A Universal Characterisation of the Closed Euclidean Interval

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Abstract

We propose a notion of *interval object* in a category with finite products, providing a universal property for closed and bounded real line segments. We test the notion in categories of interest. In the category of sets, any closed and bounded interval of real numbers is an interval object. In the category of topological spaces, the interval objects are closed and bounded intervals with the Euclidean topology. We also prove that an interval object exists in any elementary topos with natural numbers object. The universal property of an interval object provides a mechanism for defining functions on the interval. We use this to define basic arithmetic operations, and to verify equations between them. It also allows us to develop an analogue of the primitive recursive functions, yielding a natural class of computable functions on the interval.

Contents

1 Introduction

In set theory, one can implement the real numbers in many ways. For example, one can use Dedekind sections or equivalence classes of Cauchy sequences of rational numbers. But *what* is it that one is implementing? Assuming classical logic, either implementation produces a *complete Archimedian field*; moreover, any two such fields are isomorphic [29, 32]. In fact, for the purposes of classical analysis, one never uses a *particular* mathematical implementation of the reals. One relies on the *specification* of the real-number system as a complete Archimedian field and works axiomatically. The only purpose of particular implementations is to be reassured that there is at least one such field.

Unfortunately, when one tries to carry out such a programme in other foundational settings, difficulties arise. One obstacle is that the categoricity of this axiomatization relies on the principle of excluded-middle, which is not always available, particularly in settings that are relevant to the theory of computation. Further, one may also criticise the axiomatization on the grounds that, although it is aiming to characterise the real line, which is fundamentally a geometric structure, it makes essential use of abstract concepts (such as suprema of bounded sets of points) whose geometric meaning is unclear. In addition, even the field axioms arguably involve operations (such as multiplication and reciprocation) which one might rather see as derived from more primitive geometric constructions. Moreover, this axiomatization of the reals does not give rise to any direct mechanism for systematically building functions on the line.

It is our purpose to develop an axiomatization which avoids the problems identified above. Although we do not know how to do this directly for the entire real line, we shall present such an axiomatization for the closely related notion of a *closed and bounded line segment*, out of which the real line can of course be constructed. Our axiomatization is directly motivated by aiming to fulfil the following requirements: (i) it has direct computational content, (ii) it has a geometrical flavour, (iii) it applies to a variety of foundational settings, (iv) it makes as few ontological commitments as possible, and (v) it gives what one would expect in specific examples.

Regarding (iii) and (iv), we only assume a category with finite products.

Regarding (ii), we take a *midpoint* operation as the basic structure of line segments, with four axioms which correspond to intuitive geometric properties. We define an *abstract convex body* as a midpoint algebra in which the midpoint operation can be *infinitely iterated*, in a precise sense discussed in the technical development that follows. Then a closed and bounded line segment, or *interval object* for short, is defined to be a free abstract convex body over two generators, its endpoints. The free property amounts to the fact that any two points of a convex body are connected by a line segment.

Regarding (i), the free property gives a direct mechanism for defining computable functions. In particular, one can define multiplication and prove its basic properties. More generally, we develop an analogue of the primitive recursive functions for the line segment.

Regarding (v), we have: (1) In the category of sets, $[-1, 1]$ is an interval object. (2) In the category of topological spaces, $[-1, 1]$ with the Euclidean topology is an interval object. (3) In any elementary topos with natural numbers object, an interval object is given by the Cauchy completion of the interval of Cauchy reals within the Dedekind reals. For details see Section 8. In many cases this coincides with the Cauchy or Dedekind intervals; but, in general, we seem to be identifying an intriguing new (intuitionistic) notion of real number.

In summary, this paper provides a foundational axiomatization of the line segment, by means of a geometrically motivated universal property that supports the definition of computable functions on the interval. Computationally, one can view the definition as presenting the interval as an abstract data type, interpretable within many computational settings. Moreover, as will be seen, the definition involves a judicious combination of inductive and coinductive properties, making use of techniques from the coalgebra school of computer science.

Related work Higgs [13] defines *magnitude algebras* and proves that the interval $[0, \infty]$ endowed with the function $h : [0, \infty] \to [0, \infty]$ defined by $h(x) = x/2$ and the summation operation $\sum : [0, \infty]^\omega \to [0, \infty]$ is the magnitude algebra freely generated by 1. His axiomatization is purely equational and is based on binary expansions of numbers. It appears that, in the category of topological spaces, the free magnitude algebra over two generators is the interval $[0, \infty]$ with the topology of lower semicontinuity [28] rather than the Euclidean topology. In fact, the summation operation is continuous with respect to the former but not the latter. This observation shows that, in general, the Dedekind or Cauchy $[0, \infty]$ intervals in an elementary topos are not magnitude algebras (let alone free ones), as these objects do not support Higgs' $\sum$ operation (for example, in Johnstone's "topological topos" [16]).

In contrast, Pavlović and Pratt [27] consider *coalgebraic* definitions of the reals. However, they do not make connections with the computational and geometrical requirements discussed above.

Building on that, Peter Freyd [10] considers a more geometrical coalgebraic approach. In fact, he also places emphasis on midpoint algebras, although the midpoint operation is derived rather than primitive. His approach also appears to have some computational content, but this has yet to be elaborated.

Escardó and Streicher [9] consider an axiomatization of the domain of subintervals of the line segment in the category of continuous Scott domains, also using algebraic and coalgebraic methods. Although their axiomatization has strong computational content, it applies to a particular category. Moreover, it does not refer to the line segment directly.

Connections with the work of Brattka [4] remain to be investigated.

2 Abstract convex bodies and interval objects

We assume a category $\mathcal{C}$ with finite products.

Definition 2.1 (Binary algebra) A *binary algebra* is a pair (A, m) where $A \times A \xrightarrow{m} A$ is any morphism. A *homomorphism* from (A, m) to (A', m') is a morphism $A \xrightarrow{f} A'$ such that $f(m(x, y)) = m'(f(x), f(y))$.

In this definition, we follow the general style we shall adopt throughout the paper. Much of the development will involve doing algebra in categories with finite products. Our style is to write morphisms as functions, and state equations using variables. Mathematically, the variables represent generalised elements. Thus, for example, the homomorphism equation: for all generalised elements $x, y : Z \longrightarrow A$ (where Z is any object), $f \circ m \circ \langle x, y\rangle = m' \circ \langle f \circ x,\ f \circ y\rangle$. In this case, the condition simplifies to the (unquantified) equation $f \circ m = m \circ (f \times f)$.

Definition 2.2 (Midpoint algebra) A *midpoint algebra* is a binary algebra (A, m) satisfying:

1. $m(x, x) = x$ (*idempotency*)
2. $m(x, y) = m(y, x)$ (*commutativity*)
3. $m(m(x, y), m(z, w)) = m(m(x, z), m(y, w))$ (*transposition*)

A midpoint algebra is said to be *cancellative* if it satisfies:

4. $m(x, z) = m(y, z)$ implies $x = y$ (*cancellation*)

We write *MidAlg*$(\mathcal{C})$ for the category of midpoints algebras and homomorphisms between them.

We remark that midpoint algebras have previously been studied under the name *medial means*; see [20], and also [1, 11, 19].

Example 2.3 The set $\mathbb{R}^n$ is a cancellative midpoint algebra under the binary midpoint function $\oplus : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\mathbf{x} \oplus \mathbf{y} \quad = \quad (\mathbf{x} + \mathbf{y})/2$$

This yields a whole range of cancellative midpoint algebras given by subsets $A \subseteq \mathbb{R}^n$ closed under $\oplus$. We call such midpoint algebras *standard midpoint subalgebras of* $\mathbb{R}^n$. Examples are: the set of dyadic rational points; the set of rational points; the set of algebraic points; any convex set.

The intuition behind the following definition is captured by Proposition 3.1 below. For the moment, we remark that it roughly says that, for any sequence of points $x_0, x_1, x_2, \ldots$, there is a unique point $m(x_0, m(x_1, m(x_2, \ldots)))$ which arises by "infinite iteration" of the operation m over the sequence.

Definition 2.4 (Right-iterable algebra) A *right-iterable algebra* is a binary algebra (A, m) satisfying the *right-iteration axiom*: for every map $X \xrightarrow{c} A \times X$, there exists a unique $X \xrightarrow{u} A$ such that the diagram below commutes.

$$\begin{array}{ccc} A \times X & \xrightarrow{\mathsf{id} \times u} & A \times A \\ {\scriptstyle c}\uparrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow[u]{} & A \end{array}$$

In other words, (A, m) is a right-iterable algebra if it is final as an $(A \times (-))$-algebra with respect to coalgebra-to-algebra homomorphisms from $(A \times (-))$-coalgebras. There is an evident dual notion of *left-iterable* algebra, defined in terms of $((-) \times A)$ algebras and coalgebras. However, we shall be almost exclusively concerned with commutative algebras, for which the two notions coincide, hence we shall normally refer simply to *iterable algebras* and the *iteration axiom*. Examples of the dual notion, that of a coalgebra being initial with respect to coalgebra-to-algebra homomorphisms to arbitrary algebras, arise in the work of Taylor [33] and Eppendahl [7].

Definition 2.5 (Abstract convex body) An *(abstract) convex body* is a cancellative iterable midpoint algebra.

We write $Conv(\mathcal{C})$ for the full subcategory of $MidAlg(\mathcal{C})$ whose objects are convex bodies.

Although, as above, we shall usually omit the word "abstract" when referring to convex bodies, we shall take care to include the word whenever its omission might lead to confusion. For example, when we discuss subsets of $\mathbb{R}^n$, it will be important to distinguish our abstract convex bodies from the established "convex bodies" (i.e. convex sets with nonempty interior) in the literature on convex sets.

Example 2.6 Continuing from Example 2.3, any bounded convex subset of $\mathbb{R}^n$, considered as a standard midpoint subalgebra of $\mathbb{R}^n$, is an abstract convex body. Indeed, given functions $h : X \to A$ and $t : X \to X$ (where X is any set), then the unique function $u : X \to A$, determined from the coalgebra $\langle h, t \rangle : A \to A \times X$ by the iteration axiom, is:

$$u(x) \quad = \quad \sum_{i \geq 0} 2^{-(i+1)} h(t^i(x)) \tag{1}$$

An important point here is that the boundedness of A is crucial for u to be well-defined. In fact, in Section 3, we prove that a standard midpoint subalgebra of $\mathbb{R}^n$ is an abstract convex body if and only if it is a bounded convex subset of $\mathbb{R}^n$; and we also prove that, given a bounded convex subset B of $\mathbb{R}^m$, a function $f : A \to B$ is a homomorphism of abstract convex bodies (i.e. a homomorphism w.r.t. $\oplus$) if and only if it is affine.

Example 2.7 Let A be any bounded convex subset of $\mathbb{R}^n$, as above, endowed with the Euclidean topology. Then $\oplus$ also exhibits A as an abstract convex body in **Top**. Indeed, given any continuous $\langle h, t \rangle : X \to A \times X$ (where X is any space), the function defined in (1) is again the $u : X \to A$ required by the iteration axiom. The interesting fact here is that u is continuous. This example will be expanded upon in Section 7.

A *bipointed* convex body is a structure (A, m, a, b) where (A, m) is a convex body and $a, b : \mathbf{1} \longrightarrow A$ are global points. *Homomorphisms* between bipointed convex bodies are required to preserve the points as well as the binary algebra structure; i.e. $A \xrightarrow{f} A'$ is a homomorphism from (A, m, a, b) to (A', m', a', b') iff it is a homomorphism from (A, m) to (A', m') and $a' = f \circ a$ and $b' = f \circ b$. We write $BiConv(\mathcal{C})$ for the category whose objects are bipointed convex bodies and whose morphisms are homomorphisms between them.

Definition 2.8 (Interval object) An *interval object* in $\mathcal{C}$ is an initial object in $BiConv(\mathcal{C})$.

Example 2.9 In **Set** any closed proper interval $[a, b] \subseteq \mathbb{R}$ (where $a < b$) gives an interval object $([a, b], \oplus, a, b)$. Of course the choice of a and b makes no difference. For future convenience, we take the interval $\mathbb{I} = [-1, 1]$ as our standard closed interval and $(\mathbb{I}, \oplus, -1, 1)$ as our standard interval object. That $(\mathbb{I}, \oplus, -1, 1)$ really is an interval object in **Set** will be proved in Section 3.

Example 2.10 In **Top**, $(\mathbb{I}, \oplus, -1, 1)$ is again an interval object when $\mathbb{I}$ is equipped with the Euclidean topology. This will be proved in Section 7.

3 Convex bodies and interval objects in Set

In this section we study convex bodies in **Set**, and we show that the interval object in **Set** is indeed $(\mathbb{I}, \oplus, -1, 1)$, as claimed above.

Perhaps, the least familiar aspect of the definition of convex body is the notion of iterable algebra. We begin by showing that, in **Set**, iterable algebras are exactly algebras supporting an additional operation of countably-infinite arity that satisfies certain characterising properties relating it to the binary operation. In general, this reformulation provides the most straightforward method of showing that an algebra is iterable.

Proposition 3.1 *Let (A, m) be a binary algebra in* **Set**.

1. *(A, m) is right-iterable if and only if there exists a function $M : A^\omega \to A$ satisfying:*
   - *(a) $M(x_0, x_1, x_2, \ldots) = m(x_0, M(x_1, x_2, x_3, \ldots))$*
   - *(b) If $y_0 = m(x_0, y_1)$, $y_1 = m(x_1, y_2)$, $y_2 = m(x_2, y_3)$, $\ldots$ then $y_0 = M(x_0, x_1, x_2, \ldots)$.*

   Moreover if (A, m) is right-iterable then there is a unique M satisfying (a).
2. *If (A, m) and (A', m') are right-iterable algebras then any homomorphism $f : A \to A'$ is also a homomorphism with respect to the associated infinitary M and M'; i.e. for all $x_0, x_1, x_2, \ldots$, $f(M(x_0, x_1, x_2, \ldots)) = M'(f(x_0), f(x_1), f(x_2), \ldots)$.*

Below, we shall often use $M_i(x_i)$ as convenient notation for $M(x_0, x_1, x_2, \ldots)$.
Proof For statement 1, first suppose that (A, m) is right iterable. Consider the "head" and "tail" functions $h : A^\omega \to A$ and $t : A^\omega \to A^\omega$. Together these give a coalgebra $\langle h, t\rangle : A^\omega \to A \times A^\omega$ (indeed this is the final $(A \times (-))$-coalgebra). Then property (a) expresses that $M : A^\omega \to A$ is a coalgebra-to-algebra homomorphism from $\langle h, t\rangle$ to m. Thus, by the right-iteration axiom, there is indeed a unique M satisfying (a).

To show that this unique M satisfies (b), let (x_i) and (y_i) be sequences satisfying $y_i = m(x_i, y_{i+1})$. Consider the coalgebra $i \mapsto \langle x_i,\, i{+}1\rangle : \mathbb{N} \to A \times \mathbb{N}$. Then the functions $i \mapsto y_i : \mathbb{N} \to A$ and $i \mapsto M(x_i, x_{i+1}, \ldots) : \mathbb{N} \to A$ are both homomorphisms from this coalgebra to the algebra $m : A \times A \to A$, hence equal. Thus, in particular, $y_0 = M(x_0, x_1, \ldots)$ as required.

Conversely, suppose that there exists $M : A^\omega \to A$ satisfying (a) and (b). Given a coalgebra $\langle h, t\rangle : X \to A \times X$, we must show that there is a unique function $u : X \to A$ such that $u(x) = m(h(x), u(t(x)))$. Such a function is defined by:

$$u(x) \;=\; M(h(x),\, h(t(x)),\, h(t(t(x))),\, \ldots)$$

i.e. $u(x) = M_i(h(t^i(x))$. This satisfies $u(x) = m(h(x), u(t(x))$ by property (a) of M. For uniqueness, suppose there exists $u' : X \to A$ such that $u'(x) = m(h(x), u'(t(x))$. Then $u'(t(x)) = m(h(t(x)), u'(t(t(x))))$, $\ldots$, i.e. $u'(t^i(x)) = m(h(t^i(x)), u'(t^{i+1}(x))$. So, by property (b), $u'(x) = M_i(h(t^i(x))) = u(x)$, as required.

For statement 2, suppose that $f : A \to A'$ is a homomorphism. Then:

$$\begin{aligned} f(M_i(x_i)) &= f(m(x_0,\, M_{i\geq 1}(x_i))) && \text{(by property (a))} \\ &= m'(f(x_0),\, f(M_{i\geq 1}(x_i))) && \text{(as } f \text{ is a homomorphism)} \end{aligned}$$

Similarly:

$$\begin{array}{lcl} f(M_{i\geq 1}(x_i)) & = & m'(f(x_1),\, f(M_{i\geq 2}(x_i))) \\ f(M_{i\geq 2}(x_i)) & = & m'(f(x_2),\, f(M_{i\geq 3}(x_i))) \\ \text{etc.} & & \end{array}$$

So, by property (b) for M', we have $f(M_i(x_i)) = M'_i(f(x_i))$, as required. □
With an appropriate reformulation, this proposition generalises from **Set** to any category with finite products and a parameterized natural numbers objects. See Appendix A for details.

Proposition 3.1 applies to arbitrary binary algebras. However, our main interest is in midpoint algebras. For these, it is useful to identify additional equational properties satisfied by the associated infinitary operations.

Proposition 3.2 *For any iterable midpoint algebra* (A, m) *in* **Set***, with associated infinitary* $M : A^\omega \to A$*:*

1. $x = M(x, x, x, \ldots)$
2. $M_i(M_j(x_{ij})) = M_j(M_i(x_{ji}))$
3. $M_i(m(x_i, y_i)) = m(M_i(x_i), M_i(y_i))$

We omit the routine proofs.

Having obtained a reasonable understanding of iterability, we now return to Examples 2.6 and 2.9, and verify the claims made there. Recall that a subset A of $\mathbb{R}^n$ is convex if and only if it is closed under *convex combinations*, i.e., for every $\mathbf{x}_1, \ldots, \mathbf{x}_k \in A$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$,

$$\sum_{i=1}^{k} \lambda_i \mathbf{x}_i \quad \in \quad A.$$

Proposition B.4 of Appendix B shows that any bounded convex subset A is also closed under *countable convex combinations*, i.e., for every sequence $(\mathbf{x}_i)$ of points in A and sequence (λ_i) of weights in $[0, 1]$ with $\sum_{i=0}^\infty \lambda_i = 1$,

$$\sum_{i=0}^{\infty} \lambda_i \mathbf{x}_i \quad \in \quad A.$$

Proposition 3.3 *Suppose* A *is a standard midpoint subalgebra of* $\mathbb{R}^n$*, then* A *is an abstract convex body if and only if it is a bounded convex subset of* $\mathbb{R}^n$*,*

Proof Suppose $(A, \oplus)$ is an abstract convex body. Let $M : A^\omega \to A$ be the associated infinitary operation. For any two points $\mathbf{x}, \mathbf{y} \in A$ and weights $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$, let $0.d_0 d_1 d_2 \ldots$ be a binary representation of λ. For $i \geq 0$, let $\mathbf{z}_i$ be $\mathbf{x}$ if $d_i = 1$ and $\mathbf{y}$ if $d_i = 0$. Then, by Proposition 3.1.1, $\lambda\mathbf{x} + \mu\mathbf{y} = M(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, , \ldots) \in A$. So A is a convex set.

Suppose, for contradiction, that A is unbounded. By Proposition B.1 of Appendix B, A contains a ray $\{\mathbf{x}_0 + \lambda(\mathbf{x}_0 - \mathbf{x}_1) \mid \lambda \in [0, \infty)\}$, where $\mathbf{x}_0$ and $\mathbf{x}_1$ are two distinct points. Write $\mathbf{x}_\lambda$ for the point $\mathbf{x}_0 + \lambda(\mathbf{x}_1 - \mathbf{x}_0) \in A$ (this is consistent with the notation for $\mathbf{x}_0$ and $\mathbf{x}_1$). Then $\mathbf{x}_1 = \mathbf{x}_0 \oplus \mathbf{x}_2$, $\mathbf{x}_2 = \mathbf{x}_0 \oplus \mathbf{x}_4$, $\mathbf{x}_4 = \mathbf{x}_0 \oplus \mathbf{x}_8$, etc. So, by Proposition 3.1.1b, $\mathbf{x}_1 = M(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_0, \ldots) = \mathbf{x}_0$, a contradiction. Thus A is bounded.

We have shown that every abstract convex body is a bounded convex subset. For the converse, suppose A is a bounded convex subset. By Proposition B.4 of Appendix B, A is closed under countable convex combinations. Thus, in particular, it is closed under the operation $\bigoplus : A^\omega \to A$ defined by:

$$\bigoplus_i ((\mathbf{x}_i)) \quad = \quad \sum_{i \geq 0} 2^{-(i+1)} \mathbf{x}_i. \tag{2}$$

It is straightforward to establish that $\bigoplus$ satsfies (a) and (b) of Proposition 3.1.1. Thus A is indeed an abstract convex body. □

Suppose $A \subseteq \mathbb{R}^n$ and $A' \subseteq \mathbb{R}^m$ are (not necessarily bounded) convex subsets. Recall that a function $f : A \to A'$ is said to be *affine* if it preserves (finite) convex combinations, i.e., for every convex combination,

$$f(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i) \quad = \quad \sum_{i=1}^{k} \lambda_i f(\mathbf{x}_i).$$

Proposition 3.4 *Suppose $A \subseteq \mathbb{R}^n$ and $A' \subseteq \mathbb{R}^m$ are bounded convex subsets. Then an arbitrary function $f : A \to A'$ is affine if and only if it is a homomorphism with respect to $\oplus$.*

Proof It is trivial that every affine function is a homomorphism with respect to $\oplus$. Conversely, suppose that f is a homomorphism. We prove that f preserves countable convex combinations, hence finite ones.

Let (λ_i) be a sequence of elements of $[0, 1]$ with $\sum_{i=0}^{\infty} \lambda_i = 1$. Then (λ_i)-weighted countable convex combinations determine a function $\sigma : A^\omega \to A$:

$$\sigma((\mathbf{x}_i)) \quad = \quad \sum_{i=0}^{\infty} \lambda_i \mathbf{x}_i.$$

However, by expressing each weight λ_i as a sum of dyadic rationals (e.g. using its binary expansion), one can show that $\sigma((\mathbf{x}_i))$ can be expressed as

$$\sigma((\mathbf{x}_i)) \quad = \quad \bigoplus_i \sigma'_i(\mathbf{x}_0, \ldots, \mathbf{x}_{k_i}) \tag{3}$$

where each σ'_i is a k_i-ary convex combination with dyadic weights. Then each σ'_i can be expressed entirely in terms of $\oplus$. As f is a homomorphism, it preserves $\oplus$, and hence each σ'_i. Also, by Proposition 3.1.2, it preserves $\bigoplus$, hence it preserves σ. Thus f preserves arbitrary countable convex combinations. Thus f is indeed affine. □

The reduction of an arbitrary countable normalised weighted sum to (3) in the proof above, can be used to show that every abstract convex body in **Set** supports a formal notion of countable convex combination. It would be interesting to demonstrate that these combinations satisfy the expected equational properties. However, we shall not pursue this direction further in this paper.

Proposition 3.4 states a result in pure geometry. Although we are sure this result must appear somewhere in the literature, we do not know a reference. An example due to Peter Freyd [10], which uses the Axiom of Choice, can be used to show that the boundedness assumption is essential for Proposition 3.4 to hold.

The remaining goal of this section is to prove that $(\mathbb{I}, \oplus, -1, 1)$ is an interval object in **Set**. In order to do this, we need to further analyse the cancellation requirement in the definition of convex body. First, we give a simple example showing that not every iterable midpoint algebra is cancellative.

Example 3.5 Consider the midpoint algebra (A, m) in **Set** defined by:

$$\begin{aligned} A \quad &= \quad \{(x, y) \in \mathbb{I} \times \mathbb{I} \mid x = 1 \text{ or } y = 1\} \\ m((x, y), (x', y')) \quad &= \quad \begin{cases} (1, y \oplus y') & \text{if } x = x' = 1 \\ (x \oplus x', 1) & \text{otherwise} \end{cases} \end{aligned}$$

The iteration axiom is easily verified by defining M and checking properties 1a and 1b of Proposition 3.1. The failure of cancellation is shown by $m((1, x), (z, 1)) = m((1, y), (z, 1))$ whenever $z \neq 1$.

Our motivation for requiring cancellation to hold for convex bodies is that it is equivalent to an important approximation property. To formulate this, we introduce notation for some useful derived operations. For a midpoint algebra (A, m), we write m_n for the $(n+1)$-ary operation defined by: $m_0(x) = x$; and $m_n(x_0, \ldots, x_n) = m(x_0, m_{n-1}(x_1, \ldots, x_n))$ when $n \geq 1$. Thus m_1 is just m itself.

Proposition 3.6 *For an iterable midpoint algebra (A, m) in* **Set***, the following are equivalent.*

1. *(A, m) is cancellative.*
2. *The associated $M : A^\omega \to A$ satisfies the following "approximation" property.*

   If, for all $n \geq 0$, there exist $z_n, w_n \in A$ such that

$$m_n(x_0, \ldots, x_{n-1}, z_n) \quad = \quad m_n(y_0, \ldots, y_{n-1}, w_n)$$

   then $M(x_0, x_1, x_2, \ldots) = M(y_0, y_1, y_2, \ldots)$.

Proof To prove 2 implies 1, assume $M : A^\omega \to A$ satisfies approximation. To show that (A, m) is cancellative, assume $m(x, z) = m(y, z)$. We prove by induction that, for all $n \geq 0$,

$$m_n(\overbrace{x, \ldots, x}^{n \text{ times}}, z) \quad = \quad m_n(\overbrace{y, \ldots, y}^{n \text{ times}}, z) \tag{4}$$

The $n = 0$ and $n = 1$ cases are immediate. For $n \geq 2$, assume the equality holds for lower n. Write $w = m_{n-2}(x, \ldots, x, z)$. By the induction hypothesis, $w = m_{n-2}(y, \ldots, y, z)$ and $m(x, w) = m_{n-1}(x, \ldots, x, z) = m_{n-1}(y, \ldots, y, z) = m(y, w)$. So (using the idempotency and commutativity axioms of Definition 2.2 without mention):

$$\begin{array}{rcll}
m_n(x, \ldots, x, z) & = & m(x, m(x, m_{n-2}(x, \ldots, x, z))) & \\
& = & m(x, m(x, w)) & \\
& = & m(m(x, x), m(x, w)) & \\
& = & m(m(x, x), m(y, w)) & \text{(induction hypothesis)} \\
& = & m(m(x, y), m(x, w)) & \text{(transposition)} \\
& = & m(m(x, y), m(y, w)) & \text{(induction hypothesis)} \\
& = & m(m(y, y), m(x, w)) & \text{(transposition)} \\
& = & m(m(y, y), m(y, w)) & \text{(induction hypothesis)} \\
& = & m(y, m(y, w)) & \\
& = & m(y, m(y, m_{n-2}(y, \ldots, y, z))) & \\
& = & m_n(y, \ldots, y, z). &
\end{array}$$

Thus (4) holds. Hence, by approximation, $M(x, x, x, \ldots) = M(y, y, y, \ldots)$, i.e. $x = y$. This proves cancellation.

For the converse implication, assume (A, m) is cancellative. We first observe that the following "one-sided approximation" property holds: if, for all $n \geq 0$, there exists w_n such that $x = m_n(y_0, \ldots, y_{n-1}, w_n)$ then $x = M(y_0, y_1, y_2, \ldots)$. Indeed, if the premise is satisfied then, by cancellation, $w_0 = m(y_1, w_1)$, $w_1 = m(y_2, w_2)$, etc. Hence $x = M(y_0, y_1, y_2, \ldots)$, by Proposition 3.1.1b.

Next, we prove the following equality. For all (x_i), (y_i) and $n \geq 0$:

$$\begin{array}{l}
M_i(m(x_i, y_i)) \;\; = \\
\quad m(m_{n+1}(x_0, \ldots, x_n, y_n),\; M(y_0, \ldots, y_{n-1}, m(x_{n+1}, y_{n+1}), m(x_{n+2}, y_{n+2}), \ldots)).
\end{array} \tag{5}$$

The proof is by induction on n. When $n = 0$, we have:

$$\begin{array}{rcll}
\text{r.h.s.} & = & m(m(x_0, y_0), M(m(x_1, y_1), m(x_2, y_2), \ldots)) & \\
& = & M(m(x_0, y_0), m(x_1, y_1), m(x_2, y_2), \ldots) & \text{(by Proposition 3.1.1a).}
\end{array}$$

For $n > 0$, we have:

$$\begin{array}{rcl} \text{r.h.s.} & = & m(m(x_0, m_n(x_1, \ldots, x_n, y_n)),\ m(y_0, M(y_1, \ldots, y_{n-1}, m(x_{n+1}, y_{n+1}), m(x_{n+2}, y_{n+2}), \ldots))) \\ & = & m(m(x_0, y_0),\ m(m_n(x_1, \ldots, x_n, y_n)),\ M(y_1, \ldots, y_{n-1}, m(x_{n+1}, y_{n+1}), m(x_{n+2}, y_{n+2}), \ldots)) \\ & = & m(m(x_0, y_0),\ M(m(x_1, y_1), m(x_2, y_2), \ldots)) \qquad \text{(by induction hypothesis)} \\ & = & M(m(x_0, y_0), m(x_1, y_1), m(x_2, y_2), \ldots) \end{array}$$

Finally, to prove approximation, suppose that, for all $n \geq 0$, there exist z_n, w_n such that:

$$m_n(x_0, \ldots, x_{n-1}, z_n) \quad = \quad m_n(y_0, \ldots, y_{n-1}, w_n). \tag{6}$$

We show below that

$$m(M(x_0, x_1, x_2, \ldots), M(z_0, z_1, z_2, \ldots)) \quad = \quad m(M(y_0, y_1, y_2, \ldots), M(z_0, z_1, z_2, \ldots)), \tag{7}$$

from which the desired equation, $M(x_0, x_1, x_2, \ldots) = M(y_0, y_1, y_2, \ldots)$, follows by cancellation.

To verify (7), we have:

$$\begin{array}{rcll} \text{l.h.s.} & = & M_i(m(x_0, z_0), m(x_1, z_1), m(x_2, z_2), \ldots) \quad \text{(by Proposition 3.2.3)} & \\ & = & m(m_{n+1}(x_0, \ldots, x_n, z_n),\ M(z_0, \ldots, z_{n-1}, m(x_{n+1}, z_{n+1}), m(x_{n+2}, z_{n+2}), \ldots)) & \text{(by (5))} \\ & = & m(m_{n+1}(y_0, \ldots, y_n, w_n),\ M(z_0, \ldots, z_{n-1}, m(x_{n+1}, z_{n+1}), m(x_{n+2}, z_{n+2}), \ldots)) & \text{(by (6))} \\ & = & M(m(y_0, z_0), \ldots, m(y_{n-1}, z_{n-1}), m(y_n, w_n), m(x_{n+1}, z_{n+1}), m(x_{n+2}, z_{n+2}), \ldots) & \text{(by (5))} \\ & = & m_n(m(y_0, z_0), \ldots, m(y_{n-1}, z_{n-1}), w'_n), & \end{array}$$

where $w'_n = M(m(y_n, w_n), m(x_{n+1}, z_{n+1}), m(x_{n+2}, z_{n+2}), \ldots)$. But the above is true for any $n \geq 0$. Hence, by one-sided approximation, we can continue

$$\begin{array}{rcl} \text{l.h.s.} & = & M_i(m(y_0, z_0), m(y_1, z_1), m(y_2, z_2), \ldots) \\ & = & m(M(y_0, y_1, y_2, \ldots), M_i(z_0, z_1, z_2, \ldots)) \quad \text{(by Proposition 3.2.3)} \end{array}$$

as required. □

Theorem 1 *$(\mathbb{I}, \oplus, -1, 1)$ is an interval object in* **Set**.

Proof Let (A, m, a, b) be an arbitrary convex body in **Set**. We must show that there is a unique homomorphism of bipointed convex bodies from $(\mathbb{I}, \oplus, -1, 1)$ to (A, m, a, b).

Write $\mathbb{Q}_d$ for the set of dyadic rationals, and $I(\mathbb{Q}_d)$ for $\mathbb{Q}_d \cap \mathbb{I}$. We show that there is a unique homomorphism $f_d : I(\mathbb{Q}_d) \to A$ such that $f(-1) = a$ and $f(1) = b$. In fact, this holds because $(I(\mathbb{Q}_d), \oplus, -1, 1)$ is the initial bipointed midpoint algebra. We omit the straightforward argument that establishes this fact.

We define the unique homomorphism $f : \mathbb{I} \to A$ by defining, for $q_0, q_1, \ldots \in I(\mathbb{Q}_d)$:

$$f(\bigoplus_i (q_i)) \quad = \quad M_i(f_d(q_i)).$$

Clearly, for every $x \in \mathbb{I}$, there exist $q_0, q_1, \ldots \in I(\mathbb{Q}_d)$ such that $\bigoplus_i (q_i) = x$ (indeed, one can restrict to $q_i \in \{-1, 1\}$). Thus, to show f is a well-defined function, we must show that $\bigoplus_i (q_i) = \bigoplus_i (q'_i)$ implies $M_i(f_d(q_i)) = M_i(f_d(q'_i))$. Accordingly, suppose that $\bigoplus_i (q_i) = \bigoplus_i (q'_i) = x$. Then $|\oplus_n (q_0, \ldots, q_{n-1}, 0) - x| \leq 2^{-n}$ and $|\oplus_n (q'_0, \ldots, q'_{n-1}, 0) - x| \leq 2^{-n}$, for any $n \geq 0$. Thus, defining $d_n = \oplus_n (q_0, \ldots, q_{n-1}, 0) - \oplus_n (q'_0, \ldots, q'_{n-1}, 0)$, we have $|d_n| \leq 2^{-(n-1)}$. So $2^{-(n-1)} d_n \in I(\mathbb{Q}_d)$, and:

$$\oplus_n (q_0, \ldots, q_{n-1}, -2^{-(n-1)} d_n) \quad = \quad \oplus_n (q'_0, \ldots, q'_{n-1}, 2^{-(n-1)} d_n),$$

where all values are in $I(\mathbb{Q}_d)$. As f_d is a homomorphism from $\mathbb{Q}_d$ to A,

$$m_n(f_d(q_0), \ldots, f_d(q_{n-1}), f_d(-2^{-(n-1)} d_n)) \quad = \quad m_n(f_d(q'_0), \ldots, f_d(q'_{n-1}), f_d(2^{-(n-1)} d_n)),$$

and this holds for any n. Thus, by the approximation property of A (Proposition 3.6), $M_i(f_d(q_i)) = M_i(f_d(q'_i))$ as required.

Next, we show that f is a (bipointed) homomorphism. Trivially $f(-1) = f_d(-1) = a$ and $f(1) = f_d(1) = b$. For the preservation of the midpoint operation, take any $x = \bigoplus_i(q_i)$ and $y = \bigoplus_i(q'_i)$ in $\mathbb{I}$ (with each $q_i, q'_i \in I(\mathbb{Q}_d)$). Then indeed

$$\left.\begin{array}{rcll} f(x \oplus y) & = & f((\bigoplus_i q_i) \oplus (\bigoplus_i q'_i)) & \\ & = & f(\bigoplus_i (q_i \oplus q'_i)) & \\ & = & M_i(f_d(q_i \oplus q'_i)) & \text{(definition of } f) \\ & = & M_i(m(f_d(q_i),\ f_d(q'_i))) & \text{(homomorphism property of } f_d) \\ & = & m(M_i(f_d(q_i)),\ M_i(f_d(q'_i))) & \text{(Proposition 3.2.3)} \\ & = & m(f(\bigoplus_i (q_i)),\ f(\bigoplus_i (q'_i))) & \text{(definition of } f) \\ & = & m(f(x), f(y)). & \end{array}\right\} \quad (8)$$

It remains to show that f is the unique homomorphism. Suppose then that $g : \mathbb{I} \to A$ is any bipointed homomorphism. Then the restriction of g to $I(\mathbb{Q}_d)$ is also a bipointed homomorphism, so $g(q) = f_d(q)$ for any $q \in I(\mathbb{Q}_d)$ Thus, for $q_0, q_1, \ldots \in I(\mathbb{Q}_d)$,

$$\left.\begin{array}{rcll} g(\bigoplus_{i \geq n}(q_i)) & = & g(q_n \oplus (\bigoplus_{i \geq n+1}(q_i))) & \\ & = & m(g(q_n), g(\bigoplus_{i \geq n+1}(q_i))) & \text{(homomorphism property of } g) \\ & = & m(f_d(q_n), g(\bigoplus_{i \geq n+1}(q_i))), & \end{array}\right\} \quad (9)$$

and this holds for every $n \geq 0$. So:

$$\begin{array}{rcll} g(\bigoplus_{i \geq 0}(q_i)) & = & M_i(f_d(q_i)) & \text{(Proposition 3.1.1b)} \\ & = & f(\bigoplus_{i \geq 0}(q_i)) & \text{(definition of } f) \end{array} \quad (10)$$

Thus indeed $g = f$. □

4 Basic categorical properties

Having thoroughly examined convex bodies and the interval object in the category of sets, for the next two sections we turn our attention to general properties of convex bodies and interval objects arising from their categorical definitions. This general investigation will be useful in Sections 7 and 8, in which we shall study examples in categories other than **Set**.

One benefit of having simple abstract definitions of convex body and interval object is that it is easy to prove that these notions are preserved by various categorical constructions and functors. In this section, we state basic results of this nature. The proofs, which are all routine diagram chases, are omitted.

As in Section 2, let $\mathcal{C}$ be a category with finite products.

Proposition 4.1 *The forgetful functors $Conv(\mathcal{C}) \to \mathcal{C}$ and $BiConv(\mathcal{C}) \to \mathcal{C}$ create limits.*

In particular, if (A, m) and (A', m') are convex bodies then so is

$$(A \times A',\ (A \times A') \times (A \times A') \xrightarrow{\cong} (A \times A) \times (A' \times A') \xrightarrow{m \times m'} A \times A')$$

(and an analogous statement holds for bipointed convex bodies). One simple consequence of this result is that, for any interval object $(I, \oplus, a, b)$, the n-dimensional cube I^n has an induced convex body structure.

As well as being closed under limits, convex bodies are also closed under internal powers.

Proposition 4.2 *If (A, m) is a convex body then $(A^B,\ A^B \times A^B \xrightarrow{\cong} (A \times A)^B \xrightarrow{m^B} A^B)$ is a convex body, for any exponentiable object B.*

Again, the analogous result holds for bipointed convex bodies.

It is also straightforward to establish conditions under which (bipointed) convex bodies are preserved by functors. Suppose $\mathcal{D}$ is a category with finite products, and $F : \mathcal{C} \to \mathcal{D}$ preserves finite products. Then there is a functor $\overline{F} : \mathit{MidAlg}(\mathcal{C}) \to \mathit{MidAlg}(\mathcal{D})$ whose action on objects is:

$$\overline{F}(A, m) \quad = \quad (FA,\ FA \times FA \xrightarrow{\cong} F(A \times A) \xrightarrow{Fm} FA)$$

and whose action on morphisms is inherited from F.

Proposition 4.3 *If F has a left adjoint then:*

1. *$\overline{F}$ cuts down to a functor $\overline{F} : \mathit{Conv}(\mathcal{C}) \to \mathit{Conv}(\mathcal{D})$. Similarly, by extending the action of $\overline{F}$ to bipointed objects, one obtains a functor $\overline{F} : \mathit{BiConv}(\mathcal{C}) \to \mathit{BiConv}(\mathcal{D})$.*

2. *If F also has a right adjoint G then $\overline{G} : \mathit{Conv}(\mathcal{D}) \to \mathit{Conv}(\mathcal{C})$ is right adjoint to $\overline{F} : \mathit{Conv}(\mathcal{C}) \to \mathit{Conv}(D)$, and $\overline{G} : \mathit{BiConv}(\mathcal{D}) \to \mathit{BiConv}(\mathcal{C})$ is right adjoint to $\overline{F} : \mathit{BiConv}(\mathcal{C}) \to \mathit{BiConv}(D)$. Thus, in particular, $F : \mathcal{C} \to \mathcal{D}$ preserves interval objects.*

It follows from 1 above that if $\mathcal{C}$ is a full reflective subcategory of $\mathcal{D}$ and if $\mathcal{D}$ has an interval object $(I, \oplus, -1, 1)$ where I is an object of $\mathcal{C}$ then $(I, \oplus, -1, 1)$ is also an interval object in $\mathcal{C}$.

A special case of statement 2 is that interval objects are preserved by the inverse image functors of essential geometric morphisms between elementary toposes. Thus if $f : \mathcal{E} \to \mathcal{E}'$ is an essential geometric morphism and $\mathcal{E}'$ has an interval object then its image under f^* gives an interval object in $\mathcal{E}$. In particular, by Theorem 1, every presheaf topos $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has an interval object obtained as $\Delta(\mathbb{I})$ — recall that the constant presheaf functor, $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, is the inverse image functor of an essential geometric morphism [24]. More generally, in Section 8, we shall show that any elementary topos with natural numbers object has an interval object.

5 Parameterized interval objects

In this section we give some preliminary results on the power of the notion of interval object with respect to defining functions on the interval and to proving equalities between definable functions.

It is well known that Lawvere's elegant definition of a natural numbers object, which works very well in cartesian-closed categories, is not powerful enough in categories with weaker structure. Instead, a modified "parameterized" definition is needed [21, 5]. In a category with finite products, the notion of parameterized natural numbers object supports the definition of functions by primitive recursion. Moreover, in a cartesian-closed category, any ordinary (Lawvere) natural numbers objects is automatically parameterized.

Much the same situation arises for interval objects. In this section we derive some basic arithmetic operations for interval objects. To do this in an arbitrary category with finite products, we need a stronger "parameterized" notion of interval object. For cartesian-closed categories the straightforward notion of interval object will automatically be parameterized.

Definition 5.1 (Parameterized interval object) A bipointed convex body $(I, \oplus, -1, 1)$ is a *parameterized interval object* if, for any convex body (A, m) and morphisms $X \xrightarrow{f} A$ and $X \xrightarrow{g} A$ in $\mathcal{C}$, there exists a unique morphism $X \times I \xrightarrow{(\!|f,g|\!)} A$ satisfying:

$$\begin{aligned} m((\!|f,g|\!)(x,y),\ (\!|f,g|\!)(x,z)) &= (\!|f,g|\!)(x, y \oplus z) \\ (\!|f,g|\!)(x,-1) &= f(x) \\ (\!|f,g|\!)(x,1) &= g(x) \end{aligned}$$

i.e. there is a unique "right-homomorphism" (of bipointed convex bodies) from $X \times I$ to A.

By instantiating X to $\mathbf{1}$ it is easily seen that any parameterized interval object is indeed an interval object. The converse holds when $\mathcal{C}$ is cartesian closed:

Proposition 5.2 *If $\mathcal{C}$ is cartesian closed then any interval object is parameterized.*

The proof is routine.

Henceforth in this section, let $\mathcal{C}$ be a category with finite products and parameterized interval object $(I, \oplus, -1, 1)$. The basic arithmetic operations on I can be defined by:

$$\begin{aligned} \mathbf{1} \xrightarrow{0} I &= (-1) \oplus (1) \\ I \xrightarrow{-} I &= (\![1, -1]\!) \\ I \times I \xrightarrow{\times} I &= (\![-, \mathsf{id}_I]\!) \end{aligned}$$

Importantly, the universal property of I (stated in Definition 5.1) suffices to establish the basic equations between the above operations.

Proposition 5.3 $- - x = x$,
$x \times y = y \times x$,
$x \times (y \times z) = (x \times y) \times z$,
$-0 = 0$,
$x \oplus -x = 0$,
$-(x \oplus y) = (-x) \oplus (-y)$,
$x \times (y \oplus z) = (x \times y) \oplus (x \times z)$,
$x \times 0 = 0$,
$x \times -y = -(x \times y)$.

Proof To show $--x = x$, one easily establishes that $-- : I \longrightarrow I$ is a homomorphism mapping 1 to 1 and -1 to -1, and hence the identity.

Of the other equations, we just show the commutativity of multiplication, which is the most interesting. We have defined multiplication as the unique map $(x, y) \mapsto x \times y$ from $I \times I$ to I satisfying

$$\begin{aligned} x \times -1 &= -x \\ x \times 1 &= x \\ x \times (y' \oplus y'') &= (x \times y') \oplus (x \times y''). \end{aligned}$$

Thus, for commutativity, it suffices to show that the same equations are satisfied when the arguments are swapped, i.e. that

$$\begin{aligned} -1 \times y &= -y \\ 1 \times y &= y \\ (x' \oplus x'') \times y &= (x' \times y) \oplus (x'' \times y). \end{aligned}$$

For the first equation, $-1 \times y = -y$, we have that $-1 \times -1 = - - 1$ and $-1 \times 1 = -1$ and $-1 \times (y' \oplus y'') = (-1 \times y) \oplus (-1 \times y)$, all by the definition of multiplication. Thus the map $y \mapsto -1 \times y : I \longrightarrow I$ is a homomorphism satisfying the defining properties of $- : I \longrightarrow I$. Thus indeed $-1 \times y = -y$.

The second equation $1 \times y = y$ is established similarly.

To show the third equation (which says that multiplication is a left-homomorphism), we show that the maps $(x', x'', y) \mapsto (x' \oplus x'') \times y$ and $(x', x'', y) \mapsto (x' \times y) \oplus (x'' \times y)$ from $I \times I \times I$ to I are both right-homomorphisms agreeing on -1 and 1, and thus equal by the parameterized initiality of I. The map $(x', x'', y) \mapsto (x' \oplus x'') \times y$ is trivially a right-homomorphism, because multiplication is. For $(x', x'', y) \mapsto (x' \times y) \oplus (x'' \times y)$, we have:

$$\begin{aligned} & (x' \times (y' \oplus y'')) \oplus (x'' \times (y' \oplus y'')) \\ = \quad & ((x' \times y') \oplus (x' \times y'')) \oplus ((x'' \times y') \oplus (x'' \times y'')) \quad \text{(because } \times \text{ is a right-homomorphism)} \\ = \quad & ((x' \times y') \oplus (x'' \times y')) \oplus ((x' \times y'') \oplus (x'' \times y'')) \quad \text{(by transposition).} \end{aligned}$$

For agreement on -1, we have:

$$\begin{array}{rcll} (x' \times -1) \oplus (x'' \times -1) & = & -x' \oplus -x'' & \text{(definition of } \times) \\ & = & -(x' \oplus x'') & \text{(definition of } -) \\ & = & (x' \oplus x'') \times -1 & \text{(definition of } \times) \end{array}$$

And agreement on 1 is similar. □

6 Primitive interval functions

As mentioned above, any parameterized natural numbers objects support definition by primitive recursion. By analogy, it is natural to investigate what kind of definitional mechanisms are supported by parameterized interval objects.

In fact, a parameterized interval object supports two complementary styles of definition. Not only does the universal property of parameterized initiality give one useful mechanism for defining functions (used above to define negation and multiplication), but also the couniversal property of the iteration axiom offers yet another means of definition. Parameterized interval objects support any combination of these two styles. We investigate the power of such combinations for the purpose of defining functions on $\mathbb{I}$ in **Set**.

Definition 6.1 (Primitive interval functions) The *primitive interval functions* on $\mathbb{I}$ are the functions in the smallest family $\{\mathcal{F}_n \subseteq \mathbb{I}^n \to \mathbb{I}\}_{n \geq 0}$ satisfying:

(i) $-1, 1 \in \mathcal{F}_0$.

(ii) If $f \in \mathcal{F}_m$ and $g_1, \ldots, g_m \in \mathcal{F}_n$ then $f \circ \langle g_1, \ldots, g_m \rangle \in \mathcal{F}_n$.

(iii) If $f, g \in \mathcal{F}_n$ then the function h defined below is in $\mathcal{F}_{n+1}$:

$$h(\mathbf{x}, y) \quad = \quad \frac{1}{2}(1-y)f(\mathbf{x}) + \frac{1}{2}(1+y)g(\mathbf{x})$$

(iv) If $f_1, \ldots, f_n, g \in \mathcal{F}_n$ then the unique function h satisfying the equation below is in $\mathcal{F}_n$:

$$h(\mathbf{x}) \quad = \quad \frac{1}{2}g(\mathbf{x}) + \frac{1}{2}h(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

Here (iii) corresponds to the parameterized initiality of $\mathbb{I}$ with respect to $\mathbb{I}^n$ as the object of parameters, and (iv) corresponds to the iteration axiom, as induced by the coalgebra $\langle g, f_1, \ldots, f_n \rangle : \mathbb{I}^n \to \mathbb{I} \times \mathbb{I}^n$. Note that property (ii) means that tuples of primitive interval functions between finite powers of $\mathbb{I}$ form a category. This category has finite products because the projections are definable, using (iii).

The function defined by (iv) is given explicitly by

$$h(\mathbf{x}) \quad = \quad \sum_{i \geq 0} 2^{-(i+1)} g(\langle f_1, \ldots, f_n \rangle^i(\mathbf{x})).$$

A natural generalisation is to replace $(g \circ \langle f_1, \ldots, f_n \rangle^i)$ with an arbitrary sequence of (already defined) n-ary functions.

Definition 6.2 (Countably-primitive interval functions) The *countably-primitive interval functions* on $\mathbb{I}$ are the functions in the smallest family $\{\mathcal{F}_n \subseteq \mathbb{I}^n \to \mathbb{I}\}_{n \geq 0}$ satisfying (i)–(iii) of Definition 6.1 and also

(iv)′ Given $f_0, f_1, \ldots \in \mathcal{F}_n$, the function h defined below is in $\mathcal{F}_n$:

$$h(\mathbf{x}) \quad = \quad \sum_{i \geq 0} 2^{-(i+1)} f_i(\mathbf{x})$$

Clearly every primitive interval function is a countably-primitive interval function. The converse does not hold as, there are obviously continuum many countably-primitive interval functions, but only countably many primitive interval functions. Indeed, easily, every element of $\mathbb{I}$ gives a countably primitive interval function of arity 0 (i.e. a constant). Although this cannot hold for the primitive interval functions, we do at least have the following.

Proposition 6.3 *Every rational in $\mathbb{I}$ gives a primitive interval constant.*

Proof Let $q \in \mathbb{I}$ be rational. Then q has an eventually cyclic binary expansion $q = \sum_{i \geq 0} 2^{-(i+1)} d_i$ with each $d_i \in \{-1, 1\}$. Here, eventually cyclic means there exist $n \geq 0, m \geq 1$ such that $d_{n+i} = d_{n+jm+i}$, for all $0 \leq i < m$ and $j \geq 0$. Let $f_0, \ldots, f_{n+m-1}, g$ be the following $(n+m)$-ary primitive interval functions (all projections):

$$\begin{aligned} f_i(x_0, \ldots, x_{n+m-1}) &= \begin{cases} x_{i+1} & \text{if } i < n+m-1 \\ x_n & \text{if } i = n+m-1 \end{cases} \\ g(x_0, \ldots, x_{n+m-1}) &= x_0 \end{aligned}$$

Then, defining the $(n+m)$-ary primitive interval function $h(\mathbf{x})$ as in (iv), one obtains that $q = h(d_0, \ldots, d_{n+m-1})$, which is indeed a primitive interval constant. □

As earlier, we have $\oplus, -, \times$ as primitive interval functions. Thus every n-variable $\oplus$-polynomial (i.e. polynomial where $\oplus$ replaces the usual $+$) with rational coefficients is an n-ary primitive interval function.

We are not sure how much further definability can be pushed with the primitive interval functions, as we now show that even the countably-primitive interval functions are very limited.

Proposition 6.4 *Suppose that f is an n-ary countably-primitive interval function and that $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}$ $\mathbb{I}$ are such that: $f(x_0, \ldots, x_{n-1}) \in \{-1, 1\}$, and if $x_i \in \{-1, 1\}$ then $y_i = x_i$. Then $f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1})$.*

Proof A straightforward induction over the defining properties of the countably-primitive interval functions. □

Thus if f is a unary countably-primitive interval function and $f(x) \in \{-1, 1\}$ for some x in the interior $(-1, 1)$ then f is a constant function. Clearly then, the following "truncated double" function is not a countably-primitive interval function.

$$d(x) = \begin{cases} 1 & \text{if } 1/2 \leq x \\ 2x & \text{if } -1/2 \leq x \leq 1/2 \\ -1 & \text{if } x \leq -1/2 \end{cases}$$

Accordingly, define the *d-primitive interval functions* to be the smallest class of functions containing d and closed under (i)–(iv). Define the *countably-d-primitive interval functions* analogously. The reason for selecting d amongst the non-countably-primitive interval functions is:

Proposition 6.5 *The n-ary countably-d-primitive interval functions are exactly the continuous functions $\mathbb{I}^n \to \mathbb{I}$.*

Proof To show that all countably-d-primitive interval functions are continuous is a straightforward induction over the conditions defining the functions. (The result will also follow from Theorem 2, in Section 7.)

For the converse, we first show that, for any n-variable polynomial, $p(x_0, \ldots, x_{n-1})$ over $\mathbb{R}$, the restriction to $\mathbb{I}^n$ of the truncation:

$$I(p)(x_0, \ldots, x_{n-1}) = \begin{cases} 1 & \text{if } p(x_0, \ldots, x_{n-1}) \geq 1 \\ p(x_0, \ldots, x_{n-1}) & \text{if } p(x_0, \ldots, x_{n-1}) \in \mathbb{I} \\ -1 & \text{if } p(x_0, \ldots, x_{n-1}) \leq -1 \end{cases}$$

is an n-ary countably-d-primitive function. We have $p(\mathbf{x}) = \sum_{i=0}^{k} a_i x_0^{m_{i0}} \ldots x_{n-1}^{m_{i(n-1)}}$. Let λ be the maximum value of $\sum_{i=0}^{k} |a_i x_0^{m_{i0}} \ldots x_{n-1}^{m_{i(n-1)}}|$ over $\mathbf{x} \in \mathbb{I}^n$. We define $(1/\lambda) p(x_0, \ldots, x_{n-1})$ (which is a function $\mathbb{I}^n \to \mathbb{I}$) as a countably-$d$-primitive function. First, by the definition of λ, for each monomial, $(a_i/\lambda)x_0^{m_{i0}} \dots x_{n-1}^{m_{i(n-1)}}$ is a function $\mathbb{I}^n \to \mathbb{I}$. Thus $|a_i/\lambda| \leq 1$, and hence a countably-d-primitive constant, thus $(a_i/\lambda)x_0^{m_{i0}} \dots x_{n-1}^{m_{i(n-1)}}$ is countably-d-primitive. Moreover, truncated addition is d-primitive by $I(x+y) = d(x \oplus y)$. Hence,

$$(1/\lambda)p(x_0, \dots, x_{n-1}) \quad = \quad \sum_{i=0}^{k} (a_i/\lambda)x_0^{m_{i0}} \dots x_{n-1}^{m_{i(n-1)}}$$

is countably-d-primitive, as the value of each finite subsum $\sum_{i=0}^{k'}(a_i/\lambda)x_0^{m_{i0}} \dots x_{n-1}^{m_{i(n-1)}}$, for $k' \leq k$, is in $\mathbb{I}$. Finally, to define $I(p)(\mathbf{x})$, find $l \in \mathbb{N}$ such that $2^l \geq \lambda$. Then

$$I(p)(x_0, \dots, x_{n-1}) \quad = \quad d^l((\lambda/2^l)(1/\lambda)p(x_0, \dots, x_{n-1})),$$

which is countably d-primitive.

Now let $f : \mathbb{I}^n \to \mathbb{I}$ be any continuous function. By the Stone-Weierstrass approximation theorem [30], it is possible to find a sequence (p_i) of n-variable polynomials over $\mathbb{R}$ such that, for all $\mathbf{x} \in \mathbb{I}^n$, $|f(\mathbf{x}) - p_i(\mathbf{x})| \leq 2^{-(i+2)}$. Then also $|f(\mathbf{x}) - I(p_i)(\mathbf{x})| \leq 2^{-(i+2)}$, hence $|I(p_{i+1})(\mathbf{x}) - I(p_i)(\mathbf{x})| \leq 2^{-(i+1)}$. Define $g_0(\mathbf{x}) = I(p_0)(\mathbf{x})$ and $g_{i+1}(\mathbf{x}) = 2^{i+1}(I(p_{i+1})(\mathbf{x}) - I(p_i)(\mathbf{x}))$. As argued above, each $I(p_{i+1})$ is countably-d-primitive, hence, using addition (for subtraction) and d, so is each g_i. Thus $f(\mathbf{x}) = \sum_{i \geq 0} 2^{-(i+1)} g_i(\mathbf{x})$ is also countably-d-primitive, as required. □

Thus including d as a basic function enormously increases definability. It is our hope that this increase in definability also means that the d-primitive interval functions form a useful class, somewhat analogous to the primitive recursive functions on $\mathbb{N}$. Although we have yet to undertake any systematic investigation of this class, we do have one important result. Recall the standard notion of an n-ary *computable function* on $\mathbb{I}$ [34].

Proposition 6.6 *Every n-ary d-primitive interval function is an n-ary computable function on $\mathbb{I}$.*

It is possible to prove this result by showing directly that the computable functions on $\mathbb{I}$ are closed under the definitional mechanisms of the d-primitive interval functions. We give another proof at the end of Section 8.

7 Interval objects in Top

In this section we return to the claims made earlier in Examples 2.7 and 2.10, investigating convex bodies and interval objects in the category **Top** of topological spaces. Proposition 3.1 generalises to **Top** with the requirement that $M : A^\omega \to A$ be continuous with respect to the product topology. (This generalisation follows directly from Proposition A.2 in Appendix A.)

Proposition 7.1 *For any bounded convex subset $A \subseteq \mathbb{R}^n$ endowed with the Euclidean topology, $(A, \oplus)$ is an abstract convex body in* **Top**.

Proof Define $\bigoplus : A^\omega \to A$ as in (2) in the proof of Proposition 3.3. If $\bigoplus : A^\omega \to A$ is continuous (w.r.t. the product topology on A^ω) then it satisfies (a) and (b) of Proposition 3.1.1 in **Top**, because it does in **Set**. Hence $(A, \oplus)$ is an abstract convex body if $\bigoplus$ is continuous.

As A is bounded, let $r > 0$ be such that, for all $\mathbf{x} \in A$, $|\mathbf{x}| < r$. To show continuity, consider any open ϵ-ball, $B_\epsilon(\mathbf{z}) \cap A$, centred at $\mathbf{z} = \bigoplus_i(\mathbf{x}_i)$ where $(\mathbf{x}_i) \in A^\omega$. Take any n such that $2^{-n} < \epsilon/r$. Then, for all $(\mathbf{y}_i)$ in the neighbourhood $\{(\mathbf{y}_i) \in A^\omega \mid \mathbf{y}_j \in B_{\epsilon/2}(\mathbf{x}_j) \cap A$, for all $0 \leq j \leq n\}$ of $(\mathbf{x}_i)$ in A^ω, we have $\bigoplus_i \mathbf{y_i} \in B_\epsilon(\mathbf{z}) \cap A$. Thus $\bigoplus$ is indeed continuous. □

In certain cases, e.g. when $n = 1$ or when A is a Euclidean-open subset of $\mathbb{R}^n$, one can show that the Euclidean topology on A is the finest such that $(A, \oplus)$ is an abstract convex body. However, this is not the case for an arbitrary A.

Certain other basic information about convex bodies in **Top** can be inferred using Proposition 4.3. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has both a left adjoint Δ (giving the discrete toplogy) and a right adjoint ∇ (giving the indiscrete topology). Thus, both U and ∇ preserve convex bodies. As U does, we see that, by Proposition 3.3, under any topology whatsoever, for a standard midpoint subalgebra A of $\mathbb{R}^n$ to be an abstract convex body in **Top**, A must be a bounded convex set. Also, for any bounded convex set, $(A, \oplus)$ with the indiscrete topology is a convex body in **Top**. (N.b. it is *not*, in general, an abstract convex body when given the discrete topology, as then $\bigoplus$ is not continuous.)

Also, by Proposition 3.3, if an interval object exists in **Top** then U preserves it. In fact, we have already claimed in Example 2.10 that $(\mathbb{I}, \oplus, -1, 1)$ is an interval object in **Top** when given the Euclidean topology. In fact, as **Top** is not cartesian closed, it is appropriate to show that this is a parameterized interval object.

Theorem 2 $(\mathbb{I}, \oplus, -1, 1)$ *with the Euclidean topology is a parameterized interval object in* **Top***.*

Proof Let (A, m) be any convex body in **Top** and $f, g : X \to A$ be continuous functions from a space X. We must show that there is a unique right-homomorphim $h : X \times \mathbb{I} \to A$ in **Top**. By Theorem 1 and Proposition 5.2, thre is a unique such right-homomorphism h in **Set**. It suffices to show that this function h is continuous.

For each $z \in X$, write $d_z : I(\mathbb{Q}_d) \to A$, for the unique bipointed homomorphism from $I(\mathbb{Q}_d)$ to A, as in the proof of Theorem 1. Then, by the proof of Theorem 1, we have

$$h(z, \bigoplus_i (q_i)) \quad = \quad M_i(d_z(q_i)).$$

To see that this is continuous, consider the function $h' : X \times \{-1, 1\}^\omega \to A^\omega$ defined by

$$h'(z, (q_i)_i) \quad = \quad (d_z(q_i))_i.$$

It is easily verified that this is continuous with respect to the product topologies on its domain and codomain. Also, consider the function $\bigoplus : \{-1, 1\}^\omega \to \mathbb{I}$, the restriction of $\bigoplus : \mathbb{I}^\omega \to \mathbb{I}$. This is a topological quotient, as it is a surjective continuous function between compact Hausdorff spaces. Moreover, by the condition for quotients to be preserved under product from [6], for any space X, the function $\mathsf{id} \times \bigoplus : X \times \{-1, 1\}^\omega \to X \times \mathbb{I}$, is also a quotient.

Now, we have the following commuting diagram in **Set** (where M is the infinitary operation associated with (A, m)).

$$\begin{array}{ccc} X \times \{-1,1\}^\omega & \xrightarrow{h'} & A^\omega \\ {\scriptstyle \mathsf{id} \times \bigoplus} \big\downarrow & & \big\downarrow {\scriptstyle M} \\ X \times \mathbb{I} & \xrightarrow{h} & A \end{array}$$

We already know that all maps other than h are continuous. But then h is too because $\mathsf{id} \times \bigoplus$ is a quotient. □

By Proposition 4.3.1, $(\mathbb{I}, \oplus, -1, 1)$ with the Euclidean topology is a parameterized interval object in any full reflective subcategory of **Top** that contains the closed Euclidean interval. Thus, for example, it is a parameterized interval object in the category **KHaus** of compact Hausdorff spaces.

8 Interval objects in an elementary topos

Let $\mathcal{E}$ be an elementary topos with nno $\mathbf{N}$. It is a well-known fact that different constructions of the real numbers, which are equivalent to each other using classical logic, give different notions of real number when interpreted within the intuitionistic internal logic of a topos. Among the many alternatives, two are considered as being the most natural, the *Dedekind* reals and the *Cauchy* (or

Cantor) reals. Both are defined using the object of rationals $\mathbf{Q}$ and its associated ordering. We give brief reviews of the definitions. For more details see [15].

One way of defining the Dedekind reals in $\mathcal{E}$ is as an object of pairs of sets of rationals, where: each pair, (L, U), consists of two disjoint inhabited sets; L is a down-closed set, each element of which has a strictly greater element in L; U is an upper-closed set, each element of which has a strictly lower element in U; and the pair satisfies the *locatedness* property that $x < y$ in $\mathbf{Q}$ implies either $x \in L$ or $y \in U$. We write $\mathbf{R}_D$ for the object of Dedekind reals, and use standard notation for the usual arithmetic operations on it. We also identify $\mathbf{Q}$ explicitly as a subobject of $\mathbf{R}_D$ (via the embedding $q \mapsto (\{r \mid r < q\}, \{s \mid s > q\})$).

One direct way of defining the Cauchy reals is as a quotient of the object of all Cauchy sequences of rationals (where the notion of Cauchy sequence must be phrased in an appropriate constructive way — see below). For our purposes, it is more convenient to adopt an alternative equivalent definition, identifying the Cauchy reals as particular Dedekind reals. First, we recall the "correct" intuitionistic definition of a Cauchy sequence of Dedekind reals, which requires an explicit modulus of convergence. A sequence $\alpha_{(-)} \in \mathbf{R}_D{}^{\mathbf{N}}$ is *Cauchy* if:

$$\exists m \in \mathbf{N}^{\mathbf{Q}}.\ \forall \epsilon \in \mathbf{Q}.\ \forall i, j \geq m(\epsilon),\ |\alpha_i - \alpha_j| \leq \epsilon$$

We say that $x \in \mathbf{R}_D$ is the *limit* of the Cauchy sequence α if

$$\exists m \in \mathbf{N}^{\mathbf{Q}}.\ \forall \epsilon \in \mathbf{Q}.\ \forall i, j \geq m(\epsilon),\ |\alpha_i - x| \leq \epsilon$$

(Actually, here it is not necessary to require the modulus function to exist, as the modulus of α can always be used.) The Cauchy reals are defined explicitly by:

$$\mathbf{R}_C \quad = \quad \{x \in \mathbf{R}_D \mid \exists \alpha \in \mathbf{Q}^{\mathbf{N}} \text{ s.t. } \alpha \text{ is a Cauchy sequence and } x = \ \mathit{lim}\ \alpha\}$$

The reason for defining Cauchy sequences of Dedekind reals (rather than rationals) as basic, and for considering rationals and Cauchy reals as special Dedekind reals, is that it is now easy to consider the question of whether the various objects are Cauchy complete. We say a subobject $X \subseteq \mathbf{R}_D$ is *Cauchy complete* if every Cauchy sequence in $X^{\mathbf{N}}$ has a (necessarily unique) limit. It is not hard to show that $\mathbf{R}_D$ is Cauchy complete. Obviously $\mathbf{Q}$ is not Cauchy complete. $\mathbf{R}_C$ partially rectifies the non-completeness of $\mathbf{Q}$ by adding all limits of Cauchy sequences of rationals. Given the following choice principle[1]

$$(\forall n \in \mathbf{N}.\, \exists m \leq n.\ \phi(n, m))\ \mathit{implies}\ (\exists f \in \mathbf{N}^{\mathbf{N}}.\, \forall n \in \mathbf{N}.\ f(n) \leq n\ \mathit{and}\ \phi(n, f(n))), \qquad (\mathrm{AC}_{\mathbf{N} \geq \mathbf{N}})$$

which is classically provable, it holds that $\mathbf{R}_C$ itself Cauchy complete. However, it seems that, in general, $\mathbf{R}_C$ is *not* Cauchy complete, as, given a Cauchy sequence $\alpha \in \mathbf{R}_C{}^{\mathbf{N}}$, there is no available mechanism for selecting representative rational sequences from which the required limiting sequence of rationals can be extracted.

The possible failure of Cauchy completeness for $\mathbf{R}_C$ makes it natural to introduce another object of reals, namely the "Cauchy completion" of $\mathbf{Q}$ (within $\mathbf{R}_D$). This object, which we call the object of *Euclidean* reals, is defined by:

$$\mathbf{R}_E \quad = \quad \bigcap \{X \subseteq \mathbf{R}_D \mid \mathbf{Q} \subseteq X \text{ and } X \text{ is Cauchy complete}\}$$

Clearly $\mathbf{R}_E$ is Cauchy complete. In contrast to the Cauchy reals, we do not know a direct method of constructing the Euclidean reals without first going via the Dedekind reals.

So far, we have identified three objects of reals

$$\mathbf{R}_C \quad \subseteq \quad \mathbf{R}_E \quad \subseteq \quad \mathbf{R}_D$$

In the case that $\mathcal{E}$ satisfies $\mathrm{AC}_{\mathbf{N} \geq \mathbf{N}}$, both inclusions are equalities. The Grothendieck topos $\mathit{Sh}(\mathbb{R})$ (sheaves over Euclidean space $\mathbb{R}$) is a simple example in which the second inclusion is strict. To

[1] This principle of *bounded* $\mathbf{N}$-choice appears weaker than arbitrary $\mathbf{N}$-$\mathbf{N}$-choice (in which the inequalities are omitted). However, we do not know a topos in which the former holds but not the latter.

our embarrassment, we do not know an example topos in which the first inclusion is strict. (Thus we do not know if the envisaged failure of the Cauchy completeness of $\mathbf{R}_C$ is actually possible — although we are sure that it must be.)

Each notion of real number object determines a corresponding notion of interval object; for example:

$$\begin{array}{lclcl} \mathbf{I}_D & = & \{x \in \mathbf{R}_D \mid -1 \leq x \leq 1\} & & \\ \mathbf{I}_E & = & \{x \in \mathbf{R}_E \mid -1 \leq x \leq 1\} & = & \mathbf{R}_E \cap \mathbf{I}_D \end{array}$$

Our motivation for introducing the Euclidean reals is that it is the Euclidean interval that gives an interval object in $\mathcal{E}$. The remainder of this section is devoted to the proof of this fact. Note, however, that it is not obvious that $\mathbf{I}_E$ is closed under $\oplus$ in $\mathbf{I}_D$. Thus it is not immediately clear that $\mathbf{I}_E$ is even a midpoint algebra. In fact, we shall have to prove that $\mathbf{I}_E$ is closed under a number of important operations on $\mathbf{I}_D$, including $\oplus$. We next consider the various operations we require.

Observe that there is a unique function $I : \mathbf{R}_D \to \mathbf{I}_D$ satisfying:

$$I(x) \quad = \quad \begin{cases} 1 & \text{if } 1 \leq x \\ x & \text{if } -1 \leq x \leq 1 \\ -1 & \text{if } x \leq -1 \end{cases}$$

Indeed such a function is defined explicitly by $I(x) = min(1, max(x, -1))$. Its uniqueness is a routine consequence of $\mathbf{R}_D$ being $\neg\neg$-separated.

Definition 8.1 A subobject X of $\mathbf{I}_D$ is said to be a *symmetric sub(midpoint-)algebra* of $\mathbf{I}_D$ if it is closed under $\oplus : \mathbf{I}_D \times \mathbf{I}_D \to \mathbf{I}_D$ and $- : \mathbf{I}_D \to \mathbf{I}_D$.

Proposition 8.2 *For any symmetric subalgebra $X \subseteq \mathbf{I}_D$, the following are equivalent.*

1. *For all $x \in X$, $I(2x) \in X$.*
2. *For all $x, y \in X$, $I(x + y) \in X$.*
3. *For all $x, y \in X$, $I(x - y) \in X$.*

Definition 8.3 We say that a symmetric subalgebra X of $\mathbf{I}_D$ is *magnifiable* if any of the equivalent conditions of Proposition 8.2 hold.

Theorem 3

1. *$\mathbf{I}_E$ is a magnifiable symmetric subalgebra of $\mathbf{I}_D$.*
2. *$(\mathbf{I}_E, \oplus, -1, 1)$ is an interval object in $\mathcal{E}$.*

For the proof, we shall need an analysis of convex bodies in $\mathcal{E}$, identical to that carried out in Section 3. For this, it suffices to observe that Propositions 3.1, 3.2 and 3.6 all hold (internally) in $\mathcal{E}$. Indeed, the proofs we have given go through, almost as written, in the internal logic of $\mathcal{E}$. (It is only necessary to be slightly careful with the ellipses, and note that the z_n and w_n in the statement of Proposition 3.6 must be given as internal sequences.)

Crucial to the proof of Theorem 3 is an alternative description of the Euclidean interval, which is better adapted to establishing the universal property of an interval object. To motivate its definition, we first observe that $(\mathbf{I}_D, \oplus)$ is a convex body. This follows from Proposition 3.1 (in $\mathcal{E}$) on account of the function $\bigoplus : \mathbf{I}_D{}^{\mathbf{N}} \to \mathbf{I}_D$ defined by:

$$\bigoplus_i x_i \quad = \quad \sum_{i \geq 0} 2^{-(i+1)} x_i$$

The alternative description of the interval object is as the smallest subalgebra of $(\mathbf{I}_D, \oplus)$ that is closed under $\bigoplus$ and contains 1 and -1. Explicitly, define

$$\mathbf{I} \quad = \quad \bigcap \{X \subseteq \mathbf{I}_D \mid -1, 1 \in X \text{ and for all } x_{(-)} \in X^{\mathbf{N}},\ \bigoplus_i x_i \in X\}$$

Then $\mathbf{I}$ is itself closed under $\bigoplus$, hence also under binary $\oplus$, because $x \oplus y = \bigoplus(x, y, y, y, \ldots)$.

Theorem 3 is an immediate consequence of the proposition below.

Proposition 8.4

1. $\mathbf{I}$ *is a magnifiable symmetric subalgebra of* $\mathbf{I}_D$.

2. $(\mathbf{I}, \oplus, -1, 1)$ *is an interval object in* $\mathcal{E}$.

3. $\mathbf{I} = \mathbf{I}_E$.

We now embark on the proof of the above proposition. The whole proof is structured around its crucial use of Pataraia's fixed-point theorem [26].

Theorem (Pataraia's fixed-point theorem) *Internally in* $\mathcal{E}$*, every monotonic endofunction on a dcpo with least element has a least fixed point.*
See Appendix C for a precise statement and proof of this result.

Let (A, m, a, b) be a bipointed convex body in $\mathcal{E}$. Consider the internal family $\mathcal{F}_{(A,m,a,b)}$ (henceforth just $\mathcal{F}$) of all subobjects X of $\mathbf{I}$ satisfying:

1. $-1, 1 \in X$;

2. X is a magnifiable symmetric subalgebra of $\mathbf{I}_D$;

3. there is a unique homomorphism, f_X, from $(X, \oplus, -1, 1)$ to (A, m, a, b); and

4. $X \subseteq \mathbf{I}_E$.

Lemma 8.5 *Internally in* $\mathcal{E}$*,* $(\mathcal{F}, \subseteq)$ *is a dcpo with least element.*

Proof Let $\mathbf{Q}_d$ be the subobject of $\mathbf{Q}$ of dyadic rationals. We write $I(\mathbf{Q}_d)$ for $\mathbf{Q}_d \cap \mathbf{I}_D$. Then $I(\mathbf{Q}_d)$ is the least element of $\mathcal{F}$. It satisfies condition 3 in the definition of $\mathcal{F}$ because it is the free midpoint algebra on two generators.

Suppose $\mathcal{D}$ is a directed subset of $\mathcal{F}$. We show that $X = \bigcup \mathcal{D}$ is in $\mathcal{F}$. Trivially, $-1, 1 \in X$ and $X \subseteq \mathbf{I}_E$. Also, X is easily shown to be a symmetric subalgebra of $\mathbf{I}_D$ (it is closed under $\oplus$ because $\mathcal{D}$ is directed). It remains to show that there is a unique bipointed homomorphism from $(X, \oplus, -1, 1)$ to (A, m, a, b).

Define $f_X : X \to A$ by mapping $x \in X$ to $f_Y(x)$ where Y is any element of $\mathcal{D}$ containing x. This is uniquely determined because, if $x \in Y \in \mathcal{D}$ and $x \in Y' \in \mathcal{D}$, then, as $\mathcal{D}$ is directed, there exists $Z \in \mathcal{D}$ with $Y \subseteq Z \supseteq Y'$. But then $f_Z : Z \to A$ restricts to homomorphisms from Y to A and from Y' to A. But f_Y and $f_{Y'}$ are the unique such, so $f_Y(x) = f_Z(x) = f_{Y'}(x)$.

Similarly, f_X is unique, because given any homomorphism $g : X \to A$ and $x \in X$, we have $x \in Y$ for some $Y \in \mathcal{D}$. Then g restricts to a homomorphism from Y to A, and f_Y is the unique such, so indeed $g(x) = f_Y(x) = f_X(x)$. □

Define $\Phi : \mathcal{P}(\mathbf{I}) \to \mathcal{P}(\mathbf{I})$ by:

$$\Phi(X) \quad = \quad \{\textstyle\bigoplus_i x_i \mid x_{(-)} \in X^{\mathbf{N}}\}$$

Lemma 8.6 *Internally in* $\mathcal{E}$*,* Φ *restricts to a monotonic inflationary function on* $(\mathcal{F}, \subseteq)$.

Proof Trivially Φ is a monotonic and inflationary endofunction on $(\mathcal{P}(\mathbf{I}), \subseteq)$ so we just have to show that Φ restricts to a function on $\mathcal{F}$. Suppose then that $X \in \mathcal{F}$. We show that $\Phi(X) \in \mathcal{F}$.

Trivially, $1, -1 \in \Phi(X)$. For closure under $\oplus$, suppose $x = \bigoplus_i x_i$ and $y = \bigoplus_i y_i$ (here $x, y \in \Phi(X)$ and $x_i, y_i \in X$). Then $x \oplus y = \bigoplus_i (x_i \oplus y_i)$, so indeed $x \oplus y \in \Phi(X)$ because each $x_i \oplus y_i \in X$. Similarly, $-x = \bigoplus_i -x_i$, so $\Phi(X)$ is a symmetric subalgebra of $\mathbf{I}_D$.

The (long) proof that $\Phi(X)$ is magnifiable is left for Lemma 8.7 below.

That $\Phi(X) \subseteq \mathbf{I}_E$ is obvious as every element of $\Phi(X)$ is easily obtained as the limit of a Cauchy sequence in X (because approximations to $\bigoplus$ form a Cauchy sequence).

It remains to show that there is a unique homomorphism $f_{\Phi(X)} : \Phi(X) \to A$. The argument is similar to that in the proof of Theorem 1. Consider the relation

$$\{(v, a) \mid \exists x_{(-)} \in X^{\mathbf{N}} \text{ s.t. } v = \textstyle\bigoplus_i (x_i) \text{ and } a = M_i(f_X(x_i))\} \quad \subseteq \quad \Phi(X) \times A.$$

We show that this relation is functional. It is trivially total. For single-valuedness, we need that $\bigoplus_i x_i = \bigoplus_i y_i$ implies $M_i(f_X(x_i)) = M_i(f_X(y_i))$. Suppose then that $\bigoplus_i x_i = \bigoplus_i y_i = v$. So, for any $n \geq 1$, $|v - \oplus_n(x_1, \ldots, x_n, 0)| \leq 2^{-n}$ and $|v - \oplus_n(y_1, \ldots, y_n, 0)| \leq 2^{-n}$. Then, defining $d_n = \oplus_n(x_1, \ldots, x_n, 0) - \oplus_n(y_1, \ldots, y_n, 0)$, we have that $|d_n| \leq 2^{-(n-1)}$. Therefore, $d_n \in X$, because X is magnifiable and hence closed under (truncated) subtraction. Also, $2^{n-1}.d_n \in X$ and $-2^{n-1}.d_n \in X$, again because X is magnifiable. By the definition of d_n,

$$\oplus_n(x_1, \ldots, x_n, -2^{n-1}.d_n) \quad = \quad \oplus_n(y_1, \ldots, y_n, 2^{n-1}.d_n),$$

and this value is in X. So, as $f_X : X \to A$ is a homomorphism,

$$m_n(f_X(x_1), \ldots, f_X(x_n), f_X(-2^{n-1}.d_n)) \quad = \quad m_n(f_X(y_1), \ldots, f_X(y_n), f_X(2^{n-1}.d_n)),$$

and this holds for any n. Hence, by the approximation property for A (from the generalisation of Proposition 3.6 to $\mathcal{E}$), $M_i(f_X(x_i)) = M_i(f_X(y_i))$. Thus the relation is indeed functional. Define $f_{\Phi(X)} : \Phi(X) \to A$ to be the associated function.

To show that $f_{\Phi(X)}$ is a homomorphism, suppose we have $u = \bigoplus_i x_i$ and $v = \bigoplus_i y_i$ with $u, v \in \Phi(X)$ and $x_i, y_i \in X$. Then, similar to (8) in the proof of Theorem 1:

$$\begin{aligned} f_{\Phi(X)}(u \oplus v) &= M_i(f_X(x_i \oplus y_i)) \\ &= M_i(m(f_X(x_i), f_X(y_i))) \quad \text{(as } f_X \text{ is a homomorphism)} \\ &= m(f_{\Phi(X)}(u), f_{\Phi(X)}(v)) \end{aligned}$$

as required.

For uniqueness, suppose $g : \Phi(X) \to A$ is another homomorphism. Then g restricts to a homomorphism from X to A, so $g(x) = f_X(x)$ for $x \in X$. Suppose $u = \bigoplus_i x_i$ with $u \in \Phi(X)$ and $x_i \in X$. Then, as in (9) and (10) from the proof of Theorem 1, $g(u) = f_{\Phi(X)}(u)$. Thus indeed $g = f_{\Phi(X)}$. □

Lemma 8.7 *Internally in $\mathcal{E}$, if X is a magnifiable symmetric subalgebra of $\mathbf{I}_D$ then so is $\Phi(X)$.*

Proof Suppose X is a magnifiable symmetric subalgebra of $\mathbf{I}_D$. We first establish:

$$\text{for all } u \in \Phi(X) \text{ and } y \in X, \text{ it holds that } I(u + 2y) \in \Phi(X) \tag{11}$$

For this define $\psi : X^{\mathbf{N}} \times X \to X^{\mathbf{N}}$ to be the unique function satisfying:

$$\psi((x_i)_{i \geq 0},\, y) \quad = \quad I(x_0 + 4y) :: \psi((x_i)_{i \geq 1},\, I((x_0 + 4y) \oplus -I(x_0 + 4y))$$

where the double colon concatenates a value onto the head of a sequence. To verify that this is a well-defined function, first observe that, for $x_0, y \in X$ we have $I(x_0 + 4y) \in X$, because $I(x_0 + 4y) = I(8 \times (x_0/4 \oplus y))$, which is in X by magnifiability (n.b. $x_0/4 = \oplus_2(0, 0, x_0)$). Therefore also $I((x_0 + 4y) \oplus -I(x_0 + 4y)) \in X$, because we have $I((x_0 + 4y) \oplus -I(x_0 + 4y)) = I(8 \times (x_0/4 \oplus y) \oplus I(x_0 + 4y)/8)$, which is again in X by magnifiability.

To prove (11), it suffices to verify that, for $x_i, y \in X$, $I((\bigoplus_{i \geq 0} x_i) + 2y) = \bigoplus \psi((x_i)_{i \geq 0}, y)$. Below we show that the equation

$$I((x \oplus z) + 2y) \quad = \quad I(x + 4y) \;\oplus\; I(z + 2I((x + 4y) \oplus -I(x + 4y))), \tag{12}$$

holds for all $x, y, z \in \mathbf{I}_D$. By this equation, we have:

$$\begin{aligned} & I((\textstyle\bigoplus_{i \geq 0} x_i) + 2y) \\ &\quad = I((x_0 \oplus \textstyle\bigoplus_{i \geq 1} x_i) + 2y) \\ &\quad = I(x_0 + 4y) \;\oplus\; I((\textstyle\bigoplus_{i \geq 1} x_i) + 2I((x_0 + 4y) \oplus -I(x_0 + 4y))) \\ &\quad = \psi_0((x_i)_{i \geq 0}, y) \;\oplus\; I((x_1 \oplus \textstyle\bigoplus_{i \geq 2} x_i) + 2I((x_0 + 4y) \oplus -I(x_0 + 4y))) \\ &\quad = \psi_0((x_i)_{i \geq 0}, y) \;\oplus\; (\psi_1((x_i)_{i \geq 0}, y) \;\oplus\; I(\ldots)) \\ &\quad = \psi_0((x_i)_{i \geq 0}, y) \;\oplus\; (\psi_1((x_i)_{i \geq 0}, y) \;\oplus\; (\psi_2((x_i)_{i \geq 0}, y) \;\oplus\; I(\ldots))) \\ &\quad = \ldots \end{aligned}$$

Thus, by approximation, indeed $I((\bigoplus_{i\geq 0} x_i) + 2y) = \bigoplus \psi((x_i)_{i\geq 0}, y)$.

To verify (12), we use the $\neg\neg$-separatedness of $\mathbf{I}_D$, which allows us to check the equation by splitting into cases: (i) $3 \leq x + 4y$ (≤ 5), (ii) $1 \leq x + 4y \leq 3$, (iii) $-1 \leq x + 4y \leq 1$, (iv) $-3 \leq x + 4y \leq -1$, (v) $(-5 \leq)$ $x + 4y \leq -3$. The verification is routine. We check case (ii). Referring to (12) we have:

$$\begin{array}{rcll} \text{r.h.s.} & = & 1 \oplus I(z + 2I((x + 4y) \oplus -1)) & \text{(because } 1 \leq x + 4y) \\ & = & 1 \oplus I(z + 2((x + 4y) \oplus -1)) & \text{(because } x + 4y \leq 3) \\ & = & 1 \oplus I(x + z + 4y - 1) & \\ & = & 1 \oplus I(2((x \oplus z) + 2y) - 1) & \end{array}$$

There are now two subcases. If $(x\oplus z)+2y \geq 1$ then $1\oplus I(2((x\oplus z)+2y)-1) = 1 = I((x\oplus z)+2y)$ as required. If instead $0 \leq (x\oplus z)+2y \leq 1$ then $1\oplus I(2((x\oplus z)+2y)-1) = 1\oplus(2((x\oplus z)+2y)-1) = (x \oplus z) + 2y = I((x \oplus z) + 2y)$. Together with the other cases, this completes the verification of (12) and hence (11).

Finally, we show that $\Phi(X)$ is indeed magnifiable. Suppose that $u \in \Phi(X)$. Then $u = \bigoplus_{i\geq 0} x_i$ where $x_i \in X$. We must show that $I(2u) \in \Phi(X)$.

Below we verify that the equation

$$I(x + (y \oplus z)) \quad = \quad I(2x + y) \oplus I(z + 2I((2x + y) \oplus -I(2x + y))) \tag{13}$$

holds for all $x, y, z \in \mathbf{I}_D$. By this equation, we have:

$$\begin{array}{rcl} I(2u) & = & I(2 \bigoplus_{i\geq 0} x_i) \\ & = & I(x_0 + (x_1 \oplus \bigoplus_{i\geq 2} x_i)) \\ & = & I(2x_0 + x_1) \oplus I((\bigoplus_{i\geq 2} x_i) + 2I((2x_0 + x_1) \oplus -I(2x_0 + x_1))). \end{array}$$

Using the magnifiability of X, it is straightforward to show that $I(2x_0 + x_1) \in X$ and then that $I((2x_0 + x_1) \oplus -I(2x_0 + x_1)) \in X$. Hence $I((\bigoplus_{i\geq 2} x_i) + 2I((2x_0 + x_1) \oplus -I(2x_0 + x_1))) \in \Phi(X)$, by (11). Thus indeed $I(2u) \in \Phi(X)$.

It remains to verify (13). This is done by splitting into three cases: (i) $1 \leq 2x + y$ (≤ 3); (ii) $-1 \leq 2x + y \leq 1$; (iii) $(-3 \leq)$ $2x + y \leq -1$. Again, the verification is routine. This time we check case (i). Referring to (13) we have:

$$\begin{array}{rcll} \text{r.h.s.} & = & 1 \oplus I(z + 2I((2x + y) \oplus -1)) & \text{(because } 1 \leq 2x + y) \\ & = & 1 \oplus I(2x + y + z - 1) & \text{(because } 0 \leq (2x + y) \oplus -1 \leq 1) \\ & = & 1 \oplus I(2(x + (y \oplus z)) - 1) & \end{array}$$

There are now two subcases. When $x+(y\oplus z) \geq 1$ then $1\oplus I(2(x+(y\oplus z))-1) = 1 = I(x+(y\oplus z))$. If instead $0 \leq x+(y\oplus z) \leq 1$ then $1\oplus I(2(x+(y\oplus z))-1) = 1\oplus(2(x+(y\oplus z))-1) = x+(y\oplus z) = I(x + (y \oplus z))$. □

Proof of Proposition 8.4 By Pataraia's theorem $\Phi : \mathcal{F} \to \mathcal{F}$ has a fixed point $X \in \mathcal{F}$. Then $X \subseteq \mathbf{I}$, by the definition of $\mathcal{F}$, Also $X = \Phi(X)$. Therefore X is a subobject of $\mathbf{I}_D$ that contains $1, -1$ and is closed under $\bigoplus$. But $\mathbf{I}$ was defined as the smallest such subset of $\mathbf{I}_D$. Therefore $\mathbf{I} \subseteq X$. Thus $\mathbf{I} = X$ and hence $\mathbf{I} \in \mathcal{F}$. That $\mathbf{I}$ is magnifiable, is now immediate from the definition of $\mathcal{F}$.

Also by the definition of $\mathcal{F}$, there is a unique homomorphism from $(\mathbf{I}, \oplus, -1, 1)$ to (A, m, a, b). Also, to any other bipointed convex body (A', m', a', b'), the unique homomorphism from $(\mathbf{I}, \oplus, -1, 1)$ is provided by applying the above argument to $\mathcal{F}_{(A',m',a',b')}$. Thus $(\mathbf{I}, \oplus, -1, 1)$ is indeed an interval object.

It remains to show that $\mathbf{I} = \mathbf{I}_E$. First, we show that $\mathbf{I}$ is Cauchy complete. Suppose $\alpha_{(-)} \in \mathbf{I}^{\mathbf{N}}$ is a Cauchy sequence. Without loss of generality we can assume that

$$\text{for all } i,\ |\alpha_{i+1} - \alpha_i| \leq 2^{-(i+1)}.$$

Then

$$\mathit{lim}\ \alpha \quad = \quad \bigoplus(\alpha_0,\, 2(\alpha_0 - \alpha_1),\, 4(\alpha_1 - \alpha_2),\, \ldots).$$

But, by magnifiability, $\mathbf{I}$ is closed under (truncated) subtraction and multiplication by any 2^n. It is also closed under $\bigoplus$. Thus indeed $\mathit{lim}\ \alpha \in \mathbf{I}$.

To show that $\mathbf{I} = \mathbf{I}_E$, the $\subseteq$ inclusion is immediate because $\mathbf{I} \in \mathcal{F}$. For the converse, define $\mathbf{R_I} = \{2^n x \in \mathbf{R}_D \mid x \in \mathbf{I} \text{ and } n \in \mathbf{N}\}$. As every rational in $\mathbf{I}_D$ can be defined by a "binary" sequence $\bigoplus_i x_i$ where each x_i is -1 or 1, it follows that $\mathbf{R_I}$ contains all rationals. Also it follows from the Cauchy completeness of $\mathbf{I}$ that $\mathbf{R_I}$ is Cauchy complete (because, for any Cauchy sequence in $\mathbf{R_I}$, we can obtain every element of the sequence as $2^n x_i$ for a fixed n, yielding an associated Cauchy sequence (x_i) in $\mathbf{I}$). Therefore $\mathbf{R}_E \subseteq \mathbf{R_I}$. Thus, for any $y \in \mathbf{I}_E$, there exists $x \in \mathbf{I}$ such that $y = 2^n x$ for some n. But then $y \in \mathbf{I}$ because $\mathbf{I}$ is magnifiable. □

We have proved Proposition 8.4. Theorem 3 follows immediately.

We conclude this section with an application of the work in this section, the promised proof of Proposition 6.6. Consider the *relative realizability topos* $RT(\mathcal{P}\omega, \mathcal{P}\omega_{re})$, as defined in [2]. As $RT(\mathcal{P}\omega, \mathcal{P}\omega_{re})$ satisfies $\mathbf{N}$-$\mathbf{N}$-choice (hence $\mathrm{AC}_{\mathbf{N} \geq \mathbf{N}}$), we write $\mathbf{I}$ for $\mathbf{I}_C = \mathbf{I}_E = \mathbf{I}_D$. By the definition of this topos, the morphisms from $\mathbf{I}^n$ to $\mathbf{I}$ are exactly the n-ary computable functions on $\mathbb{I}$, see [3]. However, because $\mathbf{I}$ is an interval object, the computable functions on $\mathbb{I}$ are closed under the definitional mechanisms of the primitive interval functions. Moreover, the truncated double function, d, is a morphism from $\mathbf{I}$ to $\mathbf{I}$ (for two different reasons: on the one hand, because it is computable; on the other, because $\mathbf{I}$ is magnifiable). Thus the computable functions contain the d-primitive interval functions. This completes the proof of Proposition 6.6.

9 Concluding remarks

We have provided a foundational axiomatization of the line segment, by means of a geometrically motivated universal property that supports the definition of computable functions on the interval. Moreover, we have investigated this axiomatization in a number of settings.

Many other settings remain to be investigated. In the category of setoids in intuitionistic type theory [14, 25], it can be shown that any of the usual constructions of a closed real interval gives an interval object. In the category of locales over any topos, we conjecture that the standard localic interval [17] is an interval object.

The fact that the computable functions on the reals are continuous is often cited in the literature. In connection with that, we observe that our axiomatization of line segments exhibits the Euclidean topology of the line as intrinsic rather than imposed structure, because it is this topology that gives rise to an interval object.

By definition, an interval object is a free convex body over two generators. Freely generated convex bodies over different generating objects coincide with other familiar mathematical structures. Interesting examples can be shown to occur in the category of topological spaces: (1) The interval $[-1, 1]$ with the topology of lower semicontinuity [28] (which is the same as the Scott topology of the usual order) is the free convex body over Sierpinski space, (2) the set of subintervals of $[-1, 1]$ under the Scott topology, with the pointwise midpoint structure, is the free convex body over the flat domain $\{\mathrm{true}, \mathrm{false}\}_\perp$ of booleans under the Scott topology. (3) The free convex body over a finite discrete space of cardinality n is an n-simplex. In particular, the free convex body over three and four generators are the triangle and the tetrahedron. All the above examples are applications of the left adjoint to the forgetful functor from topological convex bodies to **Top**, which exists by Freyd's Adjoint Functor Theorem [23]. We do not have an explicit description of this adjoint.

There are intriging connections between midpoint algebras and the probabilistic algebras that arise in the study of probabilistic powerdomains—see the axiomatizations discussed by Heckmann [12]. It is plausible that the free convex body over a sufficiently nice domain may be nothing but the probabilistic powerdomain of normalized valuations [18].

Finally, observe that our axiomatization, unlike that of Higgs [13], is not purely equational. We wonder if it is possible to obtain an equational (i.e. algebraic) characterisation of the interval, applicable to a similar range of examples as ours.

A Convex bodies in the presence of an nno

In this appendix we generalise Proposition 3.1, the equivalence of right-iterability with the existence of a (necessarily unique) well-behaved infinitary operation, to the setting of a category $\mathcal{C}$ with finite products and a parameterized natural numbers object $(\mathbf{N}, 0, s)$.

Definition A.1 (N-ary operation) An $\mathbf{N}$-*ary operation* on an object A is given by a family of functions $M_Z : \mathcal{C}(Z \times \mathbf{N}, A) \to \mathcal{C}(Z, A)$, natural in Z (i.e. for any $Z' \times \mathbf{N} \xrightarrow{f} A$ and $Z \xrightarrow{g} Z'$, $M_Z(f \circ (g \times \mathsf{id}_{\mathbf{N}})) = M_{Z'}(f) \circ g : Z \longrightarrow A$).

In the special case that $\mathbf{N}$ is exponentiable, $\mathbf{N}$-ary operations on A are in one-to-one correspondence with morphisms $A^{\mathbf{N}} \xrightarrow{M} A$. The above definition is the appropriate generalisation to the general case of a non-exponentiable $\mathbf{N}$.

We shall incorporate $\mathbf{N}$-ary operations, $M : \mathcal{C}(Z \times \mathbf{N}, A) \to \mathcal{C}(Z, A)$, into our algebraic style of notation by treating them as binding operators. To ease readability, we shall consider morphisms $Z \times \mathbf{N} \longrightarrow A$ as representing internal $\mathbf{N}$-indexed families of generalised points $Z \longrightarrow A$. For example, we often use the notation $Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$ for such a morphism, and then write $Z \xrightarrow{x_i} A$ for the morphism given as the composite $x_{(-)} \circ \langle \mathsf{id}_Z, i\rangle$, where $Z \xrightarrow{i} \mathbf{N}$ is a generalised point of $\mathbf{N}$. Given a morphism, $Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$, we write $M_i(x_i)$ for the morphism $Z \xrightarrow{M(x_{(-)})} A$, where, syntactically, M_i binds the variable i of type $\mathbf{N}$. It is precisely the naturality of the M_Z operations in Z that licenses the use of such syntax, allowing equational reasoning using α-conversion and capture-avoiding substitution.

Proposition A.2 *Let (A, m) be a binary algebra in $\mathcal{C}$.*

1. *(A, m) is a right-iterable algebra if and only if there exists an $\mathbf{N}$-ary operation $M : \mathcal{C}(X \times \mathbf{N}, A)$ satisfying:*

   (a) *For any $Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$, $M_i(x_i) = m(x_0, M_i(x_{s(i)}))$; or equivalently:*

$$\begin{array}{ccc} X & \xrightarrow{\langle x_{(-)} \circ \langle \mathsf{id}_X, 0\rangle,\ M_X(x_{(-)} \circ (\mathsf{id}_X \times s))\rangle} & A \times A \\ & \searrow^{M_X(x_{(-)})} & \downarrow m \\ & & A \end{array}$$

   (b) *For any $Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$ and for any $Z \times \mathbf{N} \xrightarrow{y_{(-)}} A$ if $y_{(-)} = m(x_{(-)}, y_{s(-)})$ then $y_0 = M_i(x_i)$; or equivalently, if the following diagram commutes*

$$\begin{array}{ccc} X \times \mathbf{N} & \xrightarrow{\langle x_{(-)}, y_{(-)} \circ (\mathsf{id}_X \times s)\rangle} & A \times A \\ & \searrow^{y_{(-)}} & \downarrow m \\ & & A \end{array}$$

   then $y_{(-)} \circ \langle \mathsf{id}_X, 0\rangle = M_X(x_{(-)})$.

   Moreover if (A, m) is right-iterable then there is a unique M satisfying (a).

2. *If (A, m) and (A', m') are right-iterable algebras then any homomorphism $f : A \to A'$ is also a homomorphism with respect to the associated $\mathbf{N}$-ary M and M'; i.e. for any $Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$, $f(M_i(x_i)) = M'_i(f(x_i))$.*

Proof The proof is essentially an internalization of the proof of Proposition 3.1 in $\mathcal{C}$, but with suitable modifications for the proof to go through at the level of generality of this section.

For 1, as $\mathbf{N}$ is not assumed to be exponentiable, there is no analogue of the (final) coalgebra $A^\omega \to A \times A^\omega$ used in the proof of Proposition 3.1. Instead, to show the existence and properties of M_Z, use the coalgebra $\langle x, \mathsf{id} \times s\rangle : Z \times \mathbf{N} \longrightarrow A \times (Z \times \mathbf{N})$. The full argument, which is somewhat tedious, is omitted.

For 2, suppose that f is a homomorphism with respect to m and m'. One uses the definable addition morphism $+ : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$, to define $M_j(x_{(-)+j}) : Z \times \mathbf{N} \longrightarrow A$ (explicitly, this is obtained by applying $M_{Z\times\mathbf{N}}$ to the composite $Z \times \mathbf{N} \times \mathbf{N} \xrightarrow{\mathsf{id}\times +} Z \times \mathbf{N} \xrightarrow{x_{(-)}} A$). Then:

$$\begin{array}{rcll} f(M_j(x_{(-)+j})) & = & f(m(x_{(-)}, M_j(x_{(-)+s(j)}))) & \text{(property (a))} \\ & = & m'(f(x_{(-)}), f(M_j(x_{(-)+s(j)}))) & (f \text{ a homomorphism}) \\ & = & m'(f(x_{(-)}), f(M_j(x_{s(-)+j}))) & \text{(provable property of } +) \end{array}$$

So, by property (b) for M', we have $f(M_j(x_{0+j})) = M'_i(f(x_i))$, i.e. $f(M_i(x_i)) = M'_i(f(x_i))$ as required. □

We also have an analogue of Proposition 3.2.

Proposition A.3 *If (A, m) is an iterable midpoint algebra, then the associated $\mathbf{N}$-ary operation M satisfies:*

1. *For "constant" sequences $Z \times \mathbf{N} \xrightarrow{\pi_1} Z \xrightarrow{x} A$, $M_i(x) = x$.*
2. *For "doubly-indexed" sequences $Z \times \mathbf{N} \times \mathbf{N} \xrightarrow{x_{(-)(-)}} A$, $M_i(M_j(x_{ij})) = M_j(M_i(x_{ij}))$.*
3. *For $x_{(-)}, y_{(-)} : Z \times \mathbf{N} \longrightarrow A$, $M_i(m(x_i, y_i)) = m(M_i(x_i), M_i(y_i))$.*

The straightforward proof is omitted.

It is also possible to formulate the approximation property of Proposition 3.6.2 at the level of generality of this section. However, not only is the formulation somewhat involved, but also the proof that approximation is equivalent to cancellation appears not to go through at this level of generality. In order to reinstate the equivalence between cancellation and approximation, it seems that one needs to assume that $\mathcal{C}$ is a locally cartesian closed category. Because of its potential application to characterising interval objects in Martin-Löf type theory, it would be interesting to elaborate further on this situation.

B Convex sets in Euclidean space

In this appendix, we prove a couple of straightforward results about convex subsets of $\mathbb{R}^n$, which we use in the proof of Proposition 3.3. We follow standard terminology from the literature on convex sets (see, for example, [22]).

Let A be any subset of $\mathbb{R}^n$, and let $\mathcal{L} \subseteq \mathbb{R}^n$ be the smallest flat (i.e. affine subspace) containing A. The *relative interior*, $\operatorname{relint} A$, is defined to be the interior of A in the relative (Euclidean) topology on $\mathcal{L}$. If A is a nonempty convex set then $\operatorname{relint} A$ is nonempty. The *Minkowski distance function*[2] $p : \mathcal{L} \to \mathbb{R}$, with respect to $\mathbf{x}_0 \in \operatorname{relint} A$, is defined by:

$$p(\mathbf{x}) \quad = \quad \inf\{\lambda \mid \lambda > 0 \text{ and } \mathbf{x}_0 + \lambda^{-1}(\mathbf{x} - \mathbf{x}_0) \in A\}.$$

It is easy to show that the Minkowski function is convex (i.e. for $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$, $p(\lambda\mathbf{x} + \mu\mathbf{y}) \leq \lambda p(\mathbf{x}) + \mu p(\mathbf{y})$) and (hence) continuous on $\mathcal{L}$. Clearly $p(\mathbf{x}_0) = 0$. A *ray* with *vertex* $\mathbf{x}_0$ is a subset of $\mathbb{R}^n$ of the form $\{\mathbf{x}_0 + \lambda(\mathbf{x}_1 - \mathbf{x}_0) \mid \lambda \in [0, \infty)\}$ where $\mathbf{x}_1$ is distinct from $\mathbf{x}_0$. If A contains no ray with vertex $\mathbf{x}_0$ then $p(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{x}_0$ in $\mathcal{L}$.

Proposition B.1 *If A is an unbounded convex subset of $\mathbb{R}^n$ then A contains a ray.*

[2] This definition is a mild generalisation of that in [22].

Proof Let A be a nonempty convex subset of $\mathbb{R}^n$ with $\mathbf{x}_0 \in \operatorname{relint} A$. Let m be the dimension of the smallest flat $\mathcal{L}$ containing A, and let S_m be the m-sphere of unit radius centered at $\mathbf{x}_0$ in $\mathcal{L}$. S_m is compact and $p : S_m \to \mathbb{R}$ is continuous, so, at some $\mathbf{z} \in S_m$, $p(\mathbf{z})$ takes a smallest value λ. If A contains no ray then $\lambda > 0$ and $|\mathbf{x} - \mathbf{x}_0| \leq \lambda^{-1}$ for all $\mathbf{x} \in A$. Thus A is bounded. □

For any subset $A \subseteq \mathbb{R}^n$, we write $\operatorname{conv} A$ for the convex hull of A, i.e. for the smallest convex set containing A.

Lemma B.2 *For any* $\lambda_1, \ldots, \lambda_k > 0$ *with* $\sum_{i=1}^k \lambda_i = 1$,

$$\sum_{i=1}^k \lambda_i \mathbf{x}_i \quad \in \quad \operatorname{relint}(\operatorname{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}).$$

Lemma B.3 *Suppose* $0 \leq \lambda < 1$ *and* A *is a convex set with* $\mathbf{x_0} \in \operatorname{relint} A$. *Define*

$$r_\lambda(A, \mathbf{x}_0) \quad = \quad \{\mathbf{x}_0 + \lambda(\mathbf{x} - \mathbf{x}_0) \mid \mathbf{x} \in A\}$$

Then $\overline{r_\lambda(A, \mathbf{x}_0)} \subseteq A$ *(where* $\overline{r_\lambda(A, \mathbf{x}_0)}$ *is the topological closure of* $r_\lambda(A, \mathbf{x}_0)$*).*

Proof Let $\mathcal{L}$ be the smallest flat containing A, and let p be the Minkowski distance function as defined above. Then $\overline{r_\lambda(A, \mathbf{x}_0)} = \{\mathbf{x} \in \mathcal{L} \mid p(x) \leq \lambda\}$, because, on the one hand, the latter is closed by the continuity of p, and, on the other, it is clear that every point in it lies in the cloure of $r_\lambda(A, \mathbf{x}_0)$. As $\lambda < 1$, it is trivial that $\{\mathbf{x} \in \mathcal{L} \mid p(x) \leq \lambda\} \subseteq A$. □

Proposition B.4 *If* A *is a bounded convex subset of* $\mathbb{R}^n$ *then* A *is closed under countable convex combinations.*

Proof Let $(\mathbf{x}_i)$ be a sequence of points in A and (λ_i) a sequence of weights in $[0, 1]$ with $\sum_{i=0}^\infty = 1$. As A is bounded, $\mathbf{z} = \sum_{i=0}^\infty \lambda_i \mathbf{x}_i$ is defined as an element of $\mathbb{R}^n$. We must show that $\mathbf{z} \in A$.

Without loss of generality, assume each $\lambda_i > 0$. Let m be the dimension of (the smallest flat containing) $B = \operatorname{conv}\{\mathbf{x}_i \mid i \geq 0\} \subseteq A$. Then $\{\mathbf{x}_i \mid i \geq 0\}$ contains $m + 1$ affinely independent points (i.e. $m+1$ points whose convex hull has dimension m). Without loss of generality, suppose these are $\mathbf{x}_0, \ldots, \mathbf{x}_m$. For $i \geq m$ define:

$$\begin{aligned} \sigma_i &= \lambda_0 + \lambda_1 + \ldots + \lambda_i \\ \mathbf{y}_i &= {\sigma_i}^{-1}(\lambda_0 \mathbf{x}_0 + \lambda_1 \mathbf{x}_1 + \ldots + \lambda_i \mathbf{x}_i). \end{aligned}$$

Clearly $(\mathbf{y}_i)_{i \geq m}$ is a Cauchy sequence with limit $\mathbf{z}$. Also we have $\mathbf{y}_i \in r_{(1-\sigma_m)}(B, \mathbf{y}_m)$ (as defined in Lemma B.3). Thus $\mathbf{z} \in \overline{r_{(1-\sigma_m)}(B, \mathbf{y}_m)}$. Moreover, $\mathbf{y}_m \in \operatorname{relint}(\operatorname{conv}\{\mathbf{x}_0, \ldots, \mathbf{x}_m\})$, by Lemma B.2. Hence $\mathbf{y}_m \in \operatorname{relint} B$, because $\operatorname{conv}\{\mathbf{x}_0, \ldots, \mathbf{x}_m\} \subseteq B$ has dimension m. Thus $\overline{r_{(1-\sigma_m)}(B, \mathbf{y}_m)} \subseteq B$, by Lemma B.3. So indeed $\mathbf{z} \in B \subseteq A$. □

C Pataraia's fixed-point theorem

In this appendix we give a precise statement of Pataraia's Fixed-point Theorem, which was used crucially in Section 8. The results are due to D. Pataraia [26].

For this appendix, let $\mathcal{E}$ be any elementary topos. (It does not need to have an nno.) Let $(X, \leq)$ be a poset in $\mathcal{E}$ (i.e. $\leq$ is a subobject of $X \times X$ satisfying the usual axioms for a non-strict partial order, expressed in the internal logic of $\mathcal{E}$). Internally in $\mathcal{E}$, we say that a suobject $D \subseteq X$ (i.e. an element $D \in \mathcal{P}X$) is *directed* if it is inhabited (i.e. there exists some $x \in D$) and, for any $x, y \in D$ there exists $z \in D$ with $x \leq z \geq y$. (Because we are working intuitionistically, the condition of being inhabited is stronger than the condition of being nonempty.) We say that X is a *directed-complete partial order (dcpo)* if, internally in $\mathcal{E}$, every directed subobject $D \subseteq X$ has a least upper bound (lub) in X. We say that an endofunction f on a poset X is *inflationary* if it is both monotonic (i.e. $x \leq y$ implies $f(x) \leq f(y)$) and increasing (i.e. $x \leq f(x)$).

Proposition C.1 *If* X *is an inhabited dcpo then, internally in* $\mathcal{E}$, *every inflationary function on* X *has a fixed point.*

Proof Let I be the object of inflationary endofunctions on X ordered pointwise. It is easily checked that I is a dcpo: lubs can be constructed pointwise because the image of a directed set (of functions) under a monotonic function (application to an argument) is directed, and the lub so constructed is indeed inflationary. Moreover, I is itself directed: the identity is in I; and, given $f, g \in I$, it holds that $f \leq f \circ g \geq g$. Therefore I considered as a subobject of itself has a lub, i.e. I has a maximum element, t. Then, for any $f \in I$ we have that $f \circ t = t$. Thus, given any element $x \in X$, it holds that $t(x)$ is a common fixed point for all inflationary functions on X. □

Theorem (Pataraia's fixed-point theorem) *If X is a dcpo with least element then, internally in $\mathcal{E}$, every monotonic endofunction on X has a least fixed point.*
Proof Let f be a monotonic endofunction on X. Let X' be intersection of all subdcpos (in the evident sense) of X that contain the least element, $\perp$, and are closed under f. Then X' is itself an f-closed subdcpo of X containing $\perp$. Also, f is an inflationary endofunction on X', because $\{x \in X \mid x \leq f(x)\}$ is another such subdcpo. Therefore, by the proposition, f has a fixed point $a \in X'$. We show this is the least prefixed point of f. This holds because, for any prefixed point $f(y) \leq y$, the subobject $\{x \in X \mid x \leq y\}$ is an f-closed subdcpo containing $\perp$, hence a is in this set, i.e. $a \leq y$. □
Our application in this paper is one of three applications that we have, between us, discovered of this result; the others being to the construction of initial algebras of endofunctors [31, Theorem 5], and to a localic version of the Hofmann-Mislove Theorem [8]. In fact, note that the application in this paper follows directly from Proposition C.1 alone.

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