Function-space compactifications of function spaces

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Based on the paper published with the same title in
Topology and its applications, 120(2002)441-463, with some new results.

The problem

For topological spaces X and Y, the *set* of continuous maps

$$\mathrm{C}(X, Y)$$

under the compact-open topology gives rise to a function *space*

$$Y^X.$$

Even if X and Y are compact Hausdorff, Y^X fails badly to be locally compact, e.g. $X = Y = [0, 1]$. Every compact set has empty interior.

We wish to embed Y^X into a compact Hausdorff space of lower complexity than its Stone–Cech compactification.

Constraints

The function space Y^X has to be completely regular for any X, and hence the space Y itself has to be completely regular (consider $X = 1$).

In our method of proof, X has to be exponentiable, which means that Y^X obeys the laws of exponentiation.

Essentially, exponentiable $=$ locally compact. More soon.

Strategy

Construct a space D having Y as a (homeomorphically) embedded subspace.

The exponential law gives an embedding $Y^X \hookrightarrow D^X$.

Show that the topology of D^X can be refined to a compact Hausdorff topology in such a way that this remains an embedding.

NB. The space D cannot possibly be Hausdorff.

However, considering $X = 1$, the topology of D must have a compact Hausdorff refinement.

General construction of D

We modify a construction of the Stone–Cech compactification of Y.

Let I and $\underline{I}$ be $[0,1]$ respectively endowed with the usual topology and with the topology of lower semicontinuity with opens $(x,1]$.

Let D be the $\mathrm{C}(Y,I)$-fold product $\underline{I}^{\mathrm{C}(Y,I)}$.

The evaluation map that sends y to $g \mapsto g(y)$ is an embedding $Y \hookrightarrow D$.

We shall argue that it satisfies the requirements of our strategy.

Remark on the general construction of D

We have modified the construction of the Stone–Cech compactification of Y to get a compactification of Y^X.

Our construction gives a compact-Hausdorff refinement of the topology of $(\underline{I}^{\mathrm{C}(Y,I)})^X$ for which the embedding $Y^X \hookrightarrow (\underline{I}^{\mathrm{C}(Y,I)})^X$ remains an embedding.

By the exponential law, $(\underline{I}^{\mathrm{C}(Y,I))})^X \cong \underline{I}^{X \times \mathrm{C}(Y,I)}$, where the set $\mathrm{C}(Y,I)$ is regarded as a discrete space.

Hence the above embedding can be regarded as an embedding $Y^X \hookrightarrow \underline{I}^{X \times \mathrm{C}(Y,I)}$ that sends f to the functional $(x,g) \mapsto g(f(x))$.

Thus, one has a sort of parametric Stone–Cech compactification $\beta(X,Y)$.

Remainder of the talk

Outline of proof that the construction of D works.

But we must first digress to introduce the theory of continuous lattices.

Alternative constructions of D for particular types of spaces Y and corollaries.

Continuous lattices

Three independent threads that lead to them.

1. Exponentiable spaces.

2. Injective spaces.

3. Compact Hausdorff semilattices.

And the connections between them that we need for our purposes.

Continuous lattices 1 — exponentiable spaces

A homotopy

$$H\colon [0,1]\times X \to Y$$

could be seen as a path

$$\overline{H}\colon [0,1] \to \mathrm{C}(X,Y)$$

if one could topologize $\mathrm{C}(X,Y)$ in such a way that H is continuous iff $\overline{H}$ is continuous.

Hurewicz asked to Fox in the 1930's. Can we topologize $\mathrm{C}(X,Y)$ in this way? Perhaps with $[0,1]$ generalized to an arbitrary parameter space P.

The exponential law

Definition. A function space Y^X satisfies the exponential law if continuity of a function $f\colon P \times X \to Y$ is equivalent to that of $\overline{f}\colon P \to Y^X$ defined by

$$\overline{f}(p) = (x \mapsto f(p, x)).$$

Definition. X is called exponentiable if for every space Y there exists a function space Y^X for which the exponential law holds for all spaces P.

Lemma (internal exponential law). If P and X are exponentiable then so is $P \times X$; moreover, the map $f \mapsto \bar{f}$ is a homeomorphism

$$Y^{P \times X} \cong (Y^X)^P.$$

Intrinsic characterizations for some spaces

Theorem (Fox 1945).

A separable metric space is exponentiable iff it is locally compact.

In this case, the exponential topology is the compact-open topology.

Other authors extended this beyond separable metric spaces.

Intrinsic characterization for all spaces

Was eventually given in 1970 by Day and Kelley.

Isbell put it in the following form, based on work by Scott (1972) which was independent of Day and Kelley.

Definition. A space is called core-compact iff every neighbourhood U of a point x contains a neighbourhood U' of x such that every open cover of U has a finite subcover of U'.

Theorem. *A space X is exponentiable iff it is core-compact.*

The generalized compact-open topology

One writes $U' \ll U$ to mean that every open cover of U has a finite subcover of U'.

Core-compactness amounts to $U = \bigcup\{U' \mid U' \ll U\}$ for every open U.

For X core-compact, the topology of the exponential Y^X has a subbase of open sets of the form

$$N(U,V) = \{f \in \mathrm{C}(X,Y) \mid U \ll f^{-1}(V)\},$$

where $U \subseteq X$ and $V \subseteq Y$ are open.

Core-compactness as a lattice-theoretic concept

For elements u and u' of a complete lattice, define

$u' \ll u$ if every cover of u has a finite subcover of u'.

That is, for every set C with $u \leq \bigvee C$ there is a finite subset C' with $u' \leq \bigvee C'$.

Definition (Scott 1972). A continuous lattice is a complete lattice s.t. for every element u,

$$u = \bigvee \{u' \mid u' \ll u\}.$$

With this terminology. *A space is exponentiable iff its topology is a continuous lattice.*

Continuous lattices 2 — injective spaces

Definition (Scott 1972). A space D is called injective if it has the following strong extension property for D-valued continuous maps:

Every continuous map $f\colon X \to D$ extends to a continuous map $\bar{f}\colon \bar{X} \to D$ for any space $\bar{X}$ having X as an embedded subspace.

$$
\begin{tikzcd}
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, dotted, "\bar{f}"] \\
& D &
\end{tikzcd}
$$

Equivalently, D is a retract of every space of which it is a subspace.

The Scott topology of a continuous lattice

An upper set U of a complete lattice is Scott open if every cover of a member u of U has a finite subcover of member $u' \leq u$ of U.

In a continuous lattice, the sets $\twoheaduparrow u = \{v \mid u \ll v\}$ form a base.

Example. $[0,1]$ is a continuous lattice with Scott topology coinciding with the topology of lower semicontinuity: $\twoheaduparrow x = (x, 1]$.

Theorem (Scott 1972). *The injective spaces are precisely the continuous lattices under the Scott topology.*

The extension property

In a situation

$$
\begin{tikzcd}
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, dotted, "\bar{f}"] \\
& D, &
\end{tikzcd}
$$

define

$$\bar{f}(\bar{x}) = \bigvee_{\bar{x} \in \bar{U} \in \mathcal{O}\,\bar{X}} \bigwedge f[j^{-1}(\bar{U})].$$

This gives the largest Scott continuous extension in the pointwise order.

The specialization order on the points of a space

Definition. $x \leq y$ iff every neighbourhood of x is a neighbourhood of y.

Equivalently, x belongs to the closure of $\{y\}$.

For a Hausdorff space, this is the identity.

In fact, a space is T_1 iff its specialization order is the identity.

Example. For $[0, 1]$ under the topology of lower semicontinuity, specialization order $=$ natural order.

If D is an injective space, its points under the specialization order form a continuous lattice.

Relating the first two threads

Lemma. If X is exponentiable and D is injective then D^X is injective.

A simple proof uses only the definitions, without reference to the continuous-lattice characterizations.

We need the following, which does rely on the characterizations:

Corollary (Keimel and Gierz 1982). *The exponential topology of D^X is the Scott topology of the pointwise order on continuous maps $X \to D$.*

Continuous lattices 3 — topological semilattices

Definition. A topological semilattice is a semilattice L with a topology on L making

$$\begin{array}{rcl} L \times L & \to & L \\ (x, y) & \mapsto & x \wedge y \end{array}$$

into a continuous map. (In this talk we include the empty meet, a top element.)

Two crucial examples.

1. The unit interval $[0, 1]$ under its natural order and Euclidean topology.

2. The Vietoris hyperspace of a compact Hausdorff space under the *reverse* inclusion order. (So meets are unions.)

Lawson semilattices

Theorem (Lawson 1969).

TFAE for any compact Hausdorff semilattice:

1. *Its continuous homomorphisms into the unit interval separate points.*

2. *Every point has a neighbourhood base of subsemilattices.*

(Cf. Locally convex topological vector spaces over the field of reals.)

Definition. A Lawson semilattice is such a compact Hausdorff semilattice.

The Lawson topology

Theorem (Lawson 1973). *A semilattice admits at most one topology making it into a compact Hausdorff topological semilattice.*

Definition. This topology, when it exists, is called the Lawson topology.

Theorem (SCSL mid 70's). *The Lawson semilattices are precisely the continuous lattices under the Lawson topology.*

For a continuous lattice, the Lawson topology is the least refinement of the Scott topology for which principal ideals $\uparrow x = \{y \mid x \leq y\}$ are closed.

Examples of Scott and Lawson

1. $[0, 1]$ under natural order.

   Scott topology $=$ the topology of lower semicontinuity.

   Lawson topology $=$ usual topology.

2. The open sets of a compact Hausdorff space form a continuous lattice.

Hence so do the closed sets under *reverse* inclusion.

   Scott topology $=$ hit topology.

   Lawson topology $=$ hit-and-miss topology $=$ Vietoris topology.

Continuous lattices concluded

The first and second threads have been linked:

If X is exponentiable and D is injective then D^X is injective.

Hence D^X is a continuous lattice under the Scott topology.

The order is the pointwise order on continuous maps.

Linking this to the third:

Hence the topology of D^X has a compact Hausdorff refinement.

A technical lemma

We need a concrete description of the Lawson topology of D^X.

Lemma.

The Lawson topology of D^X is the least refinement of the Scott topology for which the sets of the form

$$\{f \in D^X \mid U \subseteq f^{-1}(\uparrow d)\}$$

are closed, where $U \subseteq X$ is open and $d \in D$.

Don't pay much attention to this — just observe that this shows that the Lawson topology of the function space D^X is "manageable".

Back to function-space compactifications

Easy lemma. TFAE for any embedding $j\colon Y \to D$ into an injective space.

1. j is also an embedding w.r.t. the Lawson topology of D.

2. j is also continuous w.r.t. the Lawson topology of D.

Definition. Such an embedding is called strong.

We now have all the necessary ingredients to establish:

Main Theorem. *If an embedding $Y \hookrightarrow D$ is strong, then so is the induced embedding $Y^X \hookrightarrow D^X$ for every exponentiable space X.*

This would be useless if strong embeddings didn't exist.

Strongly embedded subspaces of injective spaces

Theorem. *The strongly embedded subspaces of the injective spaces are precisely the completely regular spaces.*

One direction is immediate, because the Lawson topology is compact Hausdorff.

For the other direction, we go back to the beginning of this talk.

We strongly embed Y into the $\mathrm{C}(Y, I)$-fold product $D = \underline{I}^{\mathrm{C}(Y,I)}$ where I and $\underline{I}$ are $[0, 1]$ under the Scott and Lawson topologies respectively.

The details are omitted but the necessary ingredients have been given.

Other constructions of strong embeddings $Y \hookrightarrow D$

Are available for particular cases of interest and give interesting corollaries.

1. $Y = \mathbb{R}$ with the usual Euclidean topology.
2. Y compact Hausdorff.
3. Y metrizable.

Example 1 — the real line

$Y = \mathbb{R}$ with the usual Euclidean topology.

Let $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ be $[-\infty, \infty]$ under the topologies of lower and upper semicontinuity.

We consider the product $D = \underline{\mathbb{R}} \times \overline{\mathbb{R}}$, which is an injective space.

$x \mapsto (x, x)$ is a strong embedding $\mathbb{R} \hookrightarrow \underline{\mathbb{R}} \times \overline{\mathbb{R}}$.

But $(\underline{\mathbb{R}} \times \overline{\mathbb{R}})^X \cong \underline{\mathbb{R}}^X \times \overline{\mathbb{R}}^X$.

Hence we have a strong embedding $\mathbb{R}^X \hookrightarrow \underline{\mathbb{R}}^X \times \overline{\mathbb{R}}^X$.

We compactify by taking pairs of semicontinuous functions.

Example 2 — compact Hausdorff spaces

Assume that Y is compact Hausdorff.

Let S be the two-point Sierpinski space with a *closed* point "true" and an *open* point "false". This classifies closed subspaces.

$Y \hookrightarrow S^Y$ that sends y to $y' \mapsto [\![y = y']\!]$ is a strong embedding.

Hence we get a strong embedding $Y^X \hookrightarrow (S^Y)^X$.

By the exponential law, $(S^Y)^X \cong S^{X \times Y}$.

Hence, if X is also compact Hausdorff, the Lawson topology of $S^{X \times Y}$ makes it homeomorphic to the Vietoris hyperspace $\mathrm{V}(X \times Y)$.

This amounts to the familiar graph compactification $Y^X \hookrightarrow \mathrm{V}(X \times Y)$.

Example 3 — metric spaces

Assume that Y is locally compact and metrized by d.

Endow $[0, \infty]$ with the topology of upper semicontinuity. Think of 0 as "true" and of other points as degrees of closeness to truth.

$Y \hookrightarrow [0, \infty]^Y$ that sends y to $y' \mapsto d(y, y')$ is a strong embedding.

By the exponential law, $([0, \infty]^Y)^X = [0, \infty]^{X \times Y}$.

Hence we get a strong embedding $Y^X \hookrightarrow [0, \infty]^{X \times Y}$.

It sends f to $g(x, y) = d(f(x), y)$.

We compactify by taking "real-valued graphs".

The end

The general construction of the strong embedding $Y \hookrightarrow D$ presented here is new.

A different construction is found in

M.H. Escardó. Function-space compactifications of function spaces. *Topology and its applications*, 120(2002)441-463.

The continuous-lattice story is told in more detail in Section 2 of the above, and is developed in

Gierz, Hofmann, Keimel, Lawson, Mislove and Scott. *A compendium of continuous lattices.* Springer-Verlag, 1980.

and in the "new compendium" by the same authors:

Continuous lattices and domains. Cambridge University Press, 2003.

Addendum

Notice the interaction between the Scott and Lawson topologies in the main theorem.

We take the Lawson topology on the Scott continuous maps.