# Semantic domains, injective spaces and monads (extended abstract)

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**Abstract**

Many categories of semantic domains can be considered from an order-theoretic point of view and from a topological point of view via the Scott topology. The topological point of view is particularly fruitful for considerations of computability in classical spaces such as the Euclidean real line. When one embeds topological spaces into domains, one requires that the Scott continuous maps between the host domains fully capture the continuous maps between the guest topological spaces. This property of the host domains is known as injectivity. For example, the continuous Scott domains are characterized as the injective spaces over dense subspace embeddings (Dana Scott, 1972, 1980). From a third point of view, the continuous Scott domains arise as the algebras of a monad (Wyler, 1985). The topological characterization by injectivity turns out to follow from the algebraic characterization and general category theory (Escardó 1998). In this paper we systematically consider monads that arise in semantics and topology, obtaining new *proofs* and discovering new *characterizations* of semantic domains and topological spaces by injectivity.

**Key-words:** Domain theory, injective space, continuous lattice, continuous Scott domain, L-domain, stably locally compact space, flat embedding, Kock-Zöberlein monad.

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## Introduction

In applications of domain theory to denotational semantics, one starts by implicitly or explicitly embedding given topological spaces into appropriate domains endowed with the Scott topology. For example, the discrete space of natural numbers is embedded into the flat domain of natural numbers, the one-point compactification of the discrete space of natural numbers is embedded into the domain of lazy natural numbers, Baire space is embedded into the domain of partial endofunctions of the natural numbers, Cantor space is embedded into the domain of finite and infinite binary words ordered by prefix, and the Euclidean real line is embedded into the domain of compact real intervals ordered by reverse inclusion.

Given embeddings $j : X \to D$ and $k : Y \to E$ of topological spaces into domains, one models continuous maps $X \to Y$ by Scott continuous functions $D \to E$. It is thus natural to demand that the latter fully capture the former. Technically, the requirement amounts to the topological space $E$ being injective over the embedding $j : X \to D$ (see Definition 3.3). For a more detailed discussion of these issues see the introduction of [5]. Our main references to domain theory are [1,7] (see also [18]).

Dana Scott [15] characterized the continuous lattices endowed with the Scott topology precisely as the spaces that are injective over all subspace embeddings. From another perspective, Alan Day [2] and Oswald Wyler [22] independently characterized the continuous lattices as the algebras of a certain filter monad on the category of topological spaces. It turns out that Scott's result actually follow from Day's and Wyler's. One uses the fact that the filter monad is of the Kock-Zöberlein type, and that in any poset-enriched category with such a monad structure, the injective objects over a certain class of embeddings defined in terms of the monad structure are precisely the algebras [5]. The conclusion follows from the fact that the embeddings associated to the filter monad are exactly the topological embeddings [4]. Similarly, the continuous Scott domains are the injective spaces over dense subspace embeddings [7] and this turns out to follow from the fact that they are the algebras of the proper filter monad [24], because the embeddings associated to the monad are the dense topological embeddings [4].

In this paper we systematically consider monads that arise in semantics and topology, obtaining new *proofs* of known characterizations of semantic domains and topological spaces by injectivity, and also new *characterizations* by injectivity. We develop several examples in detail, but we only include the main steps of other examples in this extended abstract.

This extended abstract is organized in the following sections: (1) The filter monads, (2) Kock-Zöberlein monads, (3) Injective objects that are the algebras of monads of the Kock-Zöberlein type, (4) Injective spaces over subspace embeddings, (5) Injective spaces over dense subspace embeddings, (6) Injective spaces over flat embeddings, (7) Injective spaces over completely flat embeddings, (8) Injective spaces over perfect embeddings, (9) Injective spaces over locally dense embeddings, (10) Injective spaces over open embeddings, (11) Injective spaces over closed embeddings, (12) Injective spaces over semi-open embeddings.

# 1 The filter monads

The filter monads considered in this paper are defined on the category of $T_0$ topological spaces. We begin by considering the monad of all filters of open sets [2]. After considering some variations, we prove basic properties.

### *1.1 The monad of all filters of opens*

Given a space $X$, one denotes its lattice of open sets by $\Omega X$ and constructs the *filter space* $\mathrm{T}\,X$ as follows. The points are the filters of $\Omega X$. The topology is generated by the sets

$$\Box U = \{\phi \in \mathrm{T}\,X \mid U \in \phi\}, \quad U \in \Omega X,$$

which form a base as

$$\Box U \cap \Box U' = \Box(U \cap U').$$

Given a continuous map $f : X \to Y$, one defines

$$\mathrm{T}\,f(\phi) = \{V \in \Omega Y \mid f^{-1}(V) \in \phi\}.$$

Then $\mathrm{T}$ is a functor and one has natural transformations $\eta : 1 \to \mathrm{T}$ and $\mu : \mathrm{T}\,\mathrm{T} \to \mathrm{T}$ defined by

$$\eta_X(x) = \{U \in \Omega X \mid x \in U\}, \qquad \mu_X(\Phi) = \{U \in \Omega X \mid \Box U \in \Phi\},$$

which make $\mathrm{T}$ into a monad $\mathrm{T} = (\mathrm{T}, \eta, \mu)$. Notice that

$$(\mathrm{T}\,f)^{-1}(\Box V) = \Box f^{-1}(V), \qquad \eta_X^{-1}(\Box U) = U, \qquad \mu_X^{-1}(U) = \Box\,\Box\,U.$$

(It is interesting that the filter monad is formally analogous to the continuation monad. In fact, an open set can be regarded as a continuous map into Sierpinski space and a filter can be regarded as a finite-meet-preserving map into the two-point lattice, so that set-abstraction corresponds to lambda-abstraction as in

$$\mathrm{T}\,f(\phi) = \lambda V \cdot \phi(\lambda x \cdot V(f(x))),$$
$$\eta_X(x) = \lambda U \cdot U(x),$$
$$\mu_X(\Phi) = \lambda U \cdot \Phi(\lambda\phi \cdot \phi(U)),$$

which is syntactically equivalent to the definition of the continuation monad.)

**Remark 1.1** $\mu_X(\Phi) = \bigcup\{\bigcap \mathcal{U} \mid \mathcal{U} \in \Phi\}$.

This shows that our definition is equivalent to that of Day [2].

**Proof.** This follows from Theorems 2.2 and 4.3 below, but we include a direct proof. Let $U \in \mu_X(\Phi)$. Then $\Box U \in \Phi$ by our definition. We have to show that there is an open $\mathcal{U} \in \Phi$ with $U \in \bigcap \mathcal{U}$. We can take $\mathcal{U} = \Box U$, because $\bigcap \Box U = \uparrow U$, the principal filter generated by $U$. Conversely, let $U \in \bigcup\{\bigcap \mathcal{U} \mid \mathcal{U} \in \Phi\}$. Then there is an open $\mathcal{U} \in \Phi$ with $U \in \bigcap \mathcal{U}$, and hence with $U \in \phi$ for all $\phi \in \mathcal{U}$. From this we see that $\mathcal{U} \subseteq \Box U$. Hence $\Box U \in \Phi$ because filters are upper sets. Therefore $U \in \mu_X(\Phi)$ by our definition. $\Box$

### *1.2 Variations*

We now consider some variations on the filter monad, obtained by allowing only particular kinds of filters of opens. The unit and multiplication are defined in the same way as for the monad of all filters (but notice that Remark 1.1 doesn't apply to the variations). In all variations the equation $\Box U \cap \Box U' = \Box(U \cap U')$ on basic open sets holds, because filters are closed under finite meets, but more equations arise. In order to avoid unnecessary decorations, we denote all monads by the standard symbol $\mathrm{T}$, relying on the context.

The *proper filter monad* is obtained by letting $\mathrm{T}\,X$ be the set of proper filters. Notice that the improper filter (the principal filter generated by the empty set) was a top point in the specialization order. For this variation the equation $\Box\emptyset = \emptyset$ holds (for the monad of all filters we have $\Box\emptyset = \{\uparrow\emptyset\}$, but this "accidental" fact is unimportant).

The *prime filter monad* is obtained by letting $\mathrm{T}\,X$ be the set of prime filters. Recall that a filter is prime iff it is inaccessible by finite joins. Since the join of the empty set is included in this definition, we see that a prime filter is proper. For this variation the equations $\Box\emptyset = \emptyset$ and $\Box U \cup \Box U' = \Box(U \cup U')$ hold.

The *completely prime filter monad* is obtained by letting $\mathrm{T}\,X$ be the set of completely prime filters. Recall that a filter is completely prime iff it is inaccessible by arbitrary joins. For this variation the equations $\Box\emptyset = \emptyset$ and $\bigcup_i \Box U_i = \Box \bigcup_i U_i$ hold.

The *Scott-open filter monad* is obtained by letting $\mathrm{T}\,X$ be the set of Scott open filters. Recall that a filter is Scott open iff it is inaccessible by directed joins. For this variation the equation $\bigcup_i \Box U_i = \Box \bigcup_i U_i$ holds, and so does $\Box\emptyset = \emptyset$ if the improper filter is excluded.

Yet another monad of filters (of connected open sets) will be considered in Section 9.

*1.3 Basic properties*

In what follows T is any of the filter monads introduced above. Continuous maps of $T_0$ topological spaces form a poset-enriched category under the pointwise specialization order. By a simple unfolding of definitions, one sees that the pointwise specialization order is characterized by, for all continuous maps $f, g : X \to Y$,

$$f \sqsubseteq g \text{ iff } f^{-1}(V) \subseteq g^{-1}(V) \text{ for every } V \in \Omega Y.$$

**Lemma 1.2** T *is locally monotone.*

**Proof.** It is immediate that the specialization order of $\mathrm{T}\,X$ is inclusion of filters. Let $f, g : X \to Y$ with $f \sqsubseteq g$. In order to prove that $\mathrm{T}\,f \sqsubseteq \mathrm{T}\,g$, let $\phi \in \mathrm{T}\,X$ and $V \in \mathrm{T}\,f(\phi)$. This means that $f^{-1}(V) \in \phi$. Since $f^{-1}(V) \subseteq g^{-1}(V)$ and $\phi$ is an upper set, we have that $g^{-1}(V) \in \phi$. But this means that $V \in \mathrm{T}\,g(\phi)$. Therefore $\mathrm{T}\,f(\phi) \subseteq \mathrm{T}\,g(\phi)$. □

**Lemma 1.3** *The inequality $\eta_{\mathrm{T}\,X} \sqsubseteq \mathrm{T}\,\eta_X$ holds for all spaces $X$.*

**Proof.** By specializing the definitions to the appropriate types, we obtain

$$\eta_{\mathrm{T}\,X}(\phi) = \{\mathcal{U} \in \Omega\mathrm{T}\,X \mid \phi \in \mathcal{U}\}, \qquad \mathrm{T}\,\eta_X(\phi) = \{\mathcal{U} \in \Omega\mathrm{T}\,X \mid \eta_X^{-1}(\mathcal{U}) \in \phi\}.$$

Let $\mathcal{U} \in \eta_{\mathrm{T}\,X}(\phi)$. Then $\phi \in \mathcal{U}$. Since the open sets $\Box U$ form a base of $\Omega\mathrm{T}\,X$, there is $U \in \Omega X$ with $\phi \in \Box U \subseteq \mathcal{U}$. Since $U = \eta_X^{-1}(\Box U) \in \phi$, we see that $\Box U \in \mathrm{T}\,\eta_X(\phi)$. Hence $\mathcal{U} \in \mathrm{T}\,\eta_X(\phi)$ because filters are upper sets. Therefore $\eta_{\mathrm{T}\,X}(\phi) \subseteq \mathrm{T}\,\eta_X(\phi)$. □

**Remark 1.4** The components of the unit are order-monic.

**Proof.** Assume that $\eta_X(x) \sqsubseteq \eta_X(y)$ and let $U$ be an open neighborhood of $x$. Then $U \in \eta_X(x)$ and hence $U \in \eta_X(y)$ by the assumption, which means that $y \in U$. Therefore $x \sqsubseteq y$. □

## 2 Kock-Zöberlein monads

We begin by specializing the notion of a Kock-Zöberlein doctrine on a 2-category to the notion of a Kock-Zöberlein monad on a poset enriched category. For simplicity, our definition is dual (at the level of hom-posets) to that of [13].

**Definition 2.1** A monad $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ defined on a poset-enriched category $\mathcal{X}$, with $\mathrm{T} : \mathcal{X} \to \mathcal{X}$ a locally monotone functor, is of the *Kock-Zöberlein type* if $\eta_{\mathrm{T}\,X} \sqsubseteq \mathrm{T}\,\eta_X$ for all $X \in \mathcal{X}$.

By Lemmas 1.2 and 1.3, the filter monads introduced in the previous section are of the Kock-Zöberlein type.

Since $\mathcal{X}$ is poset-enriched, one can consider adjunctions of arrows. An arrow $l : X \to Y$ is *left adjoint* to an arrow $r : Y \to X$, and $r$ is *right adjoint* to $l$, if $l \circ r \sqsubseteq \mathrm{id}_Y$ and $\mathrm{id}_X \sqsubseteq r \circ l$. Such an adjunction is denoted by $l \dashv r$. It is *reflective* if $l \circ r = \mathrm{id}_Y$, and *coreflective* if $\mathrm{id}_X = r \circ l$. In these cases one writes $l \dashv_{\mathrm{r}} r$ and $l \dashv_{\mathrm{c}} r$ respectively. (In domain theory one usually refers to coreflective adjunctions as embedding-projection pairs.)

By specializing Anders Kock's results [13] from 2-categories to poset-enriched categories, one learns that

**Theorem 2.2** *If* $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ *is a Kock-Zöberlein monad then*

(i) *An arrow* $\alpha : \mathrm{T}\, X \to X$ *is the structure map of a* $\mathrm{T}$*-algebra iff* $\eta_X \dashv_{\mathrm{c}} \alpha$.

(ii) $\eta_{\mathrm{T}\, X} \dashv \mu_X$.

(iii) $\mu_X \dashv \mathrm{T}\, \eta_X$.

(See [5] for a direct proof, where it is also shown that each of these properties is in fact equivalent to the defining condition of a Kock-Zöberlein monad.)

By property (i), every object can be the underlying object of at most one algebra, and every structure map of an algebra is uniquely determined by the underlying object of the algebra (as the right adjoint of the unit of the object). Due to this reason, one can identify the algebras of a Kock-Zöberlein monad with their underlying objects, and we adopt this practice.

## 3 Injective objects that are the algebras of monads of the Kock-Zöberlein type

In what follows we work on a Kock-Zöberlein monad $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ defined on a poset-enriched category $\mathcal{X}$. The maps defined below are particular cases of the "semi-upper" maps of [20], for which the reflectivity condition is not required:

**Definition 3.1** By a $\mathrm{T}$-*embedding* we mean an arrow $j : X \to Y$ such that $\mathrm{T}\, j : \mathrm{T}\, X \to \mathrm{T}\, Y$ has a reflective left adjoint, denoted by $\mathrm{T}^* j : \mathrm{T}\, Y \to \mathrm{T}\, X$.

For example, $\eta_X : X \to \mathrm{T}\, X$ is a T-embedding with $\mathrm{T}^* \eta_X = \mu_X$, because the adjunction $\mu_X \dashv \mathrm{T}\, \eta_X$ of Theorem 2.2 is reflective by virtue of the unit law $\mu_X \circ \mathrm{T}\, \eta_X = \mathrm{id}_X$.

**Remark 3.2** The following conditions are equivalent:

(i) T-embeddings are order-monic.

(ii) The components of the unit are order-monic.

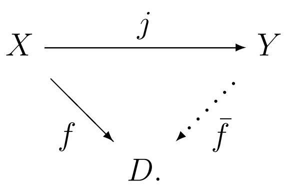
(iii) T is order-faithful.

(The conditions remain equivalent if the prefix "order-" is omitted.)

**Proof.** $(i) \implies (ii)$: Immediate. $(ii) \implies (iii)$: If $\mathrm{T}\, f \sqsubseteq \mathrm{T}\, g$ then we have that $\mathrm{T}\, f \circ \eta_X \sqsubseteq \mathrm{T}\, g \circ \eta_X$ by composition with $\eta_X$, that $\eta_Y \circ f \sqsubseteq \eta_Y \circ g$ by naturality, and that $f \sqsubseteq g$ by the assumption. $(iii) \implies (i)$: Let $j : X \to Y$ be a T-embedding and $f, g : Z \to X$ be arrows with $j \circ f \sqsubseteq j \circ g$.

Then $\mathrm{T}\, j \circ \mathrm{T}\, f \sqsubseteq \mathrm{T}\, j \circ \mathrm{T}\, g$ by local monotonicity of T. Hence $\mathrm{T}\, f \sqsubseteq \mathrm{T}\, g$ by composition with $\mathrm{T}^* j$. Therefore $f \sqsubseteq g$ by the assumption. □

**Definition 3.3** An object $D$ is *injective* over an arrow $j : X \to Y$ if every $f : X \to D$ has an extension $\bar{f} : Y \to D$ along $j : X \to Y$, in the sense that the following equation holds:




One normally assumes that $j : X \to Y$ is a monomorphism, so that the word "extension" is applied in the usual sense, but this is unimportant here.

Notice that there is nothing canonical about the extension $\bar{f}$ in the definition. But since $\mathcal{X}$ is poset-enriched, a definition with canonical choice is possible. We first recall a concept.

**Definition 3.4** A *right Kan extension* of an arrow $f : X \to D$ along an arrow $j : X \to Y$ is a (necessarily unique) arrow $f/j : Y \to D$ such that

(i) $f/j \circ j \sqsubseteq f$ and

(ii) $g \circ j \sqsubseteq f$ implies $g \sqsubseteq f/j$.

That is, $f/j$ is the greatest solution in $g$ to the inequality $g \circ j \sqsubseteq f$. When we have equality in (i), so that $f/j$ is an actual extension, we say that $f/j$ is a *right extension* of $f$ along $j$ (that is, we omit the word "Kan").

**Definition 3.5** We say that an object $D$ is *right injective* over an arrow $j : X \to Y$ if every $f : X \to D$ has a right extension along $j : X \to Y$.

The following fact was established in [5].

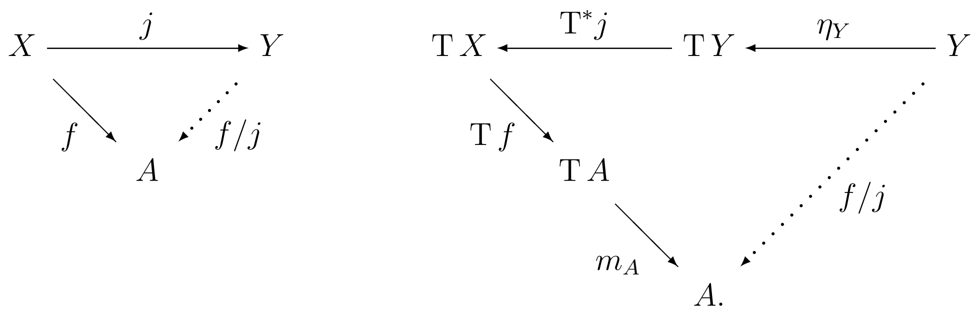
**Theorem 3.6** *The following statements are equivalent for any object $A$:*

(i) *$A$ is injective over* T*-embeddings.*

(ii) *$A$ is right injective over* T*-embeddings.*

(iii) *$A$ is a* T*-algebra.*

*In this case, if $f : X \to A$ is any arrow and $j : X \to Y$ is a* T*-embedding then $f/j : Y \to A$ is constructed as follows, where $m_A : \mathrm{T}\, A \to A$ is the unique structure map of the algebra $A$:*




**Remark 3.7** It is natural to ask whether the reflectivity condition in the definition of T-embedding is actually necessary to obtain Theorem 3.6. It is shown in [5, Theorem 4.2.2(4)] that every algebra of a Kock-Zöberlein monad is actually a right Kan object over arrows $j : X \to Y$ such that $\mathrm{T}\, j : \mathrm{T}\, X \to \mathrm{T}\, Y$ has a (not necessarily reflective) left adjoint, where an object $D$ is defined to be a *right Kan object* over an arrow $j : X \to Y$ if every $f : X \to D$ has a right Kan extension along $j : X \to Y$ (which is not assumed to be an actual extension). However, in general there may be objects that are right Kan objects over such arrows but are not algebras (see Remark 4.6).

**Lemma 3.8** *An object is a* T*-algebra iff it is a retract of a free* T*-algebra.*

This result, which is a useful tool for identifying the algebras in concrete situations, was first proved by Anders Kock [13]. It is also a corollary of Theorem 3.6, using the facts that $\eta_A$ is a T-embedding and that an injective object over T-embeddings is a retract of every object into which it is T-embedded.

## 4 Injective spaces over subspace embeddings

In this section we consider the monad of all filters of opens. We take the characterization [15, Proposition 2.4] of continuous lattices as our definition.

**Definition 4.1** A complete lattice $D$ is *continuous* if every $d \in D$ is the "lim inf" of its filter of Scott open neighborhoods, in the sense that $d = \bigsqcup\{\bigsqcap U \mid d \in U\}$, where $U$ ranges over Scott open sets.

**Lemma 4.2** $\mathrm{T}\, X$ *is an algebraic lattice endowed with the Scott topology.*

**Proof.** Since the lattice of filters is algebraic with the principal filters as the compact elements, and since $\Box U$ is the set of filters containing the principal filter $\uparrow U$, we see that $\Box U$ is a basic Scott open set. □

The following theorem was independently established by Day [2] and Wyler [23]. A simple proof based on the fact that the filter monad is of the Kock-Zöberlein type is available [4]:

**Theorem 4.3** *The algebras of the filter monad are the continuous lattices endowed with the Scott topology. Moreover, the structure map* $m_D : \mathrm{T}\, D \to D$ *of an algebra with underlying space* $D$ *is given by*

$$m_D(\phi) = \bigsqcup \left\{ \bigsqcap U \mid U \in \phi \right\}.$$

**Proof.** The algebras are continuous lattices because every algebra is a retract of a free algebra by Lemma 3.8, and retracts of algebraic lattices are continuous lattices. Conversely, let $D$ be a continuous lattice endowed with the Scott topology. It is clear that $m_D$ is monotone. Thus, in order to show that it is continuous, it is enough to show that $m_D(\phi) = \bigsqcup\{m_D(\uparrow U) \mid U \in \phi\}$, because the principal filters are compact and every filter is the directed join of the principal filters generated by its members. But this is immediate because $m_D(\uparrow U) = \bigsqcap U$. By Theorem 2.2(i), the function $m_D$ is a structure map iff $\eta_D \circ m_D \sqsubseteq \mathrm{id}_{\mathrm{T}\, D}$ and $m_D \circ \eta_D = \mathrm{id}_D$. The equation holds precisely because $D$ is continuous. In order to establish the inequality, first notice that $\eta_D \circ m_D(\phi) = \{U \in \Omega D \mid \bigsqcup\{\bigsqcap U' \mid U' \in \phi\} \in U\}$. Let $U \in \eta_D \circ m_D(\phi)$. Then $\bigsqcap U' \in U$ for some $U' \in \phi$ because $\{\bigsqcap U' \mid U' \in \phi\}$ is directed and $U$ is Scott open. Hence $U \in \phi$ because $U' \subseteq U$. Therefore $\eta_D \circ m_D(\phi) \subseteq \phi$, and the proof is concluded. □

(Day and Wyler also proved that if $D$ and $E$ are algebras then a continuous function $f : D \to E$ is an algebra homomorphism iff it preserves all meets.)

**Lemma 4.4** *The* T*-embeddings are precisely the subspace embeddings.*

**Proof.** Given a continuous map $f : X \to Y$, its frame map $f^{-1} : \Omega Y \to \Omega X$ preserves all joins and hence has a right adjoint $f_* : \Omega X \to \Omega Y$. We can thus define a map $\mathrm{T}^* f : \mathrm{T}\, Y \to \mathrm{T}\, X$ by

$$\mathrm{T}^* f(\gamma) = \{U \in \Omega X \mid f_*(U) \in \gamma\}.$$

The set $\mathrm{T}^* f(\gamma)$ is a filter because $f_*$ preserves meets. The function $\mathrm{T}^* f$ is continuous because one easily computes $(\mathrm{T}^* f)^{-1}(\Box U) = \Box f_*(U)$. That this produces a left adjoint to $\mathrm{T}\, f$ is verified as follows:

$$\mathrm{T}\, f(\mathrm{T}^* f(\gamma)) = \{V \in \Omega Y \mid f^{-1}(V) \in \mathrm{T}^* f(\gamma)\} = \{V \in \Omega Y \mid f_*(f^{-1}(V)) \in \gamma\} \supseteq \gamma,$$

because if $V \in \gamma$ then $f_*(f^{-1}(V)) \in \gamma$ as $V \subseteq f_*(f^{-1}(V))$. Similarly,

$$\mathrm{T}^* f(\mathrm{T}\, f(\phi)) = \{U \in \Omega X \mid f_*(U) \in \mathrm{T}\, f(\phi)\} = \{U \in \Omega X \mid f^{-1}(f_*(U)) \in \phi)\} \subseteq \phi.$$

Reflectiveness means that $\mathrm{T}^* f \circ \mathrm{T}\, f = \mathrm{id}_{\mathrm{T}\, X}$. So we have to check that the equation $f^{-1}(f_*(U)) = U$ holds iff $f$ is an embedding. But the equation is equivalent to saying that $f^{-1}$ is surjective. □

The following major result of Scott [15] thus appears a corollary.

**Theorem 4.5** *The injective spaces over subspace embeddings are the continuous lattices endowed with the Scott topology. Moreover, if $f : X \to D$ is a continuous map into a continuous lattice and $j : X \to Y$ is a subspace embedding, then $f$ has a greatest extension $f/j : Y \to D$ along $j$, given by*

$$f/j(y) = \bigsqcup \left\{ \bigsqcap U \mid y \in j_*(f^{-1}(U)) \right\}.$$

**Proof.** This follows from Theorem 3.6 and Lemma 4.4. The above formula is a special case of the general formula $f/j = m_A \circ \mathrm{T}\, f \circ \mathrm{T}^* j \circ \eta_Y$ of Theorem 3.6.□

(Notice that Scott's original formula [15, Proposition 3.8]

$$f/j(y) = \bigsqcup \left\{ \bigsqcap f(j^{-1}(V)) \mid y \in V \right\}$$

for the greatest extension is slightly different from ours.)

**Remark 4.6** The proof of Lemma 4.4 shows that the map $\mathrm{T}\, j : \mathrm{T}\, X \to \mathrm{T}\, Y$ has a left adjoint for any continuous map $j : X \to Y$. Hence, by Remark 3.7, the injective spaces are right Kan spaces over arbitrary continuous maps (see [5, Proposition 2.5.4] for a direct proof). But Thomas Erker [3] showed that the converse is not true. In fact, he characterized the right Kan spaces precisely as the essentially complete spaces.

## 5 Injective spaces over dense subspace embeddings

In this section we briefly consider the proper filter monad. A *continuous Scott domain* is a poset with directed joins and non-empty meets (or, equivalently, bounded joins), subject to the approximation axiom of Definition 4.1. In the algebraic case one uses the terminology *Scott domain.* The argument of Lemma 4.2 shows that $\mathrm{T}\, X$ is a Scott domain endowed with the Scott topology. The argument of Lemma 4.4 shows that the T-embeddings are precisely the dense subspace embeddings. In fact, a map $f : X \to Y$ is dense iff $f_*(\emptyset) = \emptyset$, and this is the condition for $\mathrm{T}^* f(\phi)$ as defined in Lemma 4.4 being different from $\uparrow\emptyset$ for all $\phi$ and hence $\mathrm{T}^* f$ being well-defined. Finally, the argument of Theorem 4.3 shows that the algebras of the filter monad are the continuous Scott domains, a result originally proved by Wyler [24]. Scott's result [7] that the injective spaces over dense subspace embeddings are the continuous Scott domains thus appears as a corollary.

## 6 Injective spaces over flat embeddings

In this section we consider the prime filter monad.

**Definition 6.1** A sober space is *stably compact* if it is locally compact and its compact saturated sets are closed under finite intersections [12,17]. A stably compact space is *spectral* (or coherent) if the compact open sets form a base.

Recall that a set is saturated iff it is the intersection of its neighborhoods iff it is an upper set in the specialization order. For example, compact Hausdorff spaces are stably compact. Notice that our definition of compactness does not include the Hausdorff separation axiom. The proof of the following lemma can be found in e.g. [18].

**Lemma 6.2** $\mathrm{T}\,X$ *is a spectral space.*

The following theorem is due to Simmons [16] and Wyler [23] independently (see also Flagg [6]). A simple proof using the fact that the monad is of the Kock-Zöberlein type is available, but it is omitted from this extended abstract due to lack of space.

**Theorem 6.3** *The algebras of the prime filter monad are the stably compact spaces.*

**Definition 6.4** A continuous map $f : X \to Y$ is *flat* if the right adjoint $f_* : \Omega Y \to \Omega X$ of the frame map $f^{-1} : \Omega Y \to \Omega X$ preserves finite joins.

Since the join of the empty family is included, flat embeddings are dense. This localic definition is that of [12, III.1.11]. An equivalent topological definition is that the map sending each closed subset of $X$ to the closure of its image in $Y$ preserves finite intersections [10], but we shall not use this.

**Lemma 6.5** *The* T*-embeddings are precisely the flat embeddings.*

**Proof.** It is immediate that the map $\mathrm{T}^*f : \mathrm{T}\,Y \to \mathrm{T}\,X$ constructed in Lemma 4.4 is now well-defined only if $f$ is flat. This is enough to conclude that flat embeddings are T-embeddings. Conversely, assume that $\mathrm{T}\,f : \mathrm{T}\,X \to \mathrm{T}\,Y$ has a reflective left adjoint $\mathrm{T}^*f : \mathrm{T}\,Y \to \mathrm{T}\,X$, and define

$$\Omega X \xrightarrow{\;r\;} \Omega Y \quad = \quad \Omega X \xrightarrow{\;\Box\;} \Omega \mathrm{T}\,X \xrightarrow{\;(\mathrm{T}^*f)^{-1}\;} \Omega \mathrm{T}\,Y \xrightarrow{\;\eta_Y^{-1}\;} \Omega Y.$$

The map $r$ clearly preserves finite joins. We show that $f^{-1} \dashv r$ reflectively, so that $f_* = r$ and $f$ is a flat embedding:

$$\begin{aligned}
f^{-1} \circ r(U) &= f^{-1} \circ \eta_Y^{-1} \circ (\mathrm{T}^*f)^{-1}(\Box U) && \text{by definition of } r \\
&= (\eta_Y \circ f)^{-1} \circ (\mathrm{T}^*f)^{-1}(\Box U) && \text{by contravariance of } (-)^{-1} \\
&= (\mathrm{T}\,f \circ \eta_X)^{-1} \circ (\mathrm{T}^*f)^{-1}(\Box U) && \text{by naturality of } \eta \\
&= \eta_X^{-1} \circ (\mathrm{T}\,f)^{-1} \circ (\mathrm{T}^*f)^{-1}(\Box U) && \text{by contravariance of } (-)^{-1} \\
&= \eta_X^{-1} \circ (\mathrm{T}^*f \circ \mathrm{T}\,f)^{-1}(\Box U) && \text{by contravariance of } (-)^{-1} \\
&= \eta_X^{-1}(\Box U) && \text{because } \mathrm{T}^*f \dashv \mathrm{T}\,f \text{ reflectively} \\
&= U,
\end{aligned}$$

$$\begin{aligned}
r \circ f^{-1}(V) &= \eta_Y^{-1} \circ (\mathrm{T}^* f)^{-1}(\Box f^{-1}(V)) && \text{by definition of } r \\
&= \eta_Y^{-1} \circ (\mathrm{T}^* f)^{-1} \circ (\mathrm{T}\, f)^{-1}(\Box V) && \text{because } (\mathrm{T}\, f)^{-1}(\Box V) = \Box f^{-1}(V) \\
&= \eta_Y^{-1} \circ (\mathrm{T}\, f \circ \mathrm{T}^* f)^{-1}(\Box V) && \text{by contravariance of } (-)^{-1} \\
&\supseteq \eta_Y^{-1}(\Box V) && \text{because } \mathrm{T}^* f \dashv \mathrm{T}\, f \\
&= V.
\end{aligned}$$

This method of proof is analogous to that of [20, Propositions 4.6 and 5.6]. □

Notice that the above proof is essentially localic. Johnstone [11] proved that every stably compact locale is injective over flat embeddings (without mentioning the converse). In the full version of this extended abstract, we prove that Johnstone's result follows by considering the ideal monad on the category of frames, which turns out to be of the Kock-Zöberlein type. The general extension formula of Theorem 3.6 specializes to Johnstone's. In this extended abstract we formulate a topological version of the result, which immediately follows from Lemma 6.5.

**Theorem 6.6** *The injective spaces over flat embeddings are the stably compact spaces.*

It follows that continuous lattices and continuous Scott domains endowed with the Scott topology are stably compact spaces. In fact, since there are fewer flat embeddings than embeddings and dense embeddings, there are more flatly injective spaces than injective and densely injective spaces. Of course, this has already been proved directly—see e.g. [1].

**Proposition 6.7** *The flat embeddings form the unique class of embeddings over which all stably compact spaces are right injective.*

**Proof.** Isbell [10] showed that an embedding $j : X \to Y$ is flat iff every continuous map $f : X \to D$ into a finite $T_0$ space has a right extension along $j$. Since finite $T_0$ spaces are trivially stably compact, Theorem 6.6 generalizes one half of Isbell's result (every finite $T_0$ space is right injective over flat embeddings). If we put Theorem 6.6 together with the difficult half of his, we obtain that an embedding $j : X \to Y$ is flat iff every continuous map $f : X \to D$ into a stably compact space has a right Kan extension along $j$. □

## 7 Injective spaces over completely flat embeddings

In this section we briefly consider the completely prime filter monad. Recall that a space is sober iff every completely prime filter is the neighborhood filter of a unique point [12,18]. Thus, $\mathrm{T}\, X$ is a sober space, the completely prime filter monad is the sobrification monad, the unit is the sobrification map, and the multiplication is an isomorphism (the Kock-Zöberlein property of the monad is thus of a trivial nature). Therefore the algebras of the completely prime filter monad are the sober spaces. In particular, it follows that retracts of sober spaces are sober.

Call a continuous map $f : X \to Y$ *completely flat* if the right adjoint $f_* : \Omega X \to \Omega Y$ of the frame map $f^{-1} : \Omega Y \to \Omega X$ preserves arbitrary joins. One readily checks that a subspace embedding $j : X \to Y$ is completely flat iff $j^{-1} : \Omega Y \to \Omega X$ is a frame isomorphism. The idea is that, in the ambient category of $T_0$ spaces, $j$ makes $Y$ "more sober" than $X$, but both $X$ and $Y$ have the same sobrification. The argument of Lemma 6.5 shows that the T-embeddings are precisely the completely flat embeddings. It follows that the injective spaces over completely flat embeddings are the sober spaces.

## 8 Injective spaces over perfect embeddings

In this section we consider the Scott open filter monad. Shalk [14] has shown that this monad restricts to the category of core-compact spaces (a mild generalization of locally compact spaces) and continuous maps, and that its algebras are the continuous meet-semilattices endowed with the Scott topology.

**Definition 8.1** A continuous map $f : X \to Y$ is *perfect* if the right adjoint $f_* : \Omega X \to \Omega Y$ of its frame map $f^{-1} : \Omega Y \to \Omega X$ preserves directed joins.

These maps are considered in detail in [5], where they are referred to as *finitary* maps (these maps are sometimes called *proper*). If both $X$ and $Y$ are sober and core-compact, then $f$ is perfect iff $f^{-1}(Q)$ is compact for every compact saturated $Q \subseteq Y$ [8], but we shall not use this fact. The argument of Lemma 6.5 shows that

**Lemma 8.2** *The* T*-embeddings are precisely the perfect embeddings.*

**Theorem 8.3** *The injective spaces over perfect subspace embeddings in the category of core-compact spaces are precisely the continuous meet-semilattices endowed with the Scott topology.*

The above result was obtained in [5] via the Smyth power space monad, in a slightly more laborious way. The proof via the Scott open filter monad is simpler, because the Smyth power space monad is ill-behaved for non-sober spaces. But notice that, by the Hofmann-Mislove Theorem [9,21,1,18], both monads agree on sober spaces.

## 9 Injective spaces over locally dense embeddings

Paul Taylor [19] characterized the category of $L$-domains and stable morphisms as the category of algebras for the connected open filter monad on the category of locally connected $T_0$-spaces. We omit the definition of the connected open filter monad, which is easily seen to be of the Kock-Zöberlein type.

**Definition 9.1** A continuous map $f : X \to Y$ of topological spaces is *locally dense* if $f^{-1} : \Omega Y \to \Omega X$ preserves connectedness and its right adjoint $f_* : \Omega X \to \Omega Y$ preserves disjoint unions.

If $X$ is $T_0$ and locally connected then the second condition is implied by the first.

**Lemma 9.2** *The* $\mathrm{T}$*-embeddings are precisely the locally dense subspace embeddings.*

**Theorem 9.3** *The injective spaces over locally dense embeddings in the category of locally connected* $T_0$ *spaces and continuous maps are the* $L$*-domains endowed with the Scott topology.*

## 10 Injective spaces over open embeddings

The lift monad on the category of cpos generalizes to topological spaces. Given a space $X$, one obtains a new space $\mathrm{T}\,X$ by adding a new point $\perp$ and a new open set $X \cup \{\perp\}$. We omit the routine details of the generalizations of the functor, unit, and multiplication. This monad is easily seen to be of the Kock-Zöberlein type. Its algebras are the spaces with a least point in the specialization order. The $T$-embeddings are the open subspace embeddings. It follows that the injective spaces over open embeddings are the spaces with a least point in the specialization order.

## 11 Injective spaces over closed embeddings

Instead of adding a new bottom point, one can add a new top point $\top$. The open sets of the resulting space are the opens of the given space with the top point added, together with empty set, of course. This construction also gives rise to a Kock-Zöberlein monad. We leave to the reader the verification that this gives rise to the fact that the injective spaces over closed subspace embeddings are the topological spaces with a greatest isolated point in the specialization order.

## 12 Injective spaces over semi-open embeddings

For the sake of completeness, we mention that the injective spaces over semi-open embeddings are the algebras of the Hoare power space monad, referring the reader to [5].

## Concluding remarks

On the conceptual side, we have used Kock-Zöberlein monads to show that many characterizations of semantic domains and topological spaces by injectivity hold due to the same abstract reason. This has led us to discover new characterizations. Notice that the monads are used in the abstract *proof* of the characterizations by injectivity, but are not mentioned in the *formulations* of the characterizations. On the technical side, we have also used the monads to simplify existing proofs of internal characterizations of the algebras.

Most monads considered in semantics are of the Kock-Zöberlein type. A notable counter-example is the Plotkin power domain monad (which is a particular case of the Vietoris power space monad).

We speculate that this abstract account to classical domain theory and topology via injectivity and monads may have applications to axiomatic and synthetic domain theory.

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