Algorithmic solution of higher-type equations

Martín Escardó
School of Computer Science, University of Birmingham, UK

Version of June 10, 2011

Abstract

In recent work we developed the notion of *exhaustible set* as a higher-type computational counter-part of the topological notion of *compact set*. In this paper we give applications to the computation of solutions of higher-type equations. Given a continuous functional $f\colon X \to Y$ and $y \in Y$, we wish to compute $x \in X$ such that $f(x) = y$, if such an x exists. We show that if x is unique and X and Y are subspaces of Kleene–Kreisel spaces of continuous functionals with X exhaustible, then x is computable uniformly in f, y and the exhaustibility condition. We also establish a version of this for computational metric spaces X and Y, where is X computationally complete and has an exhaustible set of Kleene–Kreisel representatives. Examples of interest include evaluation functionals defined on compact spaces X of bounded sequences of Taylor coefficients with values on spaces Y of real analytic functions defined on a compact set. A corollary is that it is semi-decidable whether a function defined on such a compact set fails to be analytic, and that the Taylor coefficients of an analytic function can be computed extensionally from the function.

Keywords and phrases. Higher-type computability, Kleene–Kreisel spaces of continuous functionals, exhaustible set, searchable set, computationally compact set, QCB space, admissible representation, topology in the theory of computation.

1 Introduction

Given a continuous functional $f\colon X \to Y$ and $y \in Y$, we consider the equation

$$f(x) = y$$

with the unknown $x \in X$. We show that if X and Y are subspaces of Kleene–Kreisel [9] spaces with X *exhaustible* [5], the solution is computable uniformly in f, y and the exhaustion functional $\forall_X\colon 2^X \to 2$, provided the solution is unique. Moreover, under the same assumptions for X and Y, it is uniformly semi-decidable whether a solution $x \in X$ fails to exist.

Kleene–Kreisel spaces are recalled in Section 1.1 (see also Section 1.2), the notion of exhaustibility, which amounts to a computational counter-part of the topological notion of compactness, and its fundamental properties are recalled in Section 2, and the uniqueness and exhaustibility requirements are examined in Section 3.

The computation of unique solutions of equations of the form

$$g(x) = h(x)$$

with $g, h\colon X \to Y$ is easily reduced to the case $f(x) = y$, because there are (abelian) computable group structures on the ground types that can be lifted componentwise to product types and pointwise to function types, and hence $x \in X$ is a solution of such an equation if and only if it is a solution of the equation $f(x) = 0$, where $f(x) = h(x) - g(x)$. And, by cartesian closedness of the category of Kleene–Kreisel spaces, the case in which g and h depend on parameters a,

$$g(a, x) = h(a, x),$$

and in which the solution can be computed uniformly in a is covered. Moreover, because the Kleene–Kreisel spaces are closed under finite products and countable powers, this includes the solution of finite and countably infinite systems of equations

$$g_i(\vec{a}_i, \vec{x}_i) = h_i(\vec{a}_i, \vec{x}_i),$$

with functionals g_i, h_i of finitely many or countably infinitely many parameters $\vec{a}_i$ and variables $\vec{x}_i$.

Similar results are known in constructive mathematics [4], but with a different level of generality and different foundational assumptions (here we develop higher-type computability in the realm of classical mathematics). In particular, whereas in constructive mathematics the emphasis has been on metric spaces, here the emphasis is in Kleene–Kreisel spaces, which fail to be metrizable, as is well known (but see Theorem 5.10). Moreover, in constructive mathematics, compactness is usually taken to mean completeness and total boundedness, but here, in the absence of a metric, the role of compactness is played by the notion of exhaustibility [5] (recalled in Section 2), which doesn't require metric assumptions (but again see Theorem 5.10).

We also consider a generalization to computational metric spaces, but still using exhaustibility as the computational manifestation of compactness, where f can be a functional and x a function (Section 5). And, as an application, we develop examples of sets of analytic functions that are exhaustible and can play the role of the space Y (Section 6).

This journal paper extends its earlier conference version [6] by the addition of

1. Section 3, which discusses the role of the uniqueness and exhaustibility assumptions in the context of higher-type computation with Kleene–Kreisel spaces.

2. The new proof of the result from [5] that exhaustible subspaces of arbitrary Kleene–Kreisel spaces are computationally homeomorphic to exhaustible subspaces of the Baire space (Theorem 4.7), derived as a corollary of our main Theorem 4.1.

3. Theorem 5.10, which partially answers Question 5.3, originally formulated in the conference version, and shows that any Kleene–Kreisel space has a metric which induces a zero-dimensional topology coarser than the original topology, but that agrees on compact sets, and, moreover, is computationally complete on exhaustible subsets. It seems, as our development indicates, that sometimes uniqueness can be replaced by completeness, but that uniqueness is more fundamental, and that this is at the heart of Question 5.3.

This paper also has a streamlining of the expository and technical development, with a number of new observations and technical clarifications.

1.1 Background

Recall that the Kleene–Kreisel spaces are inductively constructed from the discrete space $\mathbb{N}$ by iterating finite products and function spaces in a suitable cartesian closed category [7] (e.g. sequential topological spaces, filter spaces, limit spaces, compactly generated Hausdorff spaces, equilogical spaces or QCB spaces). Our main reference to computation over Kleene–Kreisel spaces of continuous functionals is Normann [9] (see also his expository and survey papers [10, 11]). The Kleene–Kreisel spaces can be constructed in a variety of ways [8], some of them alluded above. There are also a number of equivalent approaches to computation over such spaces, based e.g. on total elements of effectively given domains or on Kleene associates (which amount to admissible representations in the sense of Weihrauch [15, 16, 13]). In every such approach, one has a cartesian closed category, and hence computable maps are closed under λ-definability, which is the tool we apply to establish computability results. We reduce non-computability results to failure of continuity. In view of this, it is not strictly necessary to be acquainted with the technical notions and results in the form presented in the above references in order to be able to rigorously follow our presentation, provided the reader has some familiarity with the λ-calculus and with continuity at higher types. See e.g. [7] regarding continuity and other topological properties. We often rely on the density theorem [12], in the form that says that the Kleene–Kreisel spaces have computable dense sequences. We also use the fact that every Kleene–Kreisel space is a computable retract of a space of the form $\mathbb{N}^X$ with X a Kleene–Kreisel space (see e.g. [8] for a detailed proof).

1.2 Generality of the Results

Most of our results hold for all spaces that arise as computable retracts of spaces of the form $\mathbb{N}^X$ with X a Kleene–Kreisel space, and therefore for all computable retracts of Kleene–Kreisel spaces, and sometimes even more generally, as can be seen directly from the proofs we offer. The right ambient category for taking such computable retracts, in terms of generality, seems to be that of effectively presented QCB spaces [1], where by an effective presentation we mean an effectively admissible quotient representation in the sense of Schröder [13] (see the discussion of Section 8 of [1]), but knowledge of the theory of QCB spaces is not required for our presentation of the results. Readers familiar with QCB spaces will recognize that most of our result hold for retracts of spaces of the form $\mathbb{N}^X$ where X is a Hausdorff QCB space with a computable dense sequence and an effectively admissible quotient representation.

1.3 Contents and Organization

The main original contributions of this paper are in Sections4)–(6), which are preceded by sections that introduce further background and motivate the assumptions.

2 *Exhaustible spaces.* Background and a few new observations.

3 *Uniqueness and exhaustibility assumptions.* Such assumptions, under the understanding that exhaustibility amounts to computational compactness, are familiar from constructive mathematics, but here they are treated in the context of computability with higher-type continuous functionals, and are justified by (counter-)examples of increasing complexity.

4 *Equations over Kleene–Kreisel spaces.* This and the next section formulate and develop the main results of this this paper, and readers who are familiar with the main underlying concepts would probably prefer to start from this section, using the previous sections for reference or clarification.

5 *Equations over metric spaces.* This adapts the development of the previous section to the setting of computational metric spaces via representations, where the notion of *pseudo-metric with decidable closedness* is introduced as the main technical tool to deal with the lack of uniqueness in the intensional level.

6 *Exhaustible spaces of analytic functions.* This develops applications to real analysis.

2 Exhaustible Spaces

In previous work we investigated exhaustible sets of total elements of effectively given domains and their connections with Kleene–Kreisel spaces of continuous functionals [5]. In this section we recall and formulate these results directly in terms of Kleene–Kreisel spaces. Denote by Y^X the space of continuous functionals from X to Y, sometimes also written $(X \to Y)$. For the sake of definiteness and generality of the definitions, we work in the ambient cartesian closed category of effectively presented QCB spaces, discussed in Section 1.2.

Definition 2.1. Let $2 = \{0, 1\}$ be discrete.

1. A space K is called *exhaustible* if the universal quantification functional

$$\forall_K \colon 2^K \to 2$$

defined by $\forall_K(p) = 1$ if and only if $p(x) = 1$ for all $x \in K$ is computable.

2. It is called *searchable* if there is a computable selection function

$$\varepsilon_K \colon 2^K \to K$$

such that for all $p \in 2^K$, if there is $x \in K$ with $p(x) = 1$ then $p(\varepsilon_K(p)) = 1$.

3. A set $F \subseteq X$ is *decidable* if its characteristic map $X \to 2$ is computable. □

Equivalently, K is exhaustible if and only if the functional $\exists_K \colon 2^K \to 2$ defined by $\exists_K(p) = 1$ if and only if $p(x) = 1$ for some $x \in K$ is computable. If K is searchable, then it is exhaustible, because

$$\exists_K(p) = p(\varepsilon_K(p)).$$

The empty space is exhaustible, but not searchable, because there is no map $2^\emptyset \to \emptyset$.

Remark 2.2. Notice that the selection function ε_K computes solutions of equations of the form $p(x) = 1$ with the unknown $x \in K$ and with $Y = 2$. In this work we use ε_K (or its special cases $\exists_K, \forall_K$) to compute the solution of equations with more general Y, but still with x ranging over an exhaustible space K. In fact, [5, Section 8.2] anticipates the development of the present paper based on this observation. See also [5, Section 8.1], which regards satisfiability as equation-solving and gives an explicit formula for the solution which is not based on trial and error. □

Lemma 2.3 (Escardó [5])**.**

1. *The Cantor space $2^\mathbb{N}$ is searchable.*

2. *Searchable spaces are closed under the formation of computable images, intersections with decidable sets, and finite products.*

3. *Any exhaustible subspace K of a Kleene–Kreisel space X is compact, and moreover, if it is non-empty, it is a computable retract of X, and a computable image of the Cantor space, and hence searchable by (1) and (2).*

4. *A product $\prod_i K_i \subseteq X^\mathbb{N}$ of countably many searchable subspaces $K_i \subseteq X$ of a common Kleene–Kreisel space X is searchable uniformly in the sequence of quantification functionals $\forall_{K_i}$.*

This exhibits exhaustibility as a computational counter-part of the topological notion of compactness, at least for subspaces of Kleene–Kreisel spaces. It is important that these results are all uniform in the given data.

Remark 2.4. There is a subtle and important difference between the setting considered here and that considered in [5], already discussed in Section 8.7 of [5]. This is because the definition of exhaustibility studied there is more general and stronger, for two reasons:

1. Exhaustibility is defined for subsets K of a given domain D, with a computable functional $\exists_K : (D \to 2_\perp) \to 2_\perp$, where $2_\perp = \{0, 1, \perp\}$ is the domain with minimal element $\perp$ and maximal elements $0, 1$.

2. The relevant predicates $p \colon B \to 2_\perp$ need to be defined (that is, assume values distinct from $\perp$) on the subspace $K \subseteq D$ but are not required to be total in the ambient domain D.

For the weaker notion considered here, which is enough for our purposes, and more natural in the context of computation with total objects, the fact that exhaustible spaces are searchable is an easy consequence of the density theorem *and* the fact that K is a computable retract of its ambient Kleene–Kreisel space X. In fact, let $s \colon K \to X$ be the inclusion and $r \colon X \to K$ be the retraction with $r \circ s = \mathrm{id}$. Then a dense sequence $\delta \colon \mathbb{N} \to X$ gives rise to a dense sequence $r \circ \delta \colon \mathbb{N} \to K$, and assuming that K has a computable existential quantification functional $\exists_K : 2^K \to 2$ we can define a (total) computable search functional $\varepsilon_K : 2^K \to K$ by

$$\varepsilon(p) = r \circ \delta(\mu n. p(r \circ \delta(n)) = \exists_K(p))$$

which shows that exhaustibility implies searchability, for the sense of the notions used in this section and the present paper. □

3 Uniqueness and Exhaustibility Assumptions

We now justify the technical assumptions made in later sections and that were briefly mentioned in the introduction. Readers who don't need such a justification can jump directly to Section 4.

Given a continuous function $f \colon X \to Y$ of Kleene–Kreisel spaces, or more generally of computable retracts of Kleene–Kreisel spaces, and a point $y \in Y$, we wish to compute a point $x \in X$, uniformly in f and y, such that

$$f(x) = y.$$

We emphasize that, in this paper, including Section 5 on metric spaces, the terminology *uniform* is used in the sense of recursion theory, rather than metric topology (the notions of uniform continuity and uniform convergence don't feature in this paper). When we say that $x \in X$ is computable uniformly in $f \colon X \to Y$ and $y \in Y$, for suitable f and y, we mean that there is a computable functional s, defined on a suitable subspace of $(X \to Y) \times Y$, such that x can be computed as $s(f, y)$ for suitable f and y.

We discuss several cases for X and Y, of increasing generality, and explain why further assumptions and data are required in general. The simplest case is $X = Y = \mathbb{N}$, for which the algorithm

$$\mu x. f(x) = y$$

computes a solution if and only if a solution exists. This is subsumed by the next case.

Consider X arbitrary and $Y = \mathbb{N}$. By the Kleene–Kreisel density theorem, X has a computable dense sequence $\delta \colon \mathbb{N} \to X$, and, by continuity of f and discreteness of $\mathbb{N}$, if the equation has a solution, there is one of the form $x = \delta_n$ for some n. Hence the algorithm

$$x = \delta_{\mu n. f(\delta_n) = y}$$

computes a solution if and only if a solution exists. Moreover, in this particular case it is semi-decidable whether a solution exists, with the algorithm $\exists n. f(\delta_n) = y$. This is in contrast with the result that it is semi-decidable whether a solution fails to exists in given exhaustible subspaces (Section 4). Notice also that, by density of δ and discreteness of $\mathbb{N}$, whenever there is a non-isolated solution, there are uncountably many solutions unless $X = \mathbb{N}$, and so, in the absence of further assumptions, solutions are never unique if X is a Kleene–Kreisel space distinct from $\mathbb{N}$. In the complete metric case, working with representations, we are able to compute solutions that are unique up to equivalence (Section 5).

Now consider $X = 2 = \{0, 1\}$ discrete and $Y = \mathbb{N}^{\mathbb{N}}$. Then computing a solution to the above equation amounts to finding $x \in 2$ such that $f(x)(n) = y(n)$ for all $n \in \mathbb{N}$. In other words, under the assumption that

> $f(0) = y$ or $f(1) = y$,

we want to find $x \in 2$ such that $f(x) = y$. If the only data supplied to the desired algorithm are f, y, this is not possible, because no finite amount of information about the data can determine that one particular disjunct holds (a similar situation is worked out in detail below). However, if we instead assume that

> one of $f(0) = y$ and $f(1) = y$ holds, but not both,

that is, $f(x) = y$ has a unique solution, then we can compute $x \in 2$ as follows:

> Find the least n such that $f(0)(n) \neq y(n)$ or $f(1)(n) \neq y(n)$, and let x be the unique number such that $f(x)(n) = y(n)$.

Thus, in general, it is not possible to compute solutions unless we know that they are unique, and in this particular case one can compute unique solutions. This kind of phenomenon is well known in constructive mathematics (see e.g. [3]), and this section we explore the further subtleties that arise in higher-type computation with continuous functionals.

Next consider $X = \mathbb{N}$ and $Y = \mathbb{N}^{\mathbb{N}}$, and assume that the equation $f(x) = y$ has a unique solution. Now it is no longer possible to compute it uniformly in f and y. For suppose there existed a computable partial functional $s\colon (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, defined on some superset of

$$S = \{(f, y) \mid \text{the equation } f(x) = y \text{ has a unique solution } x_0\},$$

such that $x_0 = s(f, y)$ is the solution for $f, y \in S$. By continuity, for any $(f, y) \in S$ there is a number n such that $s(f, y) = x_0 = s(g, y)$ for every g such that $g(x)(i) = f(x)(i)$ for all $x < n$ and $i < n$. W.l.o.g. we can assume that $x_0 < n$ by replacing n by $\max(n, x_0)+1$ if necessary. Choose g defined by

$$g(x)(i) = \begin{cases} f(x)(i) & \text{if } x < n \text{ and } i < n, \\ y(i) & \text{if } x = n, \\ y(i) + 1 & \text{otherwise.} \end{cases}$$

By construction, $x = n$ is the unique solution of $g(x) = y$, and hence $s(g, y) = n$, which contradicts $s(g, y) = x_0$, and concludes the proof of the impossibility claim.

However, if we know that there is a unique solution in a finite set $K \subseteq \mathbb{N}$, then the solution can be found uniformly in f, y and a finite enumeration $e_0, \dots, e_{k-1}$ of K, as follows:

> Find n and $j < k$ such that the decidable conditions $\forall i < n. f(e_j)(i) = y(i)$ and $\forall l < k, l \neq j. \exists i < n. f(e_l)(i) \neq y(i)$ hold, and take $x = e_j$.

This generalizes the situation $X = 2$, and is a particular case of Theorem 4.1 below, which shows that unique solutions in exhaustible subsets K of Kleene–Kreisel spaces are computable uniformly in f, y and $\forall_K$.

It must be stressed, however, that uniqueness is not the only possible requirement to guarantee the uniform of computability of solutions in exhaustible domains. In fact, the solutions computed in Section 5 below are not unique at the intensional level, although they are unique up to equivalence, and completeness is the assumption that replaces strict uniqueness.

4 Equations over Kleene–Kreisel Spaces

This section proves the main result of this work:

Theorem 4.1. *If $f\colon X \to Y$ is a continuous map of subspaces of Kleene–Kreisel spaces with X exhaustible, and $y \in Y$, then, uniformly in $\forall_X$, f, and y:*

1. *It is semi-decidable whether the equation $f(x) = y$ fails to have a solution.*
2. *If $f(x) = y$ has a unique solution $x \in X$, then it is computable.*

Under the same assumptions:

Corollary 4.2. *If $f\colon X \to Y$ is a computable bijection then it has a computable inverse, uniformly in $\forall_X$ and f.*

This is a computational counter-part of the topological theorem that any continuous bijection from a compact Hausdorff space to a Hausdorff space is a homeomorphism. The following will be applied to semi-decide absence of solutions:

Lemma 4.3. *Let X be an exhaustible subspace of a Kleene–Kreisel space and $K_n \subseteq X$ be a sequence of sets that are decidable uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

emptiness of $\bigcap_n K_n$ is semi-decidable.

Proof. Because X is compact by exhaustibility, K_n is also compact as it is closed. Because X is Hausdorff, $\bigcap_n K_n = \emptyset$ if and only if there is n such that $K_n = \emptyset$. But emptiness of this set is decidable uniformly in n by the algorithm $\forall x \in X. x \notin K_n$, because the set K_n is decidable. Hence a semi-decision procedure is given by $\exists n. \forall x \in X. x \notin K_n$. □

As a preparation for a lemma that will be applied to compute unique solutions, notice that if a singleton $\{u\} \subseteq \mathbb{N}^Z$ is exhaustible, then the function u is computable, because $u(z) = \mu m. \forall v \in \{u\}. v(z) = m$. Moreover, u is computable uniformly in $\forall_{\{u\}}$, in the sense that there is a computable functional

$$U\colon S \to \mathbb{N}^Z \quad \text{with} \quad S = \{\phi \in 2^{2^{\mathbb{N}^Z}} \mid \phi = \forall_{\{v\}} \text{ for some } v \in \mathbb{N}^Z\},$$

such that $u = U\left(\forall_{\{u\}}\right)$, namely

$$U(\phi)(z) = \mu m. \phi(\lambda u. u(z) = m).$$

Lemma 4.4 below generalizes this, using an argument from [5] that was originally used to prove that non-empty exhaustible subsets of Kleene–Kreisel spaces are computable images of the Cantor space and hence searchable. Here we find additional applications and further useful generalizations.

Lemma 4.4. *Let X be an exhaustible subspace of a Kleene–Kreisel space and $K_n \subseteq X$ be a sequence of sets that are exhaustible uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

> *if $\bigcap_n K_n$ is a singleton $\{x\}$, then x is computable.*

Proof. By Lemma 2.3, X is a computable retract of its Kleene–Kreisel super-space. Because any Kleene–Kreisel space is a computable retract of a Kleene–Kreisel space of the form $\mathbb{N}^Z$, and because retractions compose, there are computable maps $s\colon X \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to X$ with $r \circ s = \mathrm{id}_X$. It suffices to show that the function $u = s(x) \in \mathbb{N}^Z$ is computable, because $x = r(u)$. The sets $L_n = s(K_n) \subseteq \mathbb{N}^Z$, being computable images of exhaustible sets, are themselves exhaustible. For any $z \in Z$, the set $U_z = \{v \in \mathbb{N}^Z \mid v(z) = u(z)\}$ is clopen and $\bigcap_n L_n = \{u\} \subseteq U_z$. Because $\mathbb{N}^Z$ is Hausdorff, because $L_n \supseteq L_{n+1}$, because each L_n is compact and because U_z is open, there is n such that $L_n \subseteq U_z$. That is, $v \in L_n$ implies $v(z) = u(z)$. Therefore, for every $z \in Z$ there is n such that $v(z) = w(z)$ for all $v, w \in L_n$. Now, the map $n(z) = \mu n.\forall v, w \in L_n.v(z) = w(z)$ is computable by the uniform exhaustibility of L_n. But $u \in L_{n(z)}$ for any $z \in Z$ and therefore u is computable by exhaustibility as $u(z) = \mu m.\forall v \in L_{n(z)}.v(z) = m$, as required. □

To build sets K_n suitable for applying these two lemmas, we use:

Lemma 4.5. *For every computable retract of a Kleene–Kreisel space, there is a family $(=_n)$ of equivalence relations that are decidable uniformly in n and satisfy*

$$\begin{aligned} x = x' &\iff \forall n.\, x =_n x', \\ x =_{n+1} x' &\implies x =_n x'. \end{aligned}$$

Proof. Let X be a Kleene–Kreisel space and $s\colon X \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to X$ be computable maps with Z a Kleene–Kreisel space and $r \circ s = \mathrm{id}_X$. By the density theorem, there is a computable dense sequence $\delta_n \in Z$. Then the definition

$$x =_n x' \iff \forall i < n.s(x)(\delta_i) = s(x')(\delta_i)$$

clearly produces an equivalence relation that is decidable uniformly in n and satisfies $x =_{n+1} x' \implies x =_n x'$. Moreover, $x = x'$ iff $s(x) = s(x')$, because s is injective, iff $s(x)(\delta_n) = s(x')(\delta_n)$ for every n, by density, iff $x =_n x'$ for every n, by definition. □

Proof of Theorem 4.1. The set $K_n = \{x \in X \mid f(x) =_n y\}$, being a decidable subset of an exhaustible space, is exhaustible. Therefore the result follows from Lemmas 4.3 and 4.4, because $x \in \bigcap_n K_n$ iff $f(x) =_n y$ for every n iff $f(x) = y$ by Lemma 4.5. □

Algorithms 4.6. In summary, the algorithm for semi-deciding non-existence of solutions is

$$\exists n.\forall x \in X.f(x) \neq_n y,$$

and that for computing the solution x_0 as a function of $\forall_X$, f, and y is:

$$\begin{aligned} \forall x \in K_n.p(x) &= \forall x \in X.f(x) =_n y \implies p(x), \\ \forall v \in L_n.q(v) &= \forall x \in K_n.q(s(x)), \\ n(z) &= \mu n.\forall v, w \in L_n.v(z) = w(z), \\ u(z) &= \mu m.\forall v \in L_{n(z)}.v(z) = m, \\ x_0 &= r(u). \end{aligned}$$

Here $r\colon \mathbb{N}^Z \to X$ is a computable retraction with section $s\colon X \to \mathbb{N}^Z$, where Z is a Kleene–Kreisel space, as constructed in the proof of Lemma 4.4. □

Of course, even in the absence of uniqueness, *approximate* solutions with precision n are trivially computable with the algorithm

$$\varepsilon_X(\lambda x.f(x) =_n y),$$

using the fact that non-empty exhaustible subsets of Kleene–Kreisel spaces are searchable. But the above unique-solution algorithm uses the quantification functional $\forall_X$ rather than the selection functional ε_X. In the next section we compute solutions as limits of approximate solutions.

We conclude this section by showing that Theorem 4.1 gives an alternative route to the following fact established in [5]:

Theorem 4.7. *Any exhaustible subspace of a Kleene–Kreisel space is computably homeomorphic to an exhaustible subspace of the Baire space* $\mathbb{N}^{\mathbb{N}}$.

Proof. Let K be an exhaustible subspace of a Kleene–Kreisel space, let $s\colon K \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to K$ be computable maps with $r \circ s = \mathrm{id}_K$ and Z a Kleene–Kreisel space, and let $\delta_n \in Z$ be a computable dense sequence. The subspace $X = s(K) \subseteq \mathbb{N}^Z$, being a computable image of an exhaustible space, is itself exhaustible. As in [5], we consider the map $X \to \mathbb{N}^{\mathbb{N}}$ that sends $u \in X$ to the sequence $u(\delta_n)$, but we argue using Theorem 4.1 instead. Let $f\colon X \to Y$ be the restriction of this map to its image $Y \subseteq \mathbb{N}^{\mathbb{N}}$. By density, f is one-to-one, and, by construction, it is onto, and hence it has a computable inverse. Therefore there is computable map $g\colon K \to Y$ defined by $g(k) = f(s(k))$ with computable inverse given by $g^{-1}(\alpha) = r(f^{-1}(\alpha))$. □

5 Equations over Metric Spaces

For the purposes of this and the following section, we can work with computational spaces in the sense of TTE [15] using Baire-space representations, or equivalently, using partial equivalence relations on representatives living in effectively given domains [2]. Our techniques apply to both, but we choose a development based on representations, where we more conveniently assume that representatives form subspaces of arbitrary Kleene–Kreisel spaces rather than just the Baire space $\mathbb{N}^{\mathbb{N}}$, and this is the approach we take. Because Kleene–Kreisel spaces are known to have Baire-space admissible quotient representations, in fact given by Kleene associates, there is no gain in terms of generality. We first formulate the main result of this section and then supply the missing notions in Definition 5.4:

Theorem 5.1. *Let X and Y be computational metric spaces with X computationally complete and having an exhaustible set of Kleene–Kreisel representatives.*

> *If $f\colon X \to Y$ is continuous and $y \in Y$, then, uniformly in f, y and the exhaustibility condition:*
>
> 1. *It is semi-decidable whether the equation $f(x) = y$ fails to have a solution.*
> 2. *If $f(x) = y$ has a unique solution $x \in X$, then it is computable.*

Under the same assumptions:

Corollary 5.2. *Any computable bijection $f\colon X \to Y$ has a computable inverse, uniformly in f and the exhaustibility condition.*

Question 5.3. Given that exhaustibility is a computational counter-part of the topological notion of compactness, and that compact metric spaces are complete, it is natural to conjecture that, at least under suitable computational conditions, the assumption of computational completeness in the above theorem is superfluous. We leave this as an open question, with a partial result in this direction given by Theorem 5.10 below. In connection with this, notice that Theorem 5.1 is analogous to a well-known result in constructive mathematics [4], with the assumptions reformulated in our higher-type computational setting. □

There is a technical difficulty in the proof of the theorem: at the intensional level, where computations take place, solutions are unique only up to equivalence of representatives. In order to overcome this, we work with pseudo-metric spaces at the intensional level and with a notion of decidable closeness for them. Recall that a *pseudo-metric* on a set X is a function $d\colon X \times X \to [0, \infty)$ such that

$$d(x, x) = 0, \quad d(x, y) = d(y, x), \quad d(x, z) \le d(x, y) + d(y, z).$$

Then d is a *metric* if it additionally satisfies $d(x, y) = 0 \implies x = y$. If d is only a pseudo-metric, then the relation $(\sim)$ defined by

$$x \sim y \iff d(x, y) = 0$$

is an equivalence relation, referred to as *pseudo-metric equivalence*. A pseudo-metric topology is Hausdorff if and only if it is T_0 if and only if the pseudo-metric is a metric. Moreover, two points are equivalent if and only if they have the same neighbourhoods. Hence any sequence has at most one limit up to equivalence.

A *computational metric space* is a computational pseudo-metric space in which the pseudo-metric is actually a metric, and hence we formulate the following definitions in the generality of pseudo-metric spaces.

Definition 5.4. We work with the standard admissible representation of the real line $\mathbb{R}$ given by fast-converging Cauchy sequences of rational numbers, that is, sequences $q_n \in \mathbb{Q}$ with $|q_n - q_{n+1}| < 2^{-n}$. We also work with the subspace $[0, \infty) \subseteq \mathbb{R}$ with the relative representation.

1. A *computational pseudo-metric space* is a computational space X endowed with a computable pseudo-metric, denoted by $d = d_X \colon X \times X \to [0, \infty)$.

2. A *fast-converging Cauchy sequence* in a computational pseudo-metric space X is a sequence $x_n \in X$ with $d(x_n, x_{n+1}) < 2^{-n}$. The subspace of $X^{\mathbb{N}}$ consisting of fast Cauchy sequences is denoted by $\mathrm{Cauchy}(X)$.

3. A computational pseudo-metric space X is called *computationally complete* if every sequence $x_n \in \mathrm{Cauchy}(X)$ has a limit uniformly in x_n.

4. A computational pseudo-metric space X has *decidable closeness* if there is a family of relations $\sim_n$ on X that are decidable uniformly in n and satisfy:

   (a) $x \sim_n y \implies d(x, y) < 2^{-n}$,
   (b) $x \sim y \implies \forall_n . x \sim_n y$.
   (c) $x \sim_{n+1} y \implies x \sim_n y$,
   (d) $x \sim_n y \iff y \sim_n x$,
   (e) $x \sim_{n+1} y \sim_{n+1} z \implies x \sim_n z$.

   The second last condition corresponds to symmetry of the metric and the last one is a counter-part of the triangle inequality. It follows from the first condition that if $x \sim_n y$ for every n, then $x \sim y$. Write

   $$[x] = \{y \in X \mid x \sim y\}, \qquad [x]_n = \{y \in X \mid x \sim_n y\}.$$

   Then the equivalence class $[x]$ is the closed ball of radius 0 centered at x.

For instance, the spaces $\mathbb{R}$ and $[0,\infty)$ are computationally complete metric spaces under the Euclidean metric, but don't have decidable closeness, although their representing spaces consisting of Cauchy sequences of rational numbers do (Lemma 5.5). $\square$

We are now ready to prove the theorem.

Lemma 5.5. *For every computational metric space X there is a canonical computable pseudo-metric $d = d_{\ulcorner X \urcorner}$ on the representing space $\ulcorner X \urcorner$ such that:*

1. *The representation map $\rho = \rho_X : \ulcorner X \urcorner \to X$ is an isometry:*

$$d(t,u) = d(\rho(t), \rho(u)).$$

   In particular:

   (a) $t \sim u \iff d(t,u) = 0 \iff \rho(t) = \rho(u)$.

   (b) If $f \colon X \to Y$ is a computable map of metric spaces, then any representative $\ulcorner f \urcorner \colon \ulcorner X \urcorner \to \ulcorner Y \urcorner$ preserves the relation $(\sim)$.

2. *If X is computationally complete, then so is $\ulcorner X \urcorner$.*

3. *The representing space $\ulcorner X \urcorner$ has decidable closeness.*

Proof. Construct $d_{\ulcorner X \urcorner} \colon \ulcorner X \urcorner \times \ulcorner X \urcorner \to [0,\infty)$ as the composition of a computable representative $\ulcorner d_X \urcorner \colon \ulcorner X \urcorner \times \ulcorner X \urcorner \to \ulcorner [0,\infty) \urcorner$ of $d_X \colon X \times X \to [0,\infty)$ with the representation map $\rho_{[0,\infty)} \colon \ulcorner [0,\infty) \urcorner \to [0,\infty)$. A limit operator for $\ulcorner X \urcorner$ from a limit operator for X is constructed in a similar manner. For given $t, u \in \ulcorner X \urcorner$, let q_n be the n-th term of the sequence $\ulcorner d_X \urcorner(t,u) \in \ulcorner [0,\infty) \urcorner \subseteq \mathrm{Cauchy}(\mathbb{Q})$ and define $I_n = [q_n - 2^{-n}, q_n + 2^{-n}]$. By the triangle inequality and the fast Cauchy property, $I_n \subseteq I_{n+1}$, and the limit of q_n is in the intersection of these intervals, and so $d_{\ulcorner X \urcorner}(t,u) \in I_n$ for every n. Hence we can define $t \sim_n u$ to mean that $0 \in I_n$, which amounts to $q_n \in [-2^{-n}, 2^{-n}]$. $\square$

Lemma 5.6. *Let Z be a subspace of a Kleene–Kreisel space with complete computational pseudo-metric structure and decidable closeness, and $K_n \subseteq Z$ be a sequence of sets that are exhaustible uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

if $\bigcap_n K_n$ is a subset of an equivalence class, then it has a computable member.

Proof. Let $z \in \bigcap_n K_n$. For any m, we have $\bigcap_n K_n \subseteq [z] \subseteq [z]_{m+1}$, and hence there is n such that $K_n \subseteq [z]_{m+1}$, because the sets K_n are compact, because $K_n \supseteq K_{n+1}$, because Z is Hausdorff and because $[z]_{m+1}$ is open. Hence for every $u \in K_n$ we have $u \sim_{m+1} z$, and so for all $u, v \in K_n$ we have $u \sim_m v$. By the exhaustibility of K_n and the decidability of $(\sim_n)$, the function $n(m) = \mu n. \forall u, v \in K_n .\, u \sim_m v$ is computable. By the searchability of K_n, there is a computable sequence $u_m \in K_{n(m)}$. Because $n(m) \leq n(m+1)$, we have that $K_{n(m)} \supseteq K_{n(m+1)}$ and hence $u_m \sim_m u_{m+1}$ and so $d(u_m, u_{m+1}) < 2^{-m}$ and u_m is a Cauchy sequence. By completeness, u_m converges to a computable point u_∞. Because $z \in K_{n(m)}$, we have $u_m \sim_m z$ for every m, and hence $d(u_m, z) < 2^{-m}$. And because $d(u_\infty, u_m) < 2^{-m+1}$, the triangle inequality gives $d(u_\infty, z) < 2^{-m} + 2^{-m+1}$ for every m and hence $d(u_\infty, z) = 0$ and therefore $u_\infty \in \bigcap_n K_n$. $\square$

The proof of the following is essentially the same as that of Theorem 4.1, but uses Lemma 5.6 rather than Lemma 4.4, and Lemma 5.5 instead of Lemma 4.5.

Lemma 5.7. *Let Z and W be subspaces of Kleene–Kreisel spaces with computational pseudo-metric structure and decidable closeness, and assume that Z is computationally complete and exhaustible.*

> *If $g\colon Z \to W$ is a computable map that preserves pseudo-metric equivalence and $w \in W$ is computable, then, uniformly in $\forall_Z$, g, and w:*
>
> 1. *It is semi-decidable whether the equivalence $g(z) \sim w$ fails to have a solution $z \in Z$.*
> 2. *If $g(z) \sim w$ has a unique solution $z \in Z$ up to equivalence, then some solution is computable.*

Proof. The set $K_n = \{z \in Z \mid g(z) \sim_n w\}$, being a decidable subset of an exhaustible space, is exhaustible. Therefore the result follows from Lemmas 4.3 and 5.6, because $z \in \bigcap_n K_n$ iff $g(z) \sim_n w$ for every n iff $g(z) \sim w$. □

Algorithm 5.8. The solution $z = u_\infty$ is then computed from $\forall_Z$, g and w as follows, where we have expanded $\forall_{K_n}$ as a quantification over Z:

$$\begin{aligned} n(m) &= \mu n.\forall u, v \in Z.\, g(u) \sim_n w \wedge g(v) \sim_n w \implies u \sim_m v, \\ u_\infty &= \lim_m \varepsilon_K(\lambda z.g(z) \sim_{n(m)} w). \end{aligned}$$

Thus, although there are common ingredients with Theorem 4.1, the resulting algorithm is different from 4.6, because it relies on the limit operator and approximate solutions. □

But, for Theorem 5.1, approximate solutions are computable uniformly in the representatives $\ulcorner f \urcorner$ and $\ulcorner y \urcorner$ only, as different approximate solutions are obtained for different representatives of f and y:

Proof of Theorem 5.1. Let $f\colon X \to Y$ and $y \in Y$ be computable. Now apply Lemma 5.7 with $Z = \ulcorner X \urcorner$, $W = \ulcorner Y \urcorner$, $g = \ulcorner f \urcorner$, $w = \ulcorner y \urcorner$, using Lemma 5.5 to fulfil the necessary hypotheses. If $f(x) = y$ has a unique solution x, then $g(z) \sim w$ has a unique solution z up to equivalence, and $x = \rho(z)$ for any solution z, and hence x is computable. Because g preserves $(\sim)$ by Lemma 5.5, if $g(z) \sim w$ has a solution z, then $x = \rho(z)$ is a solution of $f(x) = y$. This shows that $f(x) = y$ has a solution if and only if $g(z) \sim w$ has a solution, and we can reduce the semi-decision of absence of solutions of $f(x) = y$ to absence of solutions of $g(z) \sim w$. □

Before giving applications to computational real analysis, in Section 6, we clarify some aspects of the above development.

Remark 5.9. In the above definition, we don't require the representation topology of X to agree with the pseudo-metric topology generated by open balls. But notice that the metric topology is always coarser than the representation topology, because, by continuity of the metric, open balls are open in the representation topology. Hence the representation topology of any computational metric space is Hausdorff. Moreover, if X has an exhaustible Kleene–Kreisel space of representatives and the metric topology is compact, then both topologies agree, because no compact Hausdorff topology can be properly refined to another compact Hausdorff topology. □

Recall that an ultra-metric space is a metric space for which the triangle inequality holds in the stronger form $d(x, z) \leq \max(d(x, y), d(y, z))$, and that an ultra-metric topology is zero-dimensional because open balls are closed. We now again use the equivalence relations $(=_n)$ given in Lemma 4.5.

Theorem 5.10. *Any space X that is a computable retract of some Kleene-Kreisel space becomes a computational ultra-metric space when it is equipped with the metric d defined by*

$$d(x, y) = \inf\{2^{-n} \mid x =_n y\}.$$

Moreover:

1. *The metric has decidable closeness given by $(\sim_n) = (=_n)$.*
2. *The metric topology is in general strictly coarser than the intrinsic topology, but both agree on compact subsets.*
3. *Exhaustible subspaces with the relative metric are computationally complete.*

Proof. Computability of the ultra-metric and decidability of closeness are immediate.

Let $s\colon X \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to X$ be the same computable functions selected in the proof of Lemma 4.5, let δ be the same dense sequence, and for $u, v \in \mathbb{N}^Z$ define $u =_n v \iff \forall i < n.\, u(\delta_i) = v(\delta_i)$ so that $x =_n y$ iff $s(x) =_n s(y)$.

Agreement of topologies: A subbasic open set in the topology of pointwise convergence (product topology) of $\mathbb{N}^Z$ is of the form $N(z, V) = \{u \in \mathbb{N}^Z \mid u(z) \in V\}$ with $z \in Z$ and $V \subseteq \mathbb{N}$. Now $u =_n v$ iff $d(u, v) < 2^{-n}$, and hence the open ball $B_{2^{-n}}(u)$ is the intersection of the pointwise open sets $N(\delta_i, \{u(\delta_i)\})$, for $i < n$, and hence open balls are open in the topology of pointwise convergence. For a compact subspace of $\mathbb{N}^Z$, density of δ gives that the metric topology agrees with the pointwise topology. But the relative topology on compact subsets of $\mathbb{N}^Z$ coincides with the topology of pointwise convergence, because the pointwise topology is compact Hausdorff and no such topology has a proper refinement. The reduction of this to X via the retraction is easy.

Completeness: Let $x_n \in K$ be a fast Cauchy sequence in an exhaustible subset $K \subseteq X$. Then $x_n =_n x_{n+1}$ and hence $s(x_n) =_n s(x_{n+1})$. It suffices to show that the sequence $f_n = s(x_n)$ converges to a computable limit f_∞, because then the sequence $x_n = r(f_n)$ converges to the computable point $x_\infty = r(x_\infty)$ by continuity of r. The set $L = s(K)$ is exhaustible because it is a computable image of an exhaustible set. For any n, the set $L_n = \{g \in L \mid g =_n f_n\}$ is exhaustible because it is a decidable subset of an exhaustible set, and $f_n \in L_n$. By compactness, $\bigcap_n L_n \neq \emptyset$ because clearly $L_n \supseteq L_{n+1}$. If $g, h \in \bigcap_n L_n$, then $g =_n f_n =_n h$ for every n, and hence $g = h$, and so $\bigcap_n L_n = \{f_\infty\}$ for some computable f_∞ by Lemma 4.4. Because $f_\infty \in \bigcap_n L_n$, we have $f_\infty =_n f_n$ for every n. Hence if some ball $B_{2^{-k}}(h)$ is a neighbourhood of f_∞, then $h =_k f_\infty =_k f_n$ for all $n \geq k$, and hence $f_n \in B_k(h)$ for all $n \geq k$, which shows that $f_n \to f_\infty$. $\square$

In view of this theorem, Lemma 5.6 generalizes Lemma 4.4. But Lemma 4.4 cannot be eliminated, because it is used to prove the theorem.

Algorithm 5.11. Expanding Lemma 4.4, the algorithm for computing $\lim_n f_n$ for a fast Cauchy sequence $f_n \in K \subseteq \mathbb{N}^Z$ with K exhaustible is:

$$\begin{aligned} n(z) &= \mu n. \forall g, h \in K. g =_n f_n \wedge h =_n f_n \implies g(z) = h(z), \\ \lim_n f_n &= \lambda z. \mu m. \forall g \in K. g =_{n(z)} f_{n(z)} \implies g(z) = m. \end{aligned} \qquad \square$$

Independently of this, Matthias Schröder (personal communication) showed that if a QCB space X is the sequential coreflection of a zero-dimensional topology, then there is a metric d on X such that: (1) The topology induced by d is coarser than that of X and than the zero-dimensional topology. (2) On compact subsets of X, the three topologies agree. (3) The image of d is contained in $\{0\} \cup \{2^{-n} | n \in \mathbb{N}\}$. This applies to all retracts of Kleene–Kreisel spaces in particular, as their topologies satisfy the hypotheses. His construction uses countable pseudo-bases rather than dense sequences. However, at the time of writing, he hasn't proved computational versions of these statements.

6 Exhaustible Spaces of Analytic Functions

For any $\epsilon \in (0,1)$, any $x \in [-\epsilon, \epsilon]$, any $b > 0$, and any sequence $a \in [-b, b]^{\mathbb{N}}$, the Taylor series $\sum_n a_n x^n$ converges to a number in the interval $[-b/(1-\epsilon), b/(1-\epsilon)]$. The following is proved by a standard computational analysis argument:

Lemma 6.1. *Any analytic function $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$ of the form $f(x) = \sum_n a_n x^n$ is computable uniformly in any given $\epsilon \in (0,1)$, $b > 0$ and $a \in [-b, b]^{\mathbb{N}}$.*

Definition 6.2. Denote by $A = A(\epsilon, b) \subseteq \mathbb{R}^{[-\epsilon,\epsilon]}$ the subspace of such analytic functions, and by $T = T_{\epsilon,b} \colon [-b, b]^{\mathbb{N}} \to A(\epsilon, b)$ the functional that implements the uniformity condition, so that $f = T(a)$. □

The following results also hold uniformly in ϵ and b, but we omit explicit indications for the sake of brevity. The results are uniform in the exhaustibility assumptions too. Because $[-b, b]^{\mathbb{N}}$ is compact and T is continuous, the space A is compact as well. Moreover:

Theorem 6.3. *The space A has an exhaustible set of Kleene–Kreisel representatives.*

Proof. The space $[-b, b]^{\mathbb{N}}$ has an exhaustible space of representatives, e.g. using signed-digit binary representation. Because exhaustible spaces are preserved by computable images, the image of any representative $\ulcorner T \urcorner \colon \ulcorner [-b, b]^{\mathbb{N}} \urcorner \to \ulcorner A \urcorner$ of T gives an exhaustible set of representatives of A contained in the set $\ulcorner A \urcorner$ of all representatives of A. □

Hence the solution of a functional equation with a unique analytic unknown in A can be computed using Theorem 5.1 (and this applies to a number of differential equations, including important cases of Peano's Theorem). We conclude by showing that the Taylor coefficients of any $f \in A$ can be computed from f alone, without using differentiation.

Lemma 6.4. *For any non-empty space X with an exhaustible set of Kleene–Kreisel representatives, the maximum- and minimum-value functionals*

$$\max_X, \min_X \colon \mathbb{R}^X \to \mathbb{R}$$

are computable.

Of course, any $f \in \mathbb{R}^X$ attains its maximum value because it is continuous and because spaces with exhaustible sets of representatives are compact.

Proof. We discuss $\max$ only. By e.g. the algorithm given in [14], this is the case for $X = 2^{\mathbb{N}}$. Because the representing space $\ulcorner X \urcorner$, being a non-empty exhaustible subspace of a Kleene–Kreisel space, is a computable image of the Cantor space, the space X itself is a computable image of the cantor space, say with $q \colon 2^{\mathbb{N}} \to X$. Then the algorithm $\max_X(f) = \max_{2^N}(f \circ q)$ gives the required conclusion. □

Corollary 6.5. *If K is a subspace of a metric space X and K has an exhaustible set of Kleene–Kreisel representatives, then K is* computably located *in X, in the sense that the distance function $d_K \colon X \to \mathbb{R}$ defined by*

$$d_K(x) = \min\{d(x, y) \mid y \in K\}$$

is computable.

Corollary 6.6. *For any metric space X with an exhaustible set of Kleene–Kreisel representatives, the* $\max$*-metric $d(f, g) = \max\{d(f(x), g(x)) \mid x \in X\}$ on $\mathbb{R}^X$ is computable.*

Corollary 6.7. *For $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$, it is semi-decidable whether $f \notin A$.*

Proof. Because A is computationally located in $\mathbb{R}^{[-\epsilon,\epsilon]}$ as it has an exhaustible set of representatives, and because $f \notin A \iff d_A(f) \neq 0$. □ □

Another proof, which doesn't rely on the exhaustibility of a set of representatives of A, uses Theorem 5.1: $f \notin A$ if and only if the equation $T(a) = f$ doesn't have a solution $a \in [-b, b]^{\mathbb{N}}$. But this alternative proof relies on a complete metric on $[-b, b]^{\mathbb{N}}$. For simplicity, we consider a standard construction for 1-bounded metric spaces. Because we apply this to metric spaces with exhaustible sets of representatives, this is no loss of generality as the diameter of such a space is computable as $\max(\lambda x. \max(\lambda y. d(x, y)))$ and hence the metric can be computably rescaled to become 1-bounded.

Lemma 6.8. *For any computational* 1*-bounded metric space* X*, the metric on* $X^{\mathbb{N}}$ *defined by*

$$d(x, y) = \sum_n 2^{-n-1} d(x_n, y_n)$$

is computable and 1*-bounded, and it is computationally complete if* X *is.*

Proof. Use the fact that the map $[0, 1]^{\mathbb{N}} \to [0, 1]$ that sends a sequence $a \in [0, 1]^{\mathbb{N}}$ to the number $\sum_n 2^{-n-1} a_n$ is computable. Regarding completeness, it is well known that a sequence in the space $X^{\mathbb{N}}$ is Cauchy if and only if it is componentwise Cauchy in X, and in this case its limit is calculated componentwise. $\square$

Corollary 6.9. *The Taylor coefficients of any* $f \in A$ *can be computed from* f*.*

Proof. Because $[-b, b]^{\mathbb{N}}$ has an exhaustible set of representatives, the function T has a computable inverse by Theorem 5.1 and Lemma 6.8. $\square$

References

[1] I. Battenfeld, M. Schröder, and A. Simpson. A convenient category of domains. In *Computation, meaning, and logic: articles dedicated to Gordon Plotkin*, volume 172 of *Electron. Notes Theor. Comput. Sci.*, pages 69–99. Elsevier, Amsterdam, 2007.

[2] A. Bauer. A relationship between equilogical spaces and type two effectivity. *MLQ Math. Log. Q.*, 48(suppl. 1):1–15, 2002.

[3] M.J. Beeson. *Foundations of Constructive Mathematics*. Springer, 1985.

[4] E. Bishop and D. Bridges. *Constructive Analysis*. Springer, Berlin, 1985.

[5] M. Escardó. Exhaustible sets in higher-type computation. *Log. Methods Comput. Sci.*, 4(3):3:3, 37, 2008.

[6] M.H. Escardó. Computability of continuous solutions of higher-type equations. In K. Ambos-Spies, B. Löwe, and Merkle W., editors, *CiE'09 Proceedings of the 5th Conference on Computability in Europe: Mathematical Theory and Computational Practice*, volume 5635 of *LNCS*, pages 188–197. Springer, 2009.

[7] M.H. Escardó, J. Lawson, and A. Simpson. Comparing Cartesian closed categories of (core) compactly generated spaces. *Topology Appl.*, 143(1-3):105–145, 2004.

[8] J.R. Longley. On the ubiquity of certain type structures. *Mathematical Structures in Computer Science*, 17:841–953, 2007.

[9] D. Normann. *Recursion on the countable functionals*, volume 811 of *Lec. Not. Math.* Springer, 1980.

[10] D. Normann. The continuous functionals. In *Handbook of computability theory*, volume 140 of *Stud. Logic Found. Math.*, pages 251–275. North-Holland, Amsterdam, 1999.

[11] D. Normann. Computing with functionals—computability theory or computer science? *Bull. Symbolic Logic*, 12(1):43–59, 2006.

[12] D. Normann. Applications of the Kleene–Kreisel density theorem to theoretical computer science. In B. Cooper S.B., Lowë and Sorbi A., editors, *New Computational Paradigms*, pages 119–138. Springer, 2007.

[13] M. Schröder. Extended admissibility. *Theoret. Comput. Sci.*, 284(2):519–538, 2002.

[14] A. Simpson. Lazy functional algorithms for exact real functionals. *Lec. Not. Comput. Sci.*, 1450:323–342, 1998.

[15] K. Weihrauch. *Computability*. Springer, 1987.

[16] K. Weirauch. *Computable analysis*. Springer, 2000.