The patch frame of the Lawson dual of a stably continuous frame

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Abstract

Continuous maps of compact regular locales form a coreflective subcategory of the category of perfect maps of stably compact locales. The coreflection of a stably compact locale is given by the frame of Scott continuous nuclei and is referred to as its patch. We show that the patch of a stably compact locale is isomorphic to the patch of its Lawson dual.

Keywords: Frame of nuclei, Scott continuous nucleus, patch topology, Lawson dual, stably compact locale, stably continuous frame.

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1 Introduction

The Scott continuous nuclei form a subframe of the frame of all nuclei. This subframe is referred to as the *patch frame*. It is shown in [3,2] that the patch construction is the object part of the right adjoint to the (full) inclusion of the category of continuous maps of compact regular locales into the category of perfect maps of stably compact locales. This implies that the patch frame of the frame of open sets of a stably compact space is isomorphic to the patch topology. In particular, the frame of Scott continuous nuclei on the Scott topology of a continuous lattice is isomorphic to the Lawson topology.

For a stably compact space, the compact saturated sets are the closed sets of another stably compact topology, known as the cocompact topology. The specialization orderings of the given topology and the cocompact topology are dual. Moreover, iterating the construction twice one gets the original topology. This situation is analised in terms of proximity lattices in [6]. Since the patch topology is the join of the given topology and the cocompact topology, the

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patch of a stably compact topology is the same as the patch of the cocompact topology.

In this paper we consider the localic manifestation of the cocompact topology, given by the Lawson dual [7] of a stably continuous frame [5, Corollary VII.2.12]. The morphisms considered in [7,5] make Lawson dualization into a contravariant functor. There is also a covariant version that uses perfect maps as morphisms [1].

As opposed to [3], we find it more convenient to work with frames rather than locales.

2 The Lawson dual of a stably continuous frame

The *Lawson dual* of a poset L, denoted by $L^{\vartriangle}$, consists of the Scott open filters of L ordered by inclusion. The elements of the first and second dual are ranged over by the letters ϕ, γ and Φ, Γ respectively. If L is a continuous poset, so is its Lawson dual, with $\phi \ll \gamma$ iff $\phi \subseteq \uparrow u$ for some $u \in \gamma$. Moreover, L is naturally isomorphic to its second dual, with isomorphism $\varepsilon : L \to L^{\vartriangle\vartriangle}$ given by

$$\varepsilon(u) = \{\phi \in L^{\vartriangle} \mid u \in \phi\}.$$

A frame L is *continuous* if it is a continuous lattice in the sense of Dana Scott [8,4,5]. This amounts to saying that every $u \in L$ is the limit inferior of its Scott open neighbourhoods, in the sense that $u = \bigvee\{\bigwedge U \mid u \in U\}$, where U ranges over Scott open sets.

A *stably continuous* frame is a continuous frame L such that $\twoheaduparrow u$ is a filter for each $u \in L$. This is equivalent to saying that $1 \ll 1$, where 1 is the top element of L, and that the conditions $u \ll v$ and $u \ll w$ together imply $u \ll v \wedge w$. In the presence of stability, one has that every $u \in L$ is the limit inferior of the Scott open filters to which u belongs. (With countable choice, this actually holds for any continuous poset with directed joins and non-empty meets, and in particular, for any continuous frame.) In the Lawson dual of a stably continuous frame, finite meets and directed joins are given by finite intersections and directed unions (and the join of two filters ϕ and γ is generated by the filtered set $\{u \wedge v \mid u \in \phi,\ v \in \gamma\}$).

For a stably continuous frame L one has that $\twoheaduparrow u \ll \phi$ in $L^{\vartriangle}$ if $u \in \phi$. Moreover, ϕ is the directed join of the set $\{\twoheaduparrow u \in L^{\vartriangle} \mid u \in \phi\}$. In particular, the following conditions are equivalent for any function $f : L^{\vartriangle} \to M^{\vartriangle}$ of Lawson duals of stably continuous frames:

(1) f is Scott continuous.
(2) Whenever $v \in f(\phi)$, there exists $u \in \phi$ with $v \in f(\Uparrow u)$.
(3) $f(\phi) = \{v \in M \mid \exists u \in \phi. v \in f(\Uparrow u)\}$.

Stably continuous frames are closed under Lawson duality. For a general continuous poset L, the fact that $\varepsilon : L \to L^{\vartriangle\vartriangle}$ is an isomorphism seems to rely on excluded-middle and choice principles. For stably continuous frames, these can be avoided.

Proposition 2.1 *If L is a stably continuous frame then $\varepsilon : L \to L^{\vartriangle\vartriangle}$ is an isomorphism with inverse $\delta : L^{\vartriangle\vartriangle} \to L$ given by*

$$\delta(\Phi) = \bigvee\{\bigwedge \phi \mid \phi \in \Phi\}.$$

Proof One first calculates

$$\delta(\varepsilon(u)) = \bigvee\{\bigwedge \phi \mid u \in \phi\} = u,$$

where the second equality holds because L is a continuous lattice with a base of open filters for the Scott topology. One then calculates

$$\varepsilon(\delta(\Phi)) = \{\phi \in L^{\vartriangle} \mid \exists \gamma \in \Phi. \bigwedge \gamma \in \phi\},$$

because Φ is Scott open. To see that $\Phi \subseteq \varepsilon(\delta(\Phi))$, assume that $\phi \in \Phi$. Then $\Uparrow u \in \Phi$ for some $u \in \phi$ because Φ is Scott open. Since $u \leq \bigwedge \Uparrow u$ and ϕ is a filter, we have that $\bigwedge \Uparrow u \in \phi$, which shows that $\phi \in \varepsilon(\delta(\Phi))$. Conversely, to see that $\varepsilon(\delta(\Phi)) \subseteq \Phi$, assume that $\phi \in \varepsilon(\delta(\Phi))$, that is, that there is $\gamma \in \Phi$ with $\bigwedge \gamma \in \phi$. It follows that $\gamma \subseteq \phi$ because if $v \in \gamma$ then $\bigwedge \gamma \leq v$ and hence $v \in \phi$ as ϕ is filter. Since Φ is a filter, we conclude that $\phi \in \Phi$. □

3 The patch of the Lawson dual of a stably continuous frame

A nucleus on a frame is a finite-meet-preserving inflationary idempotent. If a function j preserves finite meets then it is monotone; if it is monotone, then $j(u \wedge v) \leq j(u) \wedge j(v)$; and if it is inflationary and monotone then $j(u) \leq j(j(u))$. Therefore j is a nucleus iff it satisfies the following conditions:

(1) $u \leq v$ implies $j(u) \leq j(v)$,
(2) $j(u) \wedge j(v) \leq j(u \wedge v)$,
(3) $u \leq j(u)$,
(4) $j(j(u)) \leq j(u)$.

(For more information about nuclei see [5] and the references contained therein.)

The Scott continuous nuclei on a frame L form themselves a frame, denoted by $\mathrm{Patch}\, L$ and referred to as the *patch* frame. In this section we show that the patch of a stably continuous frame is isomorphic the patch of its Lawson dual.

LEMMA 3.1 *For a Scott continuous nucleus $j : L \to L$ defined on a stably continuous frame L, the rule*

$$j^{\vartriangle}(\phi) = \{u \in L \mid j(u) \in \phi\}$$

defines a Scott continuous nucleus $j^{\vartriangle} : L^{\vartriangle} \to L^{\vartriangle}$.

PROOF Since $j^{\vartriangle}(\phi) = j^{-1}(\phi)$, the set $j^{\vartriangle}(\phi)$ is a filter because j preserves finite meets, and it is open because j is continuous. To see that $j^{\vartriangle}$ is inflationary, notice that if $u \in \phi$ then $j(u) \in \phi$ because j is inflationary and ϕ is a filter, and hence $u \in j^{\vartriangle}(\phi)$. Since finite meets and directed joins in the Lawson dual are given by finite intersections and directed unions, we have that $j^{\vartriangle}$ preserves them. In order to see that $j^{\vartriangle}(j^{\vartriangle}(\phi)) \leq j^{\vartriangle}(\phi)$, let $u \in j^{\vartriangle}(j^{\vartriangle}(\phi))$. This means that $j(u) \in j^{\vartriangle}(\phi)$, which in turn means that $j(j(u)) = j(u) \in \phi$, which in turn means that $u \in j^{\vartriangle}(\phi)$. $\square$

LEMMA 3.2 *For a Scott continuous nucleus $J : L^{\vartriangle} \to L^{\vartriangle}$ defined on the Lawson dual of a stably continuous frame L, the rule*

$$J^{\triangledown}(u) = \bigvee\{v \in L \mid u \in J(\Uparrow v)\}$$

defines a Scott continuous nucleus $J^{\triangledown} : L \to L$.

PROOF Monotonicity: Assume that $u \leq u'$. If $u \in J(\Uparrow v)$ then $u' \in J(\Uparrow v)$ because $J(\Uparrow v)$ is a filter, which shows that $J^{\vartriangle}(u) \leq J^{\vartriangle}(u')$.

Inflationarity: Assume that $v \ll u$. By the interpolation property, there is $u' \ll u$ with $v \ll u'$. Then $\Uparrow u' \subseteq \Uparrow v$ and hence $u \in \Uparrow u' \subseteq J(\Uparrow u') \subseteq J(\Uparrow v)$, because J is inflationary and monotone, which shows that $v \leq J^{\triangledown}(u)$.

Idempotency, finite-meet preservation and Scott continuity: Let $I(u)$ denote the set $\{v \in L \mid u \in J(\Uparrow v)\}$. If V is a finite subset of L, we have that $V \subseteq I(u)$ iff $u \in \bigcap_{v \in V} J(\Uparrow v) = J(\bigcap_{v \in V} \Uparrow v) = J(\Uparrow \bigvee V)$ iff $\bigvee V \in I(u)$, because J preserves finite meets. Therefore $I(u)$ is an ideal.

Idempotency: Assume that $w \ll J^{\triangledown}(J^{\triangledown}(u))$. Since $J^{\triangledown}(J^{\triangledown}(u))$ is the join of the ideal $I(J^{\triangledown}(u))$, we have that $w \in I(J^{\triangledown}(u))$, which means that $J^{\triangledown}(u) \in J(\Uparrow w)$. And since $J(\Uparrow w)$ is Scott open and $J^{\triangledown}(u)$ is the join of the ideal $I(u)$, there is some $v \in L$ with $u \in J(\Uparrow v)$ and $v \in J(\Uparrow w)$. Since $\Uparrow v \subseteq J(\Uparrow w)$ and hence $J(\Uparrow v) \subseteq J(J(\Uparrow w)) \subseteq J(\Uparrow w)$, we have that $u \in J(\Uparrow w)$, which shows that $w \leq J^{\triangledown}(u)$.

Finite-meet preservation: Assume that $v \ll J^{\triangledown}(u) \wedge J^{\triangledown}(u')$. Then $v \ll J^{\triangledown}(u)$ and $v \ll J^{\triangledown}(u')$. Hence $u \in J(\Uparrow v)$ and $u' \in J(\Uparrow v)$ because $I(u)$ and $I(u')$ are ideals, and $u \wedge u' \in J(\Uparrow v)$ because $J(\Uparrow v)$ is a filter. Therefore $v \in I(u \wedge u')$, which shows that $v \leq J^{\triangledown}(u \wedge u')$.

Scott continuity: Assume that $v \ll J^{\triangledown}(u)$. Since $I(u)$ is an ideal, we have that $u \in J(\Uparrow v)$, and since $J(\Uparrow v)$ is Scott open, there is some $u' \ll u$ with $u' \in J(\Uparrow v)$, and hence with $v \leq J^{\triangledown}(u')$. □

THEOREM 3.3 *The patch of a stably continuous frame is isomorphic to the patch of its Lawson dual, via the assignments constructed in the above two lemmas.*

PROOF Let L be a stably continuous frame. For $j \in \operatorname{Patch} L$, we have that

$$j^{\vartriangle\triangledown}(u) = \bigvee\{v \mid u \in j^{-1}(\Uparrow v)\} = \bigvee\{v \mid v \ll j(u)\} = j(u),$$

where the last equation holds by continuity of L. For $J \in \operatorname{Patch} L^{\vartriangle}$, we have that

$$J^{\triangledown\vartriangle}(\phi) = \{u \mid \bigvee\{v \mid u \in J(\Uparrow v)\} \in \phi\} = \{u \mid \exists v \in \phi. u \in J(\Uparrow v)\} = J(\phi),$$

where the first equation holds because ϕ is open and the second because J is Scott continuous. □

COROLLARY 3.4 *Any compact regular frame is isomorphic to its Lawson dual.*

PROOF Because any compact regular frame is isomorphic to its patch—see [3] or [2]. □

Of course, there are more direct routes to this corollary, which is in principle known.

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