Topologies on spaces of continuous functions

Martín Escardó

School of Computer Science, University of Birmingham

Based on joint work with Reinhold Heckmann

7th November 2002

$\boxed{\text{Path}}$

Let x_0 and x_1 be points of a space X.

A path from x_0 to x_1 is a continuous map

$$f : [0, 1] \to X$$

with

$$f(0) = x_0 \qquad \text{and} \qquad f(1) = x_1.$$

Homotopy

Let $f_0, f_1\colon X \to Y$ be continuous maps.

A homotopy from f_0 to f_1 is a continuous map

$$H\colon [0,1] \times X \to Y$$

with

$$H(0,x) = f_0(x) \qquad \text{and} \qquad H(1,x) = f_1(x).$$

$\boxed{\text{Homotopies as paths}}$

Let $\mathsf{C}(X, Y)$ denote the set of continuous maps from X to Y.

Then a homotopy

$$H\colon [0,1] \times X \to Y$$

can be seen as a path

$$\overline{H}\colon [0,1] \to \mathsf{C}(X, Y)$$

from f_0 to f_1.

A continuous deformation of f_0 into f_1.

$\boxed{\text{Really?}}$

This would be true if one could topologize $\mathsf{C}(X, Y)$ in such a way that a function

$$H\colon [0,1] \times X \to Y$$

is continuous if and only if

$$\overline{H}\colon [0,1] \to \mathsf{C}(X, Y)$$

is continuous.

This is the question Hurewicz posed to Fox in 1930.

Is it possible to topologize $\mathsf{C}(X, Y)$ in such a way that H is continuous if and only if $\bar{H}$ is continuous?

We are interested in the more general case when $[0,1]$ is replaced by an arbitrary topological space A.

Fox published a partial answer in 1945.

The transpose of a function of two variables

The transpose of a function

$$g\colon A \times X \to Y$$

is the function

$$\overline{g}\colon A \to \mathsf{C}(X, Y)$$

defined by

$$\overline{g}(a) = g_a \in \mathsf{C}(X, Y)$$

where

$$g_a(x) = g(a, x).$$

More concisely,

$$\overline{g}(a)(x) = g(a, x).$$

Weak, strong and exponential topologies

A topology on the set $\mathsf{C}(X, Y)$ is called

weak if $g\colon A \times X \to Y$ continuous $\Longrightarrow$ $\overline{g}\colon A \to \mathsf{C}(X, Y)$ continuous,

strong if $\overline{g}\colon A \to \mathsf{C}(X, Y)$ continuous $\Longrightarrow$ $g\colon A \times X \to Y$ continuous,

exponential if it is both weak and strong.

Thus a topology on $\mathsf{C}(X, Y)$ is exponential if and only if it makes transposition into a well-defined bijection

$$\begin{array}{rcl} \mathsf{C}(A \times X, Y) & \to & \mathsf{C}(A, \mathsf{C}(X, Y)) \\ g & \mapsto & \bar{g} \end{array}$$

More standard terminologies for *weak* and *strong* are *splitting* and *conjoining* respectively.

$\boxed{\text{Strong} \iff \text{evaluation continuous}}$

Lemma

A topology on $\mathsf{C}(X,Y)$ is strong if and only if it makes the evaluation map

$$\begin{array}{rrcl} \varepsilon_{X,Y}\colon & \mathsf{C}(X,Y)\times X & \to & Y \\ & (f,x) & \mapsto & f(x) \end{array}$$

into a continuous function.

Proof. The transpose $\overline{\varepsilon}\colon \mathsf{C}(X,Y)\to\mathsf{C}(X,Y)$ of the evaluation map is continuous for any topology on $\mathsf{C}(X,Y)$ because $\overline{\varepsilon}(f)(x)=f(x)$ for all x and hence $\overline{\varepsilon}(f)=f$.

This shows that evaluation is continuous if the topology on $\mathsf{C}(X,Y)$ is strong.

Conversely, assume that evaluation is continuous for a given topology on $\mathsf{C}(X,Y)$ and let $g\colon A\times X\to Y$ be a map with a continuous transpose $\overline{g}\colon A\to\mathsf{C}(X,Y)$.

Then g is also continuous because $g(a,x)=\overline{g}(a)(x)=\varepsilon(\overline{g}(a),x)=\varepsilon\circ(\overline{g}\times\mathrm{id}_X)(a,x)$ and hence g is a composition $\varepsilon\circ(\overline{g}\times\mathrm{id}_X)$ of continuous maps. Q.E.D.

Justifying our terminology

Lemma

1. *Any weak topology is weaker than any strong topology.*

2. *Any topology weaker than a weak topology is also weak.*

3. *Any topology stronger than a strong topology is also strong.*

In particular, there is at most one exponential topology.

When it exists, it is the weakest strong topology, or, equivalently, the strongest weak topology.

Proof. Only the fact that any weak topology is weaker than any strong topology is not immediate.

Endow $\mathsf{C}(X,Y)$ with a weak and a strong topology, obtaining spaces $\mathsf{W}(X,Y)$ and $\mathsf{S}(X,Y)$ respectively. Then the evaluation map $\varepsilon\colon \mathsf{S}(X,Y)\times X\to Y$ is continuous

By definition of weak topology, its transpose $\overline{\varepsilon}\colon \mathsf{S}(X,Y)\to\mathsf{W}(X,Y)$ is also continuous.

But we have seen that $\overline{\varepsilon}(f)=f$. Therefore $O=\overline{\varepsilon}^{-1}(O)\in\mathcal{O}\,\mathsf{S}(X,Y)$ for every $O\in\mathcal{O}\,\mathsf{W}(X,Y)$.
Q.E.D.

Exponentiable space

A space X is called exponentiable if the set $\mathsf{C}(X,Y)$ admits an exponential topology for *every* space Y.

In this case, the set $\mathsf{C}(X,Y)$ endowed with the exponential topology is usually denoted by

$$Y^X$$

and referred to as an exponential.

The problem tackled in this talk is to develop a criterion for exponentiability and an explicit construction of exponential topologies.

$\boxed{\text{Topologies on lattices of open sets}}$

We now reduce the exponentiability problem to a simpler problem.

There is a *single* space $\mathbb{S}$ with the property that X is exponentiable if and only if $\mathsf{C}(X, \mathbb{S})$ has an exponential topology.

Moreover, in this case, the exponential topology of $\mathsf{C}(X, Y)$ is uniquely determined by the exponential topology of $\mathsf{C}(X, \mathbb{S})$ and by the topology of Y in a simple fashion.

The Sierpinski space

The Sierpinski space is the space $\mathbb{S}$ with two points 0 and 1 such that $\{1\}$ is open but $\{0\}$ is not.

The map $f \mapsto f^{-1}(1)$ is a bijection from $\mathsf{C}(X, \mathbb{S})$ to $\mathcal{O}\,X$.

A topology on $\mathcal{O}\,X$ is exponential if it is induced by an exponential topology on $\mathsf{C}(X, \mathbb{S})$ via the bijection.

Exponential topologies on lattices of open sets

Explicitly, a topology on $\mathcal{O}\,X$ is

strong: The graph

$$\varepsilon_X = \{(U, x) \in \mathcal{O}\,X \times X \mid x \in U\}$$

of the membership relation is open.

weak: For each

$$W \in \mathcal{O}\,(A \times X),$$

the function

$$\overline{W}\colon A \to \mathcal{O}\,X$$

defined by

$$\overline{W}(a) = \{x \in X \mid (a, x) \in W\}$$

is continuous.

Induced topology

Let T be a topology on $\mathcal{O}X$.

The induced topology on $\mathsf{C}(X, Y)$ is generated by the subbasic open sets

$$N_T(O, V) = \{f \in \mathsf{C}(X, Y) \mid f^{-1}(V) \in O\},$$

where

O ranges over T

V ranges over $\mathcal{O}Y$.

Example Let T be topology on $\mathcal{O}\,X$ whose subbasic open sets are of the form

$$O_Q = \{U \in \mathcal{O}\,X \mid Q \subseteq U\}$$

for $Q \subseteq X$ compact.

Then the induced topology is the compact open topology:

$f \in N_T(O_Q, V)$

$$\iff f^{-1}(V) \in O_Q$$

$$\iff Q \subseteq f^{-1}(V)$$

$$\iff f(Q) \subseteq V.$$

Exponentiality of the induced topology

Lemma

Let X be a topological space and T be a topology on $\mathcal{O}\,X$.

1. *T is weak if and only if it induces a weak topology on $\mathsf{C}(X, Y)$ for every Y.*

2. *T is strong if and only if induces a strong topology on $\mathsf{C}(X, Y)$ for every Y.*

3. *T is exponential if and only if it induces an exponential topology on $\mathsf{C}(X, Y)$ for every Y.*

Proof. (1)($\Leftarrow$) and (2)($\Leftarrow$): Take $Y = \mathbb{S}$ and observe that $\mathsf{C}(X, \mathbb{S})$ endowed with the topology induced by T is homeomorphic to $\mathcal{O}\,X$ endowed with T.

(1)($\Rightarrow$) and (2)($\Rightarrow$): Routine calculations.

(3): Immediate consequence of (1) and (2). Q.E.D.

Reduction of the problem

Corollary
A space X is exponentiable if and only if $\mathcal{O}\,X$ has an exponential topology.

In this case, the exponential topology of $\mathsf{C}(X, Y)$ is the topology induced by the exponential topology of $\mathcal{O}\,X$.

Thus,

> In order to know how to topologize $\mathsf{C}(X, Y)$ for arbitrary Y,
>
> it suffices to know how to topologize $\mathsf{C}(X, \mathbb{S}) \cong \mathcal{O}\,X$.

This reduction is usually performed using injective spaces, combining ideas from domain theory and category theory.

Spaces with exponential topologies on the opens

The discrete and indiscrete topologies of $\mathcal{O}\,X$ are strong and weak respectively.

We begin by improving these bounds.

A strong topology

A set $O \subseteq \mathcal{O}\,X$ is called Alexandroff open if the conditions $U \in O$ and $U \subseteq V \in \mathcal{O}\,X$ together imply that $V \in O$.

Lemma

The Alexandroff topology is strong.

In particular, any open set in a weak topology is Alexandroff open.

Proof. We have to show that the membership relation $\varepsilon_X \subseteq \mathcal{O}\,X \times X$ is open.

Let $(U, x) \in \varepsilon_X$. Then $(U, x) \in \{V \in \mathcal{O}\,X \mid U \subseteq V\} \times U$, which is a product of an Alexandroff open subset of $\mathcal{O}\,X$ with an open subset of X and hence is an open rectangle.

This product is contained in ε_X, which shows that ε_X is open, as required. Q.E.D.

A weak topology

An Alexandroff open set $O \subseteq \mathcal{O}\,X$ is called Scott open if every open cover of a member of O has a finite subcover of a member of O.

Example For any $Q \subseteq X$, the Alexandroff open set $\{V \in \mathcal{O}\,X \mid Q \subseteq V\}$ is Scott open if and only if Q is compact.

(For many spaces, in fact, the Scott topology of the opens is generated by these subbasic opens. This is the case, for example, for Hausdorff spaces.)

Lemma
The Scott topology is weak.

Proof. Let $W \subseteq A \times X$ be open. We have to show that $\overline{w}\colon A \to \mathcal{O}\,X$ is continuous.

Let $a \in A$ and let $O \subseteq \mathcal{O}\,X$ be a Scott open neighbourhood of $\overline{w}(a)$.

For each $x \in \overline{w}(a)$, that is, $(x, a) \in W$, there are $U_x \in \mathcal{O}\,A$ and $V_x \in \mathcal{O}\,X$ with $(a, x) \in U_x \times V_x \subseteq W$, by openness of W in the product topology.

Since $\overline{w}(a)$ is the union of the sets V_x and since O is Scott open, the union V of finitely many such V_x belongs to O.

Let U be the intersection of the corresponding open sets U_x. Clearly, U is a neighbourhood of a.

To conclude the proof, we show that $\overline{w}(u) \in O$ for each $u \in U$.

To this end, it is enough to show that $V \subseteq \overline{w}(u)$, because O is Alexandroff open and we know that $V \in O$.

Let $v \in V$. Then $v \in V_x$ for some $x \in \overline{w}(a)$. Since $u \in U_x$, we have that $(u, v) \in U_x \times V_x \subseteq W$.

Therefore $v \in \overline{w}(u)$, as required. Q.E.D.

$\boxed{\text{Bound situation so far}}$

discrete (strong)

$\Cup$

Alexandroff (strong)

$\Cup$

exponential (if it exists)

$\Cup$

Scott (weak)

$\Cup$

indiscrete (weak)

$\boxed{\text{Closer look at strong topologies}}$

Let T be a topology on $\mathcal{O}X$.

Let $U, V \in \mathcal{O}X$.

$\uparrow U \stackrel{\text{def}}{=} \{W \in \mathcal{O}X \mid U \subseteq W\}$.

$U \prec_T V \stackrel{\text{def}}{\iff} V \in \text{int}_T \uparrow U$.

"U is a T-approximant of V."

T is approximating if every open is the union of its T-approximants.

Lemma
A topology on $\mathcal{O}X$ is strong if and only if it is approximating.

Proof. A routine (un)folding of definitions! What is difficult about this step is to come up with the notion of approximating topology.

The strongest weak topology

We use this in order to prove the following.

Lemma

The Scott topology is the intersection of the strong topologies.

Before giving the proof, let's summarize the situation so far:

exponential $\iff$ both weak and strong

Scott is weak

strong $\iff$ approximating

Scott $=$ intersection of strong.

Scott $=$ strongest weak topology

Corollary

$\mathcal{O}X$ has an exponential topology if and only if its Scott topology is approximating.

In this case, the exponential topology is the Scott topology.

Main theorem

Combining the previous corollary with the reduction performed before, we arrive at our main theorem.

The topology on $\mathrm{C}(X, Y)$ induced by the Scott topology of $\mathcal{O}X$ is known as the Isbell topology.

Theorem
A space is exponentiable if and only if the Scott topology of its lattice of open sets is approximating.

Moreover, the topology of an exponential is the Isbell topology.

Observation

The definition of exponentiability quantifies over all topological spaces.

The criterion provided by this theorem reduces exponentiability of a space X to an intrinsic property of X.

Namely the approximation property of the Scott topology of $\mathcal{O}X$.

A related criterion, which avoids considering a topology on the topology $\mathcal{O}X$ of X, will be sketched soon.

We need to prove the last lemma that gave rise to the main theorem.

Proof. Being weak, the Scott topology is contained in the intersection of the strong topologies.

Conversely, for each $\mathcal{C} \subseteq \mathcal{O}\,X$, let $T_{\mathcal{C}}$ be the set of all Alexandroff open subsets O of $\mathcal{O}\,X$ with the property that if $\mathcal{C}$ covers a member of O then $\mathcal{C}$ has a finite subcover of a member of O.

This is easily seen to be a topology on $\mathcal{O}\,X$, and, by construction, the Scott topology is the intersection of all such topologies.

To conclude the proof, it suffices to show that they are strong. We use the approximation criterion.

Assume that $x \in U \in \mathcal{O}\,X$. We show that there is some $V \in \mathcal{O}\,X$ with $x \in V \prec_{T_{\mathcal{C}}} U$.

If $x \notin \bigcup \mathcal{C}$, then $\uparrow U = \{V \in \mathcal{O}\,X \mid U \subseteq V\} \in T_{\mathcal{C}}$, and so $x \in U \prec_{T_{\mathcal{C}}} U$. Hence we can take $V = U$

If, on the other hand, $x \in U'$ for some $U' \in \mathcal{C}$, then $\{V \in \mathcal{O}\,X \mid U' \cap U \subseteq V\} \in T_{\mathcal{C}}$, whence $x \in (U' \cap U) \prec_{T_{\mathcal{C}}} U$. Hence we can take $V = U' \cap U$, and the proof is concluded. Q.E.D.

Core-compact spaces

We now avoid the use of topologies on a topology. We shall be rather brief.

Let $U, V \in \mathcal{O}X$.

$U \ll V \stackrel{\text{def}}{\iff}$ every open cover of V has a finite subcover of U.

"U is way below V"

Example $U \ll V$ if there is a compact set $Q \subseteq X$ with $U \subseteq Q \subseteq V$.

X is called core-compact if every open is the union of the opens way below it.

Local versus core compactness

This generalizes the notion of local compactness.

For every neighbourhood V of a point x of X there is a compact neighbourhood Q of x with $Q \subseteq V$.

In order to see this, notice that core-compactness can be formulated as:

For every neighbourhood V of a point x of X there is a neighbourhood U of x such that every open cover of V has a finite subcover of U.

For Hausdorff (and more generally for sober) spaces, the two notions coincide.

Moreover, in the Hausdorff case, they further coincide with: Every point has a compact neighbourhood.

$\boxed{\text{Approximation via way below}}$

The relations $U \prec_{\text{Scott}} V$ and $U \ll V$ don't coincide in general, but they do for exponentiable spaces.

Moreover,

Lemma
The Scott topology of $\mathcal{O}X$ is approximating if and only if X is core-compact.

This result belongs to domain-theory land:

A complete lattice is continuous

$\iff$ its Scott topology is approximating

$\iff$ its way-below relation is approximating.

What is interesting here is that

(1) Scott is strong iff the lattice $\mathcal{O}X$ is continuous.

(2) Scott is always weak.

Main theorem, second version

Theorem
A space is exponentiable if and only if it is core-compact.

Moreover, if X is a core-compact space and Y is any space then the topology of the exponential Y^X is generated by the sets

$$N(U,V) \stackrel{\text{def}}{=} \{f \in Y^X \mid U \ll f^{-1}(V)\},$$

where U and V range over $\mathcal{O}X$ and $\mathcal{O}Y$ respectively.

Compare this to the compact-open topology:

$$N(Q,V) \stackrel{\text{def}}{=} \{f \in Y^X \mid Q \subseteq f^{-1}(V)\},$$

where Q and V range over compact and opens of X and Y respectively.

$\boxed{\text{What is not new}}$

The result itself.

The ingredients in general.

$\boxed{\text{What is new}}$

Their organization.

The purely topological exposition.

The idea of using the generalized Isbell topology.

The idea of using approximating topologies.

(The latter, we found, occurs in disguise in Day and Kelly's paper after all.)

The End