ON TOPOLOGIES FOR GENERAL FUNCTION SPACES

MARTÍN ESCARDÓ AND REINHOLD HECKMANN

ABSTRACT. It is well-known that the exponentiable Hausdorff spaces are precisely the locally compact spaces, and that the exponential topology is the compact-open topology. Since non-Hausdorff spaces are often regarded as uninteresting and not very well-behaved, it is less well-known that among arbitrary topological spaces, the exponentiable spaces are precisely the core-compact spaces. While the spaces considered in analysis happen to be Hausdorff, interesting and quite well-behaved non-Hausdorff spaces arise frequently in applications of topology to algebra via Stone duality and to the theory of computation. As function spaces play a fundamental rôle in the theory of computation, it is important to have exponentiability criteria for general spaces. The available approaches to the general characterization are based on either category theory or continuous-lattice theory, or even both. It is the purpose of this expository note to provide a self-contained, elementary and brief development of general function spaces. The only prerequisite is a basic knowledge of topology (continuous functions, product topology and compactness).

INTRODUCTION

A topological space X is exponentiable if for every space Y there is a topology on the set Y^X of continuous maps $X \to Y$ such that for any space A there is a natural bijection from the set of continuous maps $A \times X \to Y$ to the set of continuous maps $A \to Y^X$. This is elaborated in Section 1. It is known that a space X is exponentiable if and only if it is core-compact, in the sense that any given open neighbourhood V of a point x of X contains an open neighbourhood U of x with the property that every open cover of V has a finite subcover of U. Moreover, if X is exponentiable and Y is any space, then the open sets of the function space Y^X are generated by the sets

$$\{f \in Y^X \mid \text{every open cover of } f^{-1}(V) \text{ has a finite subcover of } U\},$$

where U and V range over open subsets of X and Y respectively.

In applications of function spaces Y^X to analysis, the spaces X and Y are usually Hausdorff and X is usually locally compact, and this is the level of generality considered in books on topology [3, 10]. In this case, the exponential topology coincides with the more familiar compact-open topology. Moreover, for Hausdorff spaces, local-compactness is the same as core-compactness. However, in applications of topology to algebra via Stone duality [8] and to the theory of computation [1, 13, 14, 15, 16], non-Hausdorff spaces arise frequently. In the case of the

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theory of computation, where function spaces play a fundamental rôle, the exponential topologies have alternative descriptions [11].

The above characterization of the exponentiable spaces and the exponential topology has a long history, which is discussed in detail by Isbell [7] and goes back to at least 1945 with the work of Fox [4]. The first general characterization is implicit in the work of Day and Kelly [2], who characterized the spaces X for which the function $q \times \mathrm{id}_X : Y \times X \to Z \times X$ is a quotient map for every quotient map $q : Y \to Z$. It was known to category theorists that, by virtue of the Adjoint Functor Theorem, such spaces coincide with the exponentiable spaces. Day and Kelly's characterization amounts to the fact that the open sets of X form a continuous lattice in the sense of Scott [15]—but continuous lattices were introduced later and independently of the work of Day and Kelly. The above formulation of the characterization has been promoted by Isbell [7].

An alternative proof of the characterization is the following. For core-compact spaces, one shows directly that a certain topology known as the Isbell topology yields an exponential [7, 6]. Conversely, as observed by Johnstone and Joyal [9], if X is an exponentiable space and L is an injective space, then the exponential L^X is also an injective space; by considering the case in which L is Sierpinski space, one sees that the open sets of X have to form a continuous lattice, as Scott characterized the injective spaces as the continuous lattices endowed with the Scott topology.

It is the purpose of this expository note to provide a self-contained, elementary and brief development of the characterization of the exponentiable spaces as the core-compact spaces. In particular, we refrain from appealing to results from the theories of continuous lattices [5] and categories [12]. We would like, however, to motivate the reader to learn how this presentation relates to these theories. The only prerequisite to this note is a basic knowledge of topology (continuous functions, product topology and compactness). Separation axioms are not needed. As far as we know, there is no such development available in the literature. Although there are one or two embellishments, our methods are certainly not original.

This note is organized as follows. We formulate the exponentiability problem in Section 1. We then reduce it to a simpler problem in Section 2, which is solved in Section 3. Finally, in Section 4, we reformulate the solution obtained by the reduction process as the solution stated in the opening paragraph of this introduction.

Notation and terminology. The lattice of open sets of a topological space X is denoted by $\mathcal{O}X$. A topology T on a given set is ***weaker*** than another topology T' on the same set if $T \subseteq T'$. In this case we also say that the topology T' is ***stronger*** than the topology T.

1. Topologies on spaces of continuous functions

For topological spaces X and Y, we denote by $\mathrm{C}(X, Y)$ the set of continuous maps from X to Y. The ***transpose*** $\overline{g} : A \to \mathrm{C}(X, Y)$ of a continuous map $g : A \times X \to Y$ is defined by

$$\overline{g}(a) = g_a, \text{ where } g_a \in \mathrm{C}(X, Y) \text{ is given by } g_a(x) = g(a, x).$$

More concisely, we write the definition of the transpose as

$$\overline{g}(a)(x) = g(a, x).$$

A topology on the set $\mathrm{C}(X, Y)$ is

1. ***weak*** if continuity of $g : A \times X \to Y$ implies that of $\overline{g} : A \to \mathrm{C}(X, Y)$,
2. ***strong*** if continuity of $\overline{g} : A \to \mathrm{C}(X, Y)$ implies that of $g : A \times X \to Y$, and
3. ***even*** if it is both weak and strong.

Thus a topology on $\mathrm{C}(X, Y)$ is even iff it makes the transposition operation $g \mapsto \overline{g}$ into a well-defined bijection from $\mathrm{C}(A \times X, Y)$ to $\mathrm{C}(A, \mathrm{C}(X, Y))$. The ***evaluation map*** $\varepsilon_{X,Y} : \mathrm{C}(X, Y) \times X \to Y$ is defined by

$$\varepsilon_{X,Y}(f, x) = f(x).$$

1.1. *A topology on* $\mathrm{C}(X, Y)$ *is strong iff* $\varepsilon_{X,Y} : \mathrm{C}(X, Y) \times X \to Y$ *is continuous.*

Proof. The transpose $\overline{\varepsilon} : \mathrm{C}(X, Y) \to \mathrm{C}(X, Y)$ of the evaluation map is continuous because $\overline{\varepsilon}(f)(x) = f(x)$ for all x and hence $\overline{\varepsilon}(f) = f$. This shows that evaluation is continuous if the topology on $\mathrm{C}(X, Y)$ is strong. Conversely, assume that evaluation is continuous and let $g : A \times X \to Y$ be a map with a continuous transpose $\overline{g} : A \to \mathrm{C}(X, Y)$. Then g is also continuous because it is a composition $\varepsilon \circ (\overline{g} \times \mathrm{id}_X)$ of continuous maps as $g(a, x) = \overline{g}(a)(x) = \varepsilon(\overline{g}(a), x) = \varepsilon \circ (\overline{g} \times \mathrm{id}_X)(a, x)$, where $\mathrm{id}_X : X \to X$ is the identity. □

1.2. 1. *Any weak topology is weaker than any strong topology.*
2. *Any topology weaker than a weak topology is also weak.*
3. *Any topology stronger than a strong topology is also strong.*

In particular, there is at most one even topology; when it exists, it is the weakest strong topology, or, equivalently, the strongest weak topology.

Proof. Only (1) is not immediate. Endow $\mathrm{C}(X, Y)$ with a weak and a strong topology, obtaining spaces $\mathrm{W}(X, Y)$ and $\mathrm{S}(X, Y)$ respectively. By 1.1, the evaluation map $\varepsilon : \mathrm{S}(X, Y) \times X \to Y$ is continuous, and, by definition of weak topology, its transpose $\overline{\varepsilon} : \mathrm{S}(X, Y) \to \mathrm{W}(X, Y)$ is continuous. But we have seen that $\overline{\varepsilon}(f) = f$. Therefore $O = \overline{\varepsilon}^{-1}(O) \in \mathcal{O}\,\mathrm{S}(X, Y)$ for every $O \in \mathcal{O}\,\mathrm{W}(X, Y)$. □

A space X is ***exponentiable*** if the set $\mathrm{C}(X, Y)$ admits an even topology for every space Y. In this case, the set $\mathrm{C}(X, Y)$ endowed with the even topology is usually denoted by Y^X and referred to as an ***exponential***. The problem tackled in this note is to develop a criterion for exponentiability, and a construction of the topologies of exponentials.

2. Topologies on lattices of open sets

In this section we reduce the exponentiability problem to a simpler problem, which is solved in the next. It turns out that there is a *single* space Σ with the property that X is exponentiable iff $\mathrm{C}(X, \Sigma)$ has an even topology. Moreover, in this case, the even topology of $\mathrm{C}(X, Y)$ is uniquely determined by the even topology of $\mathrm{C}(X, \Sigma)$ and by the topology of Y in a simple fashion.

Sierpinski space is the space Σ with two points 1 and 0 such that $\{1\}$ is open but $\{0\}$ is not. It is well-known and easy to check that the map $f \mapsto f^{-1}(1)$ is a bijection from $\mathrm{C}(X, \Sigma)$ to $\mathcal{O}\,X$. A topology on $\mathcal{O}\,X$ is ***even*** if it is induced by an even topology on $\mathrm{C}(X, \Sigma)$ via the bijection. Explicitly, this means that it is ***strong*** in the sense that the graph

$$\varepsilon_X \stackrel{\text{def}}{=} \{(U, x) \in \mathcal{O}\,X \times X \mid x \in U\}$$

of the membership relation is open, and ***weak*** in the sense that for each $W \in \mathcal{O}(A \times X)$, the function $\overline{w} : A \to \mathcal{O}X$ defined by

$$\overline{w}(a) = \{x \in X \mid (a,x) \in W\}$$

is continuous.

For a topology T on $\mathcal{O}X$, the T***-induced*** topology on $\mathrm{C}(X,Y)$ is generated by the sets

$$T(O,V) \stackrel{\mathrm{def}}{=} \{f \in \mathrm{C}(X,Y) \mid f^{-1}(V) \in O\},$$

where O ranges over T and V ranges over $\mathcal{O}Y$. Notice that $\mathrm{C}(X,\Sigma)$ endowed with the T-induced topology is homeomorphic to $\mathcal{O}X$ endowed with T.

2.1. *Let T be a topology on $\mathcal{O}X$.*

1. *T is weak iff the T-induced topology on $\mathrm{C}(X,Y)$ is weak for every Y.*
2. *T is strong iff the T-induced topology on $\mathrm{C}(X,Y)$ is strong for every Y.*
3. *T is even iff the T-induced topology on $\mathrm{C}(X,Y)$ is even for every Y.*

Proof. Item (3) is an immediate consequence of (1) and (2), and the implications (1)($\Leftarrow$) and (2)($\Leftarrow$) follow from the fact that $\mathrm{C}(X,\Sigma)$ with the T-induced topology is homeomorphic to $\mathcal{O}X$ with T.

(1)($\Rightarrow$): To show that $\overline{g} : A \to \mathrm{C}(X,Y)$ is continuous for $\mathrm{C}(X,Y)$ with the T-induced topology, it is enough to show that $\overline{g}^{-1}(T(O,V))$ is open for $O \in T$ and $V \in \mathcal{O}Y$. Let $W = g^{-1}(V)$. Since T is weak, $\overline{w} : A \to \mathcal{O}X$ is continuous for $\mathcal{O}X$ with T. Thus, in order to conclude the proof, it suffices to show that $\overline{g}^{-1}(T(O,V)) = \overline{w}^{-1}(O)$. This is equivalent to saying that $\overline{g}(a) \in T(O,V)$ iff $\overline{w}(a) \in O$. But we have that $\overline{g}(a) \in T(O,V)$ iff $(\overline{g}(a))^{-1}(V) \in O$. Therefore, the chain of equivalences $x \in (\overline{g}(a))^{-1}(V) \Leftrightarrow g(a,x) \in V \Leftrightarrow (a,x) \in g^{-1}(V) \Leftrightarrow (a,x) \in W \Leftrightarrow x \in \overline{w}(a)$ concludes the proof.

(2)($\Rightarrow$): let V be an open neighbourhood of $\varepsilon_{X,Y}(f,x) = f(x)$. Then $x \in f^{-1}(V)$, which shows that $(f^{-1}(V),x) \in \varepsilon_X$. Since ε_X is open in $\mathcal{O}X \times X$ for $\mathcal{O}X$ endowed with T, there are $O \in T$ and $U \in \mathcal{O}X$ such that $(f^{-1}(V),x) \in O \times U \subseteq \varepsilon_X$. Hence $(f,x) \in T(O,V) \times U$. But if $(g,u) \in T(O,V) \times U$ then $(g^{-1}(V),u) \in O \times U \subseteq \varepsilon_X$. We thus have that $u \in g^{-1}(V)$, i.e., that $\varepsilon_{X,Y}(g,u) = g(u) \in V$. Therefore $\varepsilon_{X,Y}$ is continuous. □

We have thus obtained the promised reduction.

Corollary 2.2. *A space X is exponentiable iff $\mathcal{O}X$ has an even topology. In this case, the even topology of $\mathrm{C}(X,Y)$ is the topology induced by the even topology of $\mathcal{O}X$.*

3. Spaces with even topologies on the lattices of open sets

The discrete and indiscrete topologies on $\mathcal{O}X$ are strong and weak respectively. We begin by improving these bounds. A set $O \subseteq \mathcal{O}X$ is ***Alexandroff open*** if the conditions $U \in O$ and $U \subseteq V \in \mathcal{O}X$ together imply that $V \in O$. It is immediate that the Alexandroff open sets form a topology.

3.1. *The Alexandroff topology is strong.*

In particular, any open set in a weak topology is Alexandroff open.

Proof. If $(U, x) \in \varepsilon_X$ then $(U, x) \in \{V \in \mathcal{O}X \mid U \subseteq V\} \times U$, which is a product of an Alexandroff open subset of $\mathcal{O}X$ with an open subset of X, and this product is clearly contained in ε_X. $\square$

An Alexandroff open set $O \subseteq \mathcal{O}X$ is ***Scott open*** if every open cover of a member of O has a finite subcover of a member of O. For example, for any subset Q of X, the Alexandroff open set $\{V \in \mathcal{O}X \mid Q \subseteq V\}$ is Scott open iff Q is compact. Again, it is easy to check that the Scott open sets form a topology.

3.2. *The Scott topology is weak.*

Proof. Let $W \subseteq A \times X$ be open, let $a \in A$ and let $O \subseteq \mathcal{O}X$ be a Scott open neighbourhood of $\overline{w}(a)$. By openness of W in the product topology, for each $x \in \overline{w}(a)$ there are $U_x \in \mathcal{O}A$ and $V_x \in \mathcal{O}X$ with $(a, x) \in U_x \times V_x \subseteq W$. Since $\overline{w}(a)$ is the union of the sets V_x and since O is Scott open, the union V of finitely many such V_x belongs to O. Let U be the intersection of the corresponding open sets U_x. Clearly, U is a neighbourhood of a. To conclude the proof, we show that $\overline{w}(u) \in O$ for each $u \in U$. To this end, it is enough to show that $V \subseteq \overline{w}(u)$, because O is Alexandroff open and we know that $V \in O$. Let $v \in V$. Then $v \in V_x$ for some $x \in \overline{w}(a)$. Since $u \in U_x$, we have that $(u, v) \in U_x \times V_x \subseteq W$. Therefore $v \in \overline{w}(u)$. $\square$

Having improved the bounds, we now have a closer look at strong topologies. Let T be a topology on $\mathcal{O}X$. For opens $U, V \in \mathcal{O}X$, we write $U \prec_T V$ to mean that there exists $O \in T$ with $V \in O$ and $U \subseteq W$ for all $W \in O$. This is equivalent to saying that V belongs to the T-interior of the set $\{W \in \mathcal{O}X \mid U \subseteq W\}$. Notice that

1. $U \prec_T V$ implies $U \subseteq V$,
2. (a) $U' \subseteq U \prec_T V$ implies $U' \prec_T V$,
   (b) $U \prec_T V \subseteq V'$ implies $U \prec_T V'$, provided T is weaker than the Alexandroff topology,
3. $\emptyset \prec_T W$, and $U \prec_T W$ and $V \prec_T W$ together imply $U \cup V \prec_T W$.

Notice also that for a topology T weaker than the Alexandroff topology, $U \prec_T U$ iff the set $\{V \in \mathcal{O}X \mid U \subseteq V\}$ is open. Hence, in this case, the relation $\prec_T$ is reflexive iff T *is* the Alexandroff topology, in which case $U \prec_T V$ iff $U \subseteq V$. Therefore the following generalizes the fact that the Alexandroff topology is strong.

3.3. *A topology T on $\mathcal{O}X$ is strong iff it is **approximating**, in the sense that for every open neighbourhood V of a point x of X, there is an open neighbourhood $U \prec_T V$ of x.*

Notice that this is equivalent to saying that every open set V is the union of the opens $U \prec_T V$.

Proof. Assume that T is strong and let V be an open neighbourhood of a point x of X. Since this means that $\varepsilon_X \subseteq \mathcal{O}X \times X$ is open with respect to T and that $(V, x) \in \varepsilon_X$, there are $O \in T$ and $U \in \mathcal{O}X$ such that $(V, x) \in O \times U \subseteq \varepsilon_X$. Hence, if $(W, u) \in O \times U$ then $u \in W$. Therefore $U \subseteq W$ for every $W \in O$, which shows that $x \in U \prec_T V$. Conversely, assume that T is approximating and that $(V, x) \in \varepsilon_X$. Then $x \in V$ and there is $U \prec_T V$ with $x \in U$. Let $O \in T$ with $V \in O$ and $U \subseteq W$ for all $W \in O$. Then $(V, x) \in O \times U \subseteq \varepsilon_X$, which shows that ε_X is open and hence that T is strong. $\square$

3.4. *The Scott topology is the intersection of the strong topologies.*

Therefore it is the strongest weak topology.

Proof. Being weak, it is contained in the intersection. Conversely, for each $\mathcal{C} \subseteq \mathcal{O}\, X$, let $T_{\mathcal{C}}$ be the set of all Alexandroff open subsets O of $\mathcal{O}\, X$ with the property that if $\mathcal{C}$ covers a member of O then $\mathcal{C}$ has a finite subcover of a member of O. This is easily seen to be a topology on $\mathcal{O}\, X$, and, by construction, the Scott topology is the intersection of all such topologies. To conclude the proof, it suffices to show that they are strong. Assume that $x \in U \in \mathcal{O}\, X$. If $x \notin \bigcup \mathcal{C}$, then $\{V \in \mathcal{O}\, X \mid U \subseteq V\} \in T_{\mathcal{C}}$, and so $x \in U \prec_{T_{\mathcal{C}}} U$. If $x \in U'$ for some $U' \in \mathcal{C}$, then $\{V \in \mathcal{O}\, X \mid U' \cap U \subseteq V\} \in T_{\mathcal{C}}$, whence $x \in (U' \cap U) \prec_{T_{\mathcal{C}}} U$. Therefore $T_{\mathcal{C}}$ is strong by 3.3. □

Theorem 3.5. *A space has an even topology on its lattice of open sets iff the Scott topology of its lattice of open sets is approximating, in which case the even topology is the Scott topology.*

The topology on $\mathrm{C}(X, Y)$ induced by the Scott topology of $\mathcal{O}\, X$ is referred to as the ***Isbell topology***. Combining Corollary 2.2 with Theorem 3.5 we obtain the following characterization of exponentiable spaces.

Corollary 3.6. *A space is exponentiable iff the Scott topology of its lattice of open sets is approximating. Moreover, the topology of an exponential is the Isbell topology.*

4. Core-compact spaces

Our next goal is to avoid explicit references to the Scott topology in the criterion for exponentiability given in Corollary 3.6. For U and V open sets, one writes $U \ll V$ to mean that every open cover of V has a finite subcover of U. For example, this is the case if there is a compact set Q with $U \subseteq Q \subseteq V$. Notice that this relative notion of compactness enjoys the following properties:

1. $U \ll V$ implies $U \subseteq V$,
2. $U' \subseteq U \ll V \subseteq V'$ implies $U' \ll V'$,
3. $\emptyset \ll W$, and $U \ll W$ and $V \ll W$ together imply $U \cup V \ll W$.

Although the relations $U \prec_{\mathrm{Scott}} V$ and $U \ll V$ don't coincide in general, they do for exponentiable spaces.

4.1. *For U and V open subsets of a space X,*

1. $U \prec_{\mathrm{Scott}} V$ *implies* $U \ll V$,
2. *if the Scott topology of $\mathcal{O}\, X$ is approximating then $U \ll V$ implies $U \prec_{\mathrm{Scott}} V$.*

Proof. Assume that $U \prec_{\mathrm{Scott}} V$. Then there is a Scott open neighbourhood O of V such that $U \subseteq W$ for all $W \in O$. Since O is Scott open, any open cover of V has a finite subcover of a member of O and hence of U. Therefore $U \ll V$. Conversely, assume that the Scott topology of $\mathcal{O}\, X$ is approximating and that $U \ll V$. Since V is the join of the opens $V' \prec_{\mathrm{Scott}} V$, we have that $U \subseteq W$ where W is a union of finitely many $V' \prec_{\mathrm{Scott}} V$. Since $W \prec_{\mathrm{Scott}} V$, we conclude that $U \prec_{\mathrm{Scott}} V$. □

A space X is ***core-compact*** if every open neighbourhood V of a point x of X contains an open neighbourhood $U \ll V$ of x. Again, this is equivalent to saying that every open V is the union of the opens $U \ll V$.

4.2. *Let X be a core-compact space.*

1. *If $U \ll W$ in $\mathcal{O}\,X$ then $U \ll V \ll W$ for some $V \in \mathcal{O}\,X$.*
2. *The set $\Uparrow U \stackrel{\text{def}}{=} \{V \in \mathcal{O}\,X \mid U \ll V\}$ is Scott open.*
3. *If $O \subseteq \mathcal{O}\,X$ is Scott open and $V \in O$ then $U \ll V$ for some $U \in O$.*
4. *The sets $\Uparrow U$ for $U \in \mathcal{O}\,X$ form a base of the Scott topology of $\mathcal{O}\,X$.*
5. *If $U \ll V$ then $U \prec_{\text{Scott}} V$.*

Proof. (1): The open set W is the union of the open sets $V \ll W$, and, in turn, each open set $V \ll W$ is the union of the open sets $V' \ll V$. Hence W is the union of the collection $\mathcal{C}$ of open sets V' for which there exists an open set V with $V' \ll V \ll W$. Since $\mathcal{C}$ is closed under the formation of finite unions, we have that $U \subseteq V'$ for some $V' \in \mathcal{C}$. By definition of $\mathcal{C}$, there is an open V with $V' \ll V \ll W$ and hence with $U \ll V \ll W$. (2): The set $\Uparrow U$ is clearly Alexandroff open. If $W \in \Uparrow U$ then $V \in \Uparrow U$ for some $V \ll W$ by (1), which shows that every open cover of a member of $\Uparrow U$ has a finite subcover of a member of $\Uparrow U$. (3): The open set V is the union of the open sets $U \ll V$, and such open sets are closed under the formation of finite unions. (4): This is an immediate consequence of (2) and (3). (5): $\Uparrow U$ is a Scott open set with $V \in \Uparrow U$ and $U \subseteq W$ for all $W \in \Uparrow U$. □

Theorem 4.3. *A space is exponentiable iff it is core-compact. Moreover, if X is a core-compact space and Y is any space then the topology of the exponential Y^X is generated by the sets*

$$\{f \in Y^X \mid U \ll f^{-1}(V)\},$$

where U and V range over $\mathcal{O}\,X$ and $\mathcal{O}\,Y$ respectively.

Proof. If the space is exponentiable then its lattice of open sets has an even topology and hence it is core-compact by Corollary 3.6 and by 4.1. Conversely, if it is core-compact then its lattice of open sets has an even topology by Corollary 3.6 and by 4.2(5), and hence it is exponentiable. For the second part, it is easy to see that if T is a topology on $\mathcal{O}\,X$ with a base B then the T-induced topology on $\mathrm{C}(X,Y)$ has as a subbase the sets $T(O,V)$ for O in B. The result then follows from the fact that the sets $\Uparrow U$ for $U \in \mathcal{O}\,X$ form a base of the Scott topology of $\mathcal{O}\,X$ if X is core-compact, and from the fact that the Isbell topology is induced by the Scott topology. □

We finish this note with a remark on exponentiable Hausdorff spaces. It can be shown [5] that for open subsets U and V of a Hausdorff space, if $U \ll V$ then $U \subseteq Q \subseteq V$ for some compact set Q. It follows that a Hausdorff space is core-compact iff every neighbourhood of a point contains a compact neighbourhood of the point, which means that the space is locally compact. Moreover, if X is a locally compact Hausdorff space, then the exponential topology of Y^X is generated by the sets

$$\{f \in Y^X \mid f(Q) \subseteq V\},$$

where Q and V range over compact subsets of X and open subsets of Y respectively. That is, the exponential topology is the compact-open topology [3, 4, 10]. The reason is that in this case the Scott topology of $\mathcal{O}\,X$ has the sets $\{U \in \mathcal{O}\,X \mid Q \subseteq U\}$ as a base and that the condition $Q \subseteq f^{-1}(V)$ is equivalent to $f(Q) \subseteq V$.

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Department of Computer Science, St Andrews University, North Haugh, St Andrews, Fife KY16 9SS, Scotland

E-mail address: mhe@dcs.st-and.ac.uk

Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ, England

E-mail address: rh@doc.ic.ac.uk