{S}^Y$ is homeomorphic to $\mathcal{O}\,Y$ under the Scott topology, for any exponential space Y. These observations together with those of Example 3.4 yield the following.

3.10 *For any compact Hausdorff space Y, the hyperspaces $\mathrm{U}_0\,Y$ and $\mathrm{V}_0\,Y$ are homeomorphic to the function spaces $\mathbb{S}^Y$ and $\mathrm{Patch}\,\mathbb{S}^Y$ respectively.*

Generalized Vietoris hyperspaces In view of this, we generalize the Vietoris hyperspace construction by defining, for every exponential space Y and every densely injective space I,

$$\mathrm{V}_I\,Y = \mathrm{Patch}\,I^Y.$$

Notice that if Y is strongly embedded into I^Y then Y, being embedded into the Hausdorff space $\mathrm{V}_I\,Y$, is a Hausdorff space. Under the assumption of exponentiality of Y, which amounts to core-compactness, this is equivalent to saying that Y is locally compact Hausdorff.

3.11 Proposition *If X, Y and I are spaces with X exponential, Y locally compact Hausdorff and I densely injective, then the function $E : Y^X \to \mathrm{V}_I(X \times Y)$ defined by*

$$E(f)(x, y) = \eta(f(x))(y)$$

is an embedding for any strong embedding $\eta : Y \to I^Y$.

Proof By Theorem 3.3, the function space Y^X is strongly embedded into $(I^Y)^X$, which is homeomorphic to $I^{X\times Y}$ by (2.37). Hence, by definition of strong embedding, the function space Y^X is embedded into $\mathrm{Patch}\,I^{X\times Y}$, which is $\mathrm{V}_I(X \times Y)$ by definition. Chasing the embeddings and homeomorphisms, the function E is obtained. $\square$

In what follows, it is useful to regard strong embeddings $Y \to I^Y$ as exponential transposes of continuous maps $Y \times Y \to I$. Under the translation (3.10), the singleton embedding of a compact Hausdorff space Y into the hyperspace $\mathrm{U}_0\, Y$ of Example 3.4 becomes the transpose of the function $d : Y \times Y \to \mathbb{S}$ defined $d(y, y') = 1$ iff $y \neq y'$. Translating the resulting embedding $E : Y^X \to \mathrm{V}_{\mathbb{S}}(X \times Y)$ back, the graph map is obtained. For a core-compact space Y, let $\mathrm{U}_0\, Y$ and $\mathrm{V}_0\, Y$ denote the closed sets under the Scott and Lawson topologies of the reverse-inclusion order, so that $\mathrm{U}_0\, Y \cong \mathbb{S}^Y$ and $\mathrm{V}_0\, Y \cong \mathrm{V}_{\mathbb{S}}\, Y = \mathrm{Patch}\, \mathbb{S}^Y$ as in (3.10). This generalization of the hyperspace construction from compact Hausdorff spaces to core-compact spaces has to be taken with caution. Firstly, notice that the empty closed set is an isolated point iff Y is compact. More importantly, as it is proved in the paragraph that follows Proposition 3.13, $\bar{d} : Y \to \mathbb{S}^Y$ is an embedding if and only if Y is compact Hausdorff. Therefore we are not able to relax the assumption that Y is compact Hausdorff in the following corollary. However, we can assume that X is just exponential, so that the generalization is not useless.

3.12 Corollary *If X is an exponential space and Y is a compact Hausdorff space, then the graph map is an embedding of the function space Y^X into the relation space* $\mathrm{V}_0(X \times Y)$.

Proposition 3.9 is a special case of this, because the image of the graph map is contained in $\mathrm{V}(X \times Y)$.

3.13 Proposition *Let $d : Y \times Y \to I$ be a continuous map with Y locally compact Hausdorff and I densely injective, and for $y_0 \in Y$ and $r \in I$, define*

$$B_r(y_0) = \{y \mid r \ll d(y, y_0)\}, \qquad \bar{B}_r(y_0) = \{y \mid r \leq d(y, y_0)\}.$$

1. *The transpose $\bar{d} : Y \to I^Y$ is an embedding if and only if it is one-to-one and the open sets $\bigcap\{B_r(q) \mid q \in Q\}$, for $Q \subseteq Y$ compact and $r \in I$, constitute a subbase of $\mathcal{O}\, Y$.*

2. *The transpose $\bar{d} : Y \to I^Y$ is continuous with respect to the dual topology of I^Y if and only if the sets $\bigcap\{\bar{B}_r(u) \mid u \in U\}$, for $U \in \mathcal{O}\, Y$ and $r \in I$, are closed in Y.*

Proof Let $Q \subseteq Y$ be compact and $r \in I$. Then the set $V = \{h \in I^Y \mid Q \subseteq h^{-1}(\Uparrow r)\}$ is a typical subbasic open, and the relation $y \in \bar{d}^{-1}(V)$ holds iff $\bar{d}(y) \in V$ iff $Q \subseteq (\bar{d}(y))^{-1}(\Uparrow r)$ iff $\bar{d}(y)(q) \in \Uparrow r$ for all $q \in Q$ iff $y \in B_r(q)$ for all $q \in Q$ iff $y \in \bigcap\{B_r(q) \mid q \in Q\}$, which shows that the trace of the topology of I^Y in Y is generated by the open sets of the form considered in statement (1). Let $U \in \mathcal{O}\, Y$ and $r \in I$. Then $C = \{h \in I^Y \mid U \subseteq h^{-1}(\uparrow r)\}$ is a typical subbasic closed set in the dual topology by (2.43) and (2.44), and $\bar{d}^{-1}(C) = \bigcap\{\bar{B}_r(u) \mid u \in U\}$ by a similar calculation, which establishes statement (2). $\square$

The map $d : Y \times Y \to \mathbb{S}$ defined by $d(y, y') = 1$ iff $y \neq y'$, being the characteristic function of the complement of the diagonal, is continuous iff Y is a Hausdorff space. For Y compact Hausdorff, as it is discussed before Proposition 3.12, the transpose $\bar{d} : Y \to \mathbb{S}^Y$ corresponds to the singleton embedding of Y into $\mathrm{U}_0\, Y$ of Example 3.4 via the translation (3.10). For Y locally compact Hausdorff, Proposition 3.13(1) asserts that the transpose is an embedding iff the sets $\bigcap\{B_1(q) \mid q \in Q\}$ with $Q \subseteq Y$ compact generate the topology of Y. A point y is a member of such an intersection iff $y \in B_1(q)$ for all $q \in Q$ iff $1 \ll d(y, q)$ for all $q \in Q$ iff $y \neq q$ for all $q \in Q$ iff $y \notin Q$ iff $y \in Y \setminus Q$. But the complements of the compact sets generate the topology of a locally compact Hausdorff space Y iff Y is compact. Therefore the transpose is an embedding iff Y is compact Hausdorff. In view of (3.10), Example 3.4 states that if Y is compact Hausdorff, then the embedding is strong. This follows from Proposition 3.13(2), because $\bigcap\{\bar{B}_1(U) \mid u \in U\} = Y \setminus U$.

Generalized Vietoris hyperspaces of locally compact metric spaces For any metric space Y, its distance function $d : Y \times Y \to [0, \infty)$ is continuous with respect to the Euclidean topology of $[0, \infty)$ and hence with respect to any weaker topology. We regard it as a continuous map $d : Y \times Y \to D$ with $D = [0, \infty]$ under the topology of upper semicontinuity. Since this is the Scott topology of the opposite of the natural order, which makes $[0, \infty]$ into a continuous lattice (with bottom element ∞ and top element 0), the space D is injective. By virtue of the reversal of the natural order, the sets $B_r(y_0)$ and $\bar{B}_r(y_0)$ of Proposition 3.13 are the open and closed balls of the metric. Considering singleton compact sets, Proposition 3.13(1) shows that the transpose $\bar{d} : Y \to D^Y$ is an embedding because it is clearly one-to-one. Since arbitrary intersections of closed balls are closed sets, Proposition 3.13(2) shows that this embedding is strong.

3.14 COROLLARY *For any locally compact Hausdorff space Y metrized by d and any exponential space X, the function $E : Y^X \to \mathrm{V}_D(X \times Y)$ defined by $E(f)(x, y) = d(f(x), y)$ is an embedding.*

3.4 Compactifications of spaces of real-valued continuous functions by spaces of pairs of semicontinuous functions

We finish by resuming the discussion of the opening paragraph of the introduction. Let $\mathbb{R}$ be the Euclidean line, $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ be the extended real line with the topologies of lower and upper semicontinuity respectively, and $\mathcal{R}$ be the topological product $\underline{\mathbb{R}} \times \overline{\mathbb{R}}$.

3.15 LEMMA *The map $r \mapsto (r, r)$ is a strong embedding of $\mathbb{R}$ into the injective space $\mathcal{R}$.*

PROOF Since injective spaces are closed under the formation of products, $\mathcal{R}$ is injective. Since the Euclidean topology is the join of the topologies of lower and upper semicontinuity, the map $k(r) = (r, r)$ is an embedding of $\mathbb{R}$ into $\mathcal{R}$. For $(a, b) \in \mathcal{R}$, we have that $r \in k^{-1}(\uparrow(a, b))$ iff $(a, b) \leq k(r)$ iff $(a, b) \leq (r, r)$ iff $a \leq r$ and $b \geq r$ iff $r \in [a, b]$, with the convention that $[a, b]$ denotes the empty interval if $a \not\leq b$, which shows that $k^{-1}(\uparrow(a, b)) = [a, b]$ and hence that k is a strong embedding. $\square$

3.16 COROLLARY *For any exponential space X, the function space $\mathbb{R}^X$ is embedded into the compact Hausdorff function space* $\mathrm{Patch}\,\mathcal{R}^X$.

Although, as we mentioned in the introduction, the relative topology of $\mathrm{Patch}\,\underline{\mathbb{R}}^X$ on the points of $\mathbb{R}^X$ is strictly weaker than the topology of $\mathbb{R}^X$, we have the following.

3.17 COROLLARY *For any exponential space X, the topology of the function space $\mathbb{R}^X$ is the join of the relative topologies of the spaces $\mathrm{Patch}\,\underline{\mathbb{R}}^X$ and $\mathrm{Patch}\,\overline{\mathbb{R}}^X$.*

PROOF By general properties of exponentials, $\mathcal{R}^X$ is homeomorphic to $\underline{\mathbb{R}}^X \times \overline{\mathbb{R}}^X$. Since the patch construction is a coreflection, it preserves categorical products, which topological products are. Therefore $\mathrm{Patch}\,\mathcal{R}^X = \mathrm{Patch}\,\underline{\mathbb{R}}^X \times \mathrm{Patch}\,\overline{\mathbb{R}}^X$. $\square$

Coming back to Corollary 3.16, we observe that a distinguished subspace of $\mathcal{R}$ has already appeared in a different guise, which is useful for discussing the closure of $\mathbb{R}^X$ in $\mathrm{Patch}\,\mathcal{R}^X$.

3.18 PROPOSITION *The closure of the image of the embedding $r \mapsto (r, r)$ of $\mathbb{R}$ into $\mathcal{R}$ consists of the points (x, y) with $x \leq y$.*

Proof   Any closed set is a lower set in the specialization order, and the lower set of the image consists of such points because $(x, y) \leq (r, r)$ holds for some r iff $x \leq r$ and $r \leq y$ hold for some r iff $x \leq y$. This set, being closed under formation of directed joins, is closed in the Scott topology, which coincides with the topology of $\mathcal{R}$. $\square$

We have seen in Examples 2.8, 2.19 and 2.29 that the non-empty closed intervals of the extended Euclidean line form a continuous Scott domain under the reverse-inclusion order. By the *interval space*, denoted by $\mathrm{I}\,\mathbb{R}$, we mean the interval domain under the Scott topology.

3.19 Proposition *$\mathrm{I}\,\mathbb{R}$ is homeomorphic to the subspace of points (x, y) of $\mathcal{R}$ with $x \leq y$.*

Proof   The map $\mathbf{x} \mapsto (\inf \mathbf{x}, \sup \mathbf{x})$ is easily seen to be a homeomorphism. $\square$

3.20 Corollary *The map $x \mapsto \{x\}$ is a strong dense embedding of $\mathbb{R}$ into $\mathrm{I}\,\mathbb{R}$.*

3.21 Corollary *For every exponential space X, the function space $\mathbb{R}^X$ is embedded into the compact Hausdorff space $\mathrm{Patch}(\mathrm{I}\,\mathbb{R})^X$.*

It is natural to wonder whether Theorem 3.3 and its corollaries generalize to the case in which the space X is compactly generated. Of course, the main difficulty is that the lattice of open sets of such a space is not continuous unless the space is already exponential. The theory developed in [8] may be relevant to this question.

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Martín H. Escardó
School of Computer Science
University of St Andrews, North Haugh
St Andrews, Fife KY16 9SS, Scotland

New address from September 2000:

School of Computer Science
University of Birmingham
Birmingham B15 2TT, England
`M.Escardo@cs.bham.ac.uk`
`http://www.cs.bham.ac.uk/~mhe/`