The geometry of constancy

(in HoTT and in `cubicaltt`)

Martín Hötzel Escardó
(Joint with Thierry Coquand)

University of Birmingham, UK

HoTT/UF at RDP in Warsaw, 2015

Exiting propositional truncations

Often we have $\|X\| \to X$, even when we don't know whether X is empty or inhabited.

E.g. For any $f : \mathbb{N} \to \mathbb{N}$, we have $\| \sum_{n:\mathbb{N}} fn = 0\| \to \sum_{n:\mathbb{N}} fn = 0$.

If there is a root of f, then we can find one.

Exiting propositional truncations

However, global choice

$$\prod_{X:U} \|X\| \to X$$

implies that all types have decidable equality.

(And even $X + \neg X$ for all $X : U$ if we have quotients.)

By Hedberg's theorem, if every type has decidable equality then every type is a set and hence global choice negates univalence.

So global choice is both a constructive and a homotopy type theory taboo.

Exiting propositional truncations

Theorem (with Nicolai, Thierry and Thorsten):

> A type X has a choice function $\|X\| \to X$ iff it has a constant endomap $X \to X$.

Question:

> *Can we eliminate $\|X\| \to A$ using a constant map $X \to A$?*

Two answers: Yes (Nicolai Kraus) and no (Mike Shulman).

Nicolai considers coherently constant functions.
Mike considers arbitrary constant functions.

Constancy

There are many notions of constancy.

We investigate the following:

1. A function $f : X \to A$ is constant if any two of its values are equal.

$$\mathrm{constant}\, f \stackrel{\mathrm{def}}{=} \prod_{x,y:X} fx = fy.$$

2. This is data rather than property, unless A is a set.

   Called a modulus of constancy of f.

   A function can have zero, one or more moduli of constancy.

3. E.g. the function $f : 1 \to S^1$ with definitional value base has $\mathbb{Z}$-many moduli of constancy $\kappa_n : \mathrm{constant}\, f$:

$$\kappa_n(x)(y) \stackrel{\mathrm{def}}{=} \mathrm{loop}^n\,.$$

Set-valued constant functions

1. For any proposition P, by definition of truncation:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \|X\| \arrow[d, dashed, "\bar{f}"] \\ & P \end{tikzcd}$$

proposition (so f constant)

2. Can replace P by a set A:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[ddrr, "f"'] & & \|X\| \arrow[dl, dashed] \arrow[dd, dashed, "\bar{f}"] \\ & \operatorname{Im} f \arrow[dr] & \\ & & A \end{tikzcd}$$

a proposition
because A is a set
and f is constant

constant by assumption
(but then in a unique way
because A is a set)

set by assumption

Propositional truncation as a set quotient

1. I.e. $\|X\|$ is the set-quotient of X by the chaotic relation:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"', "\substack{\text{constant}\\ \text{by assumption}}"' near end] & \|X\| \arrow[d, dashed, "\bar{f}"] \\ & A \;\; \text{set by assumption} \end{tikzcd}$$

2. Can we replace A by an arbitrary type?

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"', "\substack{\text{constant}\\ \text{by assumption}}"' near end] & \|X\| \arrow[d, dashed, "\bar{f} \quad ?"] \\ & A \;\; \text{arbitrary} \end{tikzcd}$$

No, not in general (Shulman, `http://homotopytypetheory.org/2015/06/11/not-every-weakly-constant-function-is-conditionally-constant/`)

When do we get a factorization of a constant function?

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"', "\text{constant}"'] & \|X\| \arrow[d, dashed, "\bar{f}\quad ?"] \\ & A \ \text{arbitrary} \end{tikzcd}$$

The factorization is possible if any of the following conditions holds:

1. X is empty.
2. X has a given point.
3. We have a function $\|X\| \to X$.
4. We have a function $A \to X$.
5. A is a set.

   What other sufficient conditions?

   And what about necessary conditions?

Also, given any factorization, we can construct another one for which the triangle commutes judgementally.

How to construct a counter example

cannot have
a known point
or be empty

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \|X\| \arrow[d, dashed, "\text{want: no } \bar{f} \text{ possible}"] \\ & A \end{tikzcd}$$

constant
by assumption

cannot be a set

Natural attempt to get a counter-example

Let $s : S^1$ be an arbitrary point of the circle.

Let A be an arbitrary type.

Let $f : s = \mathrm{base} \to A$ be constant.

We can't know a point of the path space $s = \mathrm{base}$ in general.

But we know it is inhabited, that is, $\| s = \mathrm{base} \|$

Hence $\| s = \mathrm{base} \| = 1$ by propositional univalence/extensionality.

$$
\begin{tikzcd}
s = \mathrm{base} \arrow[rr] \arrow[dr] \arrow[ddrr, "{\substack{f \\ \text{constant} \\ \text{by assumption}}}"'] & & \| s = \mathrm{base} \| \arrow[dd, dashed] \\
& 1 \arrow[ur, equal] \arrow[dr, dashed, "?"] & \\
& & A \quad \text{arbitrary}
\end{tikzcd}
$$

Attempt to get a counter-example

$$\begin{tikzcd}
s = \mathrm{base} \arrow[rr] \arrow[dr] \arrow[ddrr, "f"'] & & \|s = \mathrm{base}\| \arrow[dl, equal] \arrow[dd, dashed] \\
& 1 \arrow[dr, dashed, "?"] & \\
& & A
\end{tikzcd}$$

constant
by assumption

arbitrary

Can we expect to be able to get a point of an arbitrary type A,

from any given constant function $f : s = \mathrm{base} \to A$,

even though we can't expect to get a point of $s = \mathrm{base}$ in general?

To our surprise, we can.

The attempt fails.

Theorem/Construction

$$
\begin{tikzcd}
s = \mathrm{base} \arrow[rr] \arrow[dr] \arrow[ddrr, "\text{constant } f"'] & & \| s = \mathrm{base} \| \arrow[dd, dashed] \\
& 1 \arrow[ur, equal] \arrow[dr, dashed, "a"] & \\
& & A \quad \text{arbitrary}
\end{tikzcd}
\qquad
\begin{tikzcd}
s = \mathrm{base} \arrow[r] \arrow[dr, "\text{constant } f"'] & 1 \arrow[d, dashed, "a"] \\
& A
\end{tikzcd}
$$

For any $s : S^1$ and any constant function $f : s = \mathrm{base} \to A$ into an arbitrary type, we can find $a : A$ such that $fp = a$ for all $p : s = \mathrm{base}$.

$$\prod_{s:S^1} \prod_{A:U} \prod_{f:s=\mathrm{base}\to A} \mathrm{constant}\, f \to \sum_{a:A} \prod_{p:s=\mathrm{base}} fp = a.$$

Proof outline

1. First show that for any given family of constant functions

$$f : \prod_{s:S^1} s = \mathrm{base} \to A(s),$$

   each of them factors through $\mathbf{1}$. We get $\bar{f} : \prod_{s:S^1} A(s)$

   This allows us to use induction on the circle and on paths.

2. For any type X, consider the universal constant map on X, $\beta_X : X \to S(X)$, constructed as a HIT.

3. By (1) applied to the family $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ given by (2), we get a function $\bar{\beta} : \prod_{s:S^1} S(s = \mathrm{base})$.

4. Now, given a single constant function $f : s = \mathrm{base} \to A$, it factors through the universal constant map $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ as $f' : S(s = \mathrm{base}) \to A$ by (2), and hence we get the required point of A as using (3), as $f'(\bar{\beta}(s))$.

Step 1

For any $f : \prod_{s:S^1} s = \mathrm{base} \to A(s)$, with $f\ \mathrm{base}$ constant, there is $\bar{f} : \prod_{s:S^1} A(s)$ such that $f\,s\,p = \bar{f}\,s$ for all $p : s = \mathrm{base}$.

1. Lemma Any transport of a value of f is a value of f:

$$\prod_{b,b':S^1} \prod_{r:b=b} \prod_{l:b=b'} \sum_{q:b'=b} \mathrm{transport}\, l\, (f\, b\, r) = f\, b'\, q.$$

   This doesn't depend on the fact that S^1 is the circle or on the constancy of $f\ \mathrm{base}$, and has a direct proof by based path induction.

2. We are interested in this particular case:

$$\sum_{q:\mathrm{base}=\mathrm{base}} \mathrm{transport\, loop}\, (f\ \mathrm{base}\, (\mathrm{refl\, base})) = f\ \mathrm{base}\ q.$$

3. Then the constancy of $f\ \mathrm{base}$ gives

$$\mathrm{transport\, loop}\, (f\ \mathrm{base}\, (\mathrm{refl\, base})) = f\ \mathrm{base}\, (\mathrm{refl\, base}),$$

   which makes S^1-induction work.

Step 2

For any type X, consider the universal constant map on X,

$$\beta : X \to S(X),$$

defined as a HIT with higher constructor

$$\ell : \prod_{x,y:X} \beta x = \beta y.$$

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[dr, "\text{constant } f" ', "\text{with modulus } k : \prod_{x,y} fx = fy" '] & SX \arrow[d, dashed, "\bar{f} \text{ such that } \operatorname{ap} \bar{f}(\ell x y) = kxy"] \\
& A
\end{tikzcd}$$

When X is the terminal type 1, we get the circle S^1.

Universal property of the constancy HIT

$$\begin{aligned} \beta &: X \to S(X), \\ \ell &: \prod_{x,y:X} \beta x = \beta y. \end{aligned}$$

There is an equivalence

$$\begin{aligned} SX \to A &\cong \sum_{f:X\to A} \text{constant } f \\ g &\mapsto (g \circ \beta,\ \lambda xy.\, \mathrm{ap}\, g\, (\ell xy)). \end{aligned}$$

This generalizes the universal property of the circle

$$\begin{aligned} S^1 \to A &\cong \sum_{a:A} a = a \\ &\cong \sum_{f:1\to A} \text{constant } f. \end{aligned}$$

Side remark

(Not used in the proof, at least not explicitly.)

1. The universal constant map $\beta_X : X \to S(X)$ is a surjection.

2. The type $S(X)$ is conditionally connected, meaning that

$$\prod_{s,t:S(X)} \|s = t\|.$$

   ("Conditionally" because it is empty if (and only if) X is empty.)

`cubicaltt` proof

Demonstrate and discuss some fragments of the `geometryOfConstancy.ctt` file (on my papers web page).

The constant factorization problem

Because the universal map $X \to \|X\|$ into a proposition is constant (in a unique way), the universal property of $S(X)$ gives a function

$$\prod_{X:U} S(X) \to \|X\|.$$

The existence of a function in the other direction,

$$\prod_{X:U} \|X\| \to S(X),$$

is equivalent to the statement that all constant functions $f : X \to A$ factor through $X \to \|X\|$.

But we know that this is not the case, by Shulman's construction.

However, this does hold for $X = (s = \mathrm{base})$ and all A.

Step 3

By (1) applied to the family $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ of constant functions given by (2), we get a function

$$\bar{\beta} : \prod_{s:S^1} S(s = \mathrm{base}).$$

This is perhaps surprising, because we don't have, of course,

$$\prod_{s:S^1} s = \mathrm{base},$$

as that would mean that that the circle is contractible.

How come we are able to pick a point of the generalized circle $S(s = \mathrm{base})$, without being able to pick a point of the path space $s = \mathrm{base}$, naturally in $s : S^1$?

Step 4

Now, given a single constant function $f : s = \mathrm{base} \to A$, it factors through the universal constant map $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ as $f' : S(s = \mathrm{base}) \to A$ by (2), and hence we get the required point of A using (3), as

$$a \stackrel{\mathrm{def}}{=} f'(\bar{\beta}(s)).$$

Theorem

$$\begin{tikzcd} s = \mathrm{base} \arrow[r] \arrow[dr, "\text{constant } f"'] & 1 \arrow[d, dashed, "a"] \\ & A \end{tikzcd}$$

Conjecture

In a type theory with $\| - \|$ and (hence) function extensionality.

All constant functions $f : X \to A$ of any two types factor through $X \to \|X\|$ if and only if all types are sets (zero-truncated).

And hence univalence fails if all constant functions factor through the truncations of their domains.

(Shulman's construction exhibits a family of constant functions such that if all of them factor through the truncation of their domain, then univalence fails.)