

# The proof of the interpolation property of the way-below relation of a continuous poset

(NOTE)

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It is fair to say that there is only one proof of the interpolation property of the way-below relation of a continuous poset. Two versions can be found in [1, pages 46–47] and [2, pages 289–290]. Here is another.

1 *If  $x \ll y$  holds in a continuous poset, then for every directed set  $D$  with  $y \leq \bigvee D$  there is  $d \in D$  with  $x \ll d$ .*

PROOF By continuity of the poset, the set  $I \stackrel{\text{def}}{=} \bigcup \{\downarrow d \mid d \in D\}$  has the same join as  $D$ . Being a directed union of ideals,  $I$  is an ideal. By definition of the way-below relation, we conclude that  $x \in I$ , which gives the result.  $\square$

2 *If  $x \ll y$  holds in a continuous poset, then there is an element  $b$  with  $x \ll b \ll y$ .*

PROOF The element  $y$  is the directed join of the elements  $b \ll y$ .  $\square$

## References

- [1] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.
- [2] P.T. Johnstone. *Stone Spaces*. Cambridge University Press, Cambridge, 1982.

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