Searchable Sets, Dubuc-Penon Compactness, Omniscience Principles, and the Drinker Paradox

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Abstract. We show that a number of contenders for an abstract and general notion of compactness, applicable in particular to computability theory and constructive mathematics, coincide in some well known frameworks. We consider compactness of sets rather than of spaces, where we replace topologies by the restriction to constructive reasoning, as in the work by a number of authors, including Penon, Dubuc, Taylor and Escardó. Sets here are conceived in a very liberal way, including types of HA^ω and Martin Löf type theory, and objects of toposes, among others. Some of the equivalences require instances of the axiom of choice, which are available in some of the above frameworks but not all, as is well known. We relate the instances of the axiom of choice applied in the above equivalences to the topological notion of total separatedness.

1 Introduction

The motivation for this paper comes from our papers [3,5,8] and ongoing joint work that will be reported elsewhere (and that is briefly sketched at the end of the paper [8]). In the papers [3,5], we considered computational manifestations of the topological notion of compactness, using respectively semi-decision and decision procedures to formulate them. In the paper based on *semi-decision* procedures [3], the first author wrote that he felt that there must be connections with Dubuc and Penon's notion of compactness for objects of toposes [2]. But, perhaps surprisingly, it turns out that the direct connection of Dubuc-Penon compactness is with the later paper based on *decision* procedures [5], which is what we report in this paper. The main computational manifestation of the notion of compactness investigated in [5] is that of a *searchable set*, which will be recalled below. In the papers [5,8], we begun to investigate how one can build searchable sets in computational and constructive settings, and in particular we proved a Tychonoff theorem for them.

Taylor [9] has performed intriguing work in the direction of understanding topological notions in a set-theoretically free way via constructive mathematics, as already extensively discussed in [3]. His approach is related to that of [3] based on semi-decision procedures. The notions discussed here, based on decision procedures, seem to be related to Taylor's notion of compact overt space. But we emphasize that they are not the same, because his notion is based on the Sierpinski space and ours on the discrete booleans [5], and there is no continuous/constructive way of converting a Sierpinski valued predicate into a boolean valued predicate, as would be required to show that our notions imply overtness in his sense.

In Section 2 we recall and briefly discuss some contenders for a logical notion of compactness as discussed above, and formulate an equivalence theorem, where the equivalence with Dubuc-Penon compactness requires an instance of the axiom of choice. In Section 3 we prove the theorem and single out Dubuc–Penon compactness as the strongest notion in the absence of choice. In Section 4 we show, from a realisability point of view, that these compactness notions are preserved by countable products, providing a logical and computational counterpart of the well-known Tychonoff theorem. We also relate this to our previous work on products of selection functions [5,8,6,7]. Finally, in Section 5, we discuss the relation between the instance of choice applied in the previous sections and a logical counter-part of the topological notion of total separatedness, in connection with the investigation carried out in [5].

2 Logical notions of compactness

A classical principle that generally fails intuitionistically may hold in particular situations. As discussed in the introduction, we are interested in principles that have a flavour of the topological notion of compactness, where topological data is manifestly absent but implicitly present by working with a constructive or intuitionistic underlying logic.

Drinker paradox. In every pub there is a person a such that if a drinks then everybody drinks. Formally, a set X satisfies the *drinker paradox* iff

$$\forall p \in X \to \Omega\ \exists a \in X\, (p(a) \implies \forall x \in X(p(x)))\,,$$

where Ω is the set of truth values. In classical logic, a set satisfies this condition if and only if it is non-empty.

Boolean drinker paradoxes. We say that X satisfies the *boolean drinker paradox* iff

$$\forall\text{-BDP}(X) :\iff \forall p \in X \to 2\ \exists a \in X\, (pa = 0 \implies \forall x \in X(px = 0))\,,$$

where $2 = \{0, 1\}$ is the set of decidable truth values or booleans. This is clearly implied by the drinker paradox.

Another version of the boolean drinker paradox says that in any pub there is a person a such that if somebody drinks then a drinks:

$$\exists\text{-BDP}(X) :\iff \forall p \in X \to 2\ \exists a \in X\, (\exists x \in X(px = 1) \implies pa = 1)\,,$$

Searchable sets. We say that X is *searchable* iff

$$\text{searchable}(X) :\iff \forall p \in X \to 2\ \exists a \in X\, (\neg\neg\exists x \in X(px = 1) \implies pa = 1)\,.$$

We remark that a stronger definition is considered in [5] in the context of higher-type computability, namely

$$\exists \varepsilon \in (X \to 2) \to X\ \forall p \in X \to 2\, (\neg\neg\exists x \in X(px = 1) \implies p(\varepsilon p) = 1)\,.$$

The axiom of choice, which is validated in realisability interpretations, gives the stronger version from the weaker one. In this paper it is more natural to take the weaker definition as the official one.

Principle of omniscience. We say that X satisfies the *principle of omniscience* iff

$$\text{PO}(X) :\iff \forall p \in X \to 2\ (\exists x \in X(px = 1) \vee \forall x \in X(px = 0)).$$

Dubuc-Penon compactness. We say that X is *Dubuc-Penon compact* iff

$$\text{DPC}(X) :\iff \forall A \in \Omega\ \forall B \in X \to \Omega\ (\forall x \in X(A \vee B(x)) \implies A \vee \forall x \in X(B(x))).$$

If our underlying logic does not allow quantification over propositions, we consider the above as an axiom scheme with parameters A and B. This notion was introduced in [2] for objects of toposes, interpreted in the internal language. Dubuc and Penon proved that for certain gros toposes built out of categories of spaces, a representable object is Dubuc-Penon compact if and only if its representing space is compact in the usual topological sense.

Boolean Dubuc-Penon compactness. We say that X is *boolean Dubuc-Penon compact* iff

$$\mathrm{BDPC}(X) :\iff \forall A \in \Omega\ \forall B \in X \to 2\ (\forall x \in X(A \vee B(x)) \implies A \vee \forall x \in X(B(x))).$$

This is clearly implied by Dubuc-Penon compactness. Notice that the type of A has not changed. If we had stipulated $A \in 2$, then this principle would hold intuitionistically for any X.

We emphasize that all our results are developed in the context of intuitionistic logic. For example, the following theorem is classically trivial, simply because each principle is true in the presence of excluded middle.

Theorem 1. *The following are equivalent for any inhabited set X:*

1. *X is searchable.*
2. *X is boolean Dubuc-Penon compact.*
3. *X satisfies the boolean drinker paradox.*
4. *X satisfies the principle of omniscience.*

Moreover, Dubuc-Penon compactness of X implies these conditions, and if the instance of the axiom of choice

$$\mathrm{AC}(X, 2) :\iff \forall x \in X\ \exists y \in 2\ A(x, y) \implies \exists p \in X \to 2\ \forall x \in X\ A(x, px),$$

holds, then the converse is true.

In particular, this theorem holds in realisability over system T and in Martin Löf type theory (and we have developed it in Agda [1], a well-known implementation of type theory). This theorem is proved in the next section, which provides more information. Further information and questions about the role of choice are discussed in Section 5.

3 Proof of Theorem 1 with further information

Lemma 1.

1. *If* $\forall$-BDP(X) *then X is inhabited.*
2. DPC$(\emptyset)$.

Proof Considering any predicate, say $p(x) = 0$, we get $a \in X$ by definition. The left disjunct of the DP-compactness conclusion $A \vee \forall x \in X(B(x))$ holds vacuously when X is empty. □

Lemma 2. searchable$(X) \implies \exists$-BDP(X).

Proof $\exists$-BDP(X) has a stronger premise and hence is weaker. □

Lemma 3. $\exists$-BDP$(X) \implies \forall$-BDP(X).

Proof For any given $p \in X \to 2$, the assumption produces $a \in X$ that satisfies $\exists x \in X(px = 1) \implies p(a) = 1$, and hence $p(a) = 0 \implies \forall x \in X(px = 0)$, and so $\forall$-BDP(X) holds. □

Lemma 4. $\forall$-BDP$(X) \implies$ PO(X).

Proof For any $p \in X \to 2$, the assumption produces $a \in X$ such that $p(a) = 0 \implies \forall x \in X(px = 0)$. Because $p(a) = 0$ is decidable, we can reason by cases. If it holds, then $\forall x \in X(px = 0)$. Otherwise $p(a) = 1$ and hence $\exists x \in X(p(x) = 1)$. Therefore PO$(X)$ holds. □

Lemma 5. *For X inhabited,* $\mathrm{PO}(X) \implies \mathrm{searchable}(X)$.

Proof Let $p \in X \to 2$. By $\mathrm{PO}(X)$, either $\exists x \in X(px = 1)$ or else $\forall x \in X(px = 0)$. In the first case we take any a with $pa = 1$, and $\neg\neg\exists x \in X(px = 1) \implies pa = 1$ holds simply because the conclusion is true and so $\mathrm{searchable}(X)$ holds. In the second case we have that $\neg\neg\exists x \in X(px = 1)$ is impossible, and hence the implication $\neg\neg\exists x \in X(px = 1) \implies pa = 1$ holds for any $a \in X$, which can be found by inhabitedness of X, and again $\mathrm{searchable}(X)$ holds. □

Lemma 6. $\mathrm{BDPC}(X) \implies \mathrm{PO}(X)$.

Proof Let $p \in X \to 2$ and define $A = \exists x \in X(px = 1)$ and $B(x) = (px = 0)$. Then $A \vee B(x)$ holds for any $x \in X$. In fact, because $B(x)$ is decidable, we can reason by cases. If $B(x)$ holds, then $A \vee B(x)$. Otherwise, $px = 1$ and hence A holds, and so does $A \vee B(x)$. Hence $A \vee \forall x \in X(B(x))$ holds by DP-compactness of X, which amounts to PO. □

Lemma 7. $\mathrm{PO}(X) \implies \mathrm{BDPC}(X)$.

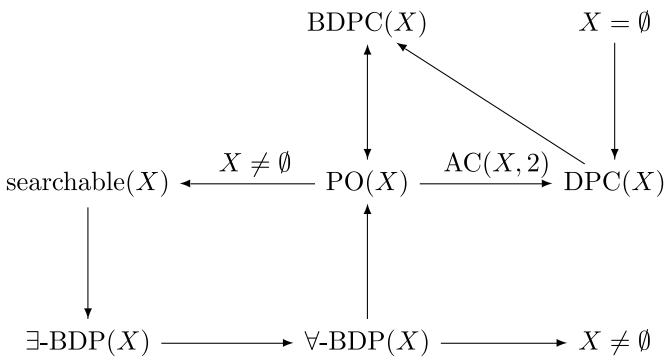
Proof By PO, either $\exists x \in X(\neg Bx)$ or else $\forall x \in X(Bx)$. In the first case A holds, and hence in both cases $A \vee \forall x \in X(B(x))$ holds, which is the conclusion of boolean DP-compactness. □

If B is not decidable, then one cannot apply PO to B, and hence the above argument cannot be used in order to show that $\mathrm{PO}(X) \implies \mathrm{DPC}(X)$. The following lemma instead applies PO to a suitable predicate constructed with the axiom of choice.

Lemma 8. $\mathrm{PO}(X) \implies \mathrm{DPC}(X)$ *if the axiom of choice* $\mathrm{AC}(X, 2)$ *holds.*

Proof Let $x \in X$ and assume that $A \vee B(x)$. Then, reasoning by cases, there is $y \in 2$ such that $(y = 1 \implies A) \wedge (y = 0 \implies B(x))$. By the axiom of choice, there is $p \in X \to 2$ such that $(px = 1 \implies A) \wedge (px = 0 \implies B(x))$. Now assume the premise $\forall x \in X(A \vee B(x))$ of DP-compactness. By PO, either $\exists x \in X(px = 1)$ or else $\forall x \in X(px = 0)$. In the first case A holds, and in the second case $\forall x \in X(B(x))$ holds, and hence in both cases $A \vee \forall x \in X(B(x))$ holds, which is the conclusion of DP-compactness. □

Hence in the absence of the axiom of choice, DP-compactness is the strongest notion, for inhabited sets, among those considered here. In summary, the following implications have been established, where labels in the arrows indicate assumptions, and hence Theorem 1 is proved:



This chain of implications, formulated and proved in type theory in Agda notation [1], is available at `http://www.cs.bham.ac.uk/~mhe/papers/DP/`, where we have not considered BDPC as the axiom of choice is available.

Remark 1. Because existential quantification over 2 is disjunction, the axiom of choice $\mathrm{AC}(X,2)$ amounts to

$$\forall x \in X(A(x,0) \vee A(x,1)) \implies \exists p \in X \to 2(\forall x \in X(A(x,p(x)))).$$

Hence another way of writing $\mathrm{AC}(X,2)$ is

$$A_0 \cup A_1 = X \implies \exists B_0 \subseteq A_0, B_1 \subseteq A_1(B_0 \cap B_1 = \emptyset \wedge B_0 \cup B_1 = X),$$

considering $B_i = p^{-1}(i)$. Thus $\mathrm{AC}(X,2)$ is a rather strong requirement from an intuitionistic point of view, although it does hold in some models such as realisability over system T, and is provable in intuitionistic systems such as (intensional) type theory. Section 5 below discusses the role of choice in the above development in connection with the paper [5].

After seeing the above development, Steve Vickers mentioned the condition in the following proposition, which he applied to Kuratowski finite objects X of toposes, without being aware of the notion of Dubuc-Penon compactness.

Proposition 1. *A set X is Dubuc-Penon compact if and only if*

$$\forall C, B \in X \to \Omega(\forall x \in X(C(x) \vee B(x)) \implies \exists x \in X(C(x)) \vee \forall x \in X(B(x))).$$

Proof That the condition implies DP compactness can be seen by considering $C(x) = A$. To see that it is implied by DP compactness, consider the proposition $A = \exists x \in X(C(x))$. Then $\forall x \in X(C(x) \vee B(x))$ implies $\forall x \in X(A \vee B(x))$, which DP-compactness transforms into $A \vee \forall x \in X(B(x))$, as required. □

4 Tychonoff theorem

The Tychonoff theorem in topology says that compact sets are closed under arbitrary products. It is shown in [5] that, in the context of higher-type computability, searchable sets are closed under countable products. Moreover, the product of selection functions $\bigotimes$, defined in [5] and in more generality in [8], provides the construction which witnesses this fact. The following theorem reformulates this in logical terms via realisability, where $x \;\mathsf{mr}\; A$ means that x modified realises A.

Theorem 2. *The countable product of selection functions $\bigotimes$ modified realises the proposition that if a sequence of sets X_i satisfy the existential boolean drinker paradox then so does their product $\Pi_i X_i$, that is,*

$$\bigotimes \quad \mathsf{mr} \quad \forall i(\exists\text{-BDP}(X_i)) \implies \exists\text{-BDP}(\Pi_i X_i).$$

Proof The key point is that $\exists$-BDP is of the form $\forall p \exists a(B(p,a))$, where $B(p,a)$ is a Harrop formula, which is then devoid of computational content in terms of realisability. Hence a realiser of $\exists\text{-BDP}(X_i)$ amounts to a selection function $\varepsilon \colon (X_i \to 2) \to X_i$. Thus, a realiser of the statement of the theorem amounts to a construction that given a sequence of selection functions for the sets X_i produces a single selection function $(\prod_i X_i \to 2) \to \prod_i X_i$ for their product. The proof that $\bigotimes$ is such a realiser, with minor adaptations to discard and introduce realisers without computational content, is the same as that of [5, Theorem 4.6]. □

Theorem 2 also holds for ∀-BDP instead of ∃-BDP with the same proof, modified to invoke the paper [8] in the last step (a countable product of attainable quantifiers is attainable, where an attainable quantifier is one that possesses a selection function). Moreover, via the equivalence between ∃-BDP and the other compactness notions (Theorem 1), we know that all of them are closed under countable products.

We have also shown in the companion papers [6,7] that the countable product of selection functions is a direct realiser for other logical principles, including a certain *J-shift* principle that generalises the well-known double-negation shift. It is thus natural to ask whether there is a single, general principle that is realised by the product of selection functions and has as logical consequences the particular principles that we have considered. We leave this as an open problem.

5 The axiom of choice and total separatedness

We now formulate a logical counter-part of the topological notion of total separatedness and investigate its relation with the instance of the axiom of choice invoked above. We first motivate the development by recalling the role of total separatedness in the investigation of searchable spaces carried out in [5].

This reference shows that, in the model of continuous maps of retracts of spaces of Kleene–Kreisel functionals, a space is searchable if and only if it is topologically compact, and for example the Cantor space $2^{\mathbb{N}}$ is searchable. The proof relies crucially on the fact that such spaces are totally separated. This means that the clopens, or equivalently the continuous maps $X \to 2$, separate the points. Moreover, as also observed in that paper, in a topological model such as compactly generated spaces, any inhabited connected space X is trivially searchable, and in particular $X = \mathbb{R}$ is searchable but of course not compact. In fact, because every continuous map $p \in X \to 2$ is constant, the search witness $a \in X$ can be taken to be any point, independently of p, since there is $x \in X$ with $p(x) = 1$ if and only if $p(a) = 1$, and hence the constant selection function $\varepsilon(p) = a$ does the job. Thus an assumption such as total separatedness is necessary in order to be able to conclude that searchable sets are topologically compact in topological models such as the above.

In summary, total separatedness of X requires the existence of plenty of maps into 2, and in this section we show that $\mathrm{AC}(X, 2)$ provides a means of constructing them. Moreover, the role of total separatedness in [5] can be seen to be played by this instance of the axiom of choice in Section 3.

Totally separated sets. We say that X is *totally separated* if

$$\forall x, y \in X (\forall p \in X \to 2 (p(x) = p(y)) \implies x = y).$$

Connected sets. It is natural to define a set X to be *connected* if all maps $X \to 2$ are constant. If X is both connected and totally separated, then it has at most one point. Hence total separatedness can be seen as a strong notion of disconnectedness.

Totally separated apartness relations. To discuss a positive version of total separatedness, we consider apartness relations. We say that an apartness relation $\sharp$ on X is *totally separated* if

$$\forall x, y \in X (x \sharp y \implies \exists p \in X \to 2 (p(x) \neq p(y))).$$

Recall that an apartness relation on a set X is a binary relation $\sharp$ such that

1. $\neg(x \sharp x)$ (irreflexivity),
2. $x \sharp y \implies y \sharp x$ (symmetry),
3. $x \sharp y \implies z \sharp x \vee z \sharp y$ (co-transitivity),

and that an apartness relation $\sharp$ is called sharp if

$$\neg(x \sharp y) \implies x = y.$$

For example, it is well known that (1) the empty relation is an apartness relation that fails to be sharp but is totally separated in a trivial way, (2) if X has decidable equality then the negation $\neq$ of equality is a sharp apartness relation, (3) the reals have a sharp apartness relation, and (4) a sharp apartness relation on the Cantor space $2^{\mathbb{N}}$ is given by

$$\alpha \sharp \beta \iff \exists i \in \mathbb{N}(\alpha_i \neq \beta_i).$$

Moreover, it is immediate this relation on the Cantor space is totally separated, by considering $p(\gamma) = \gamma_i$ where i is an apartness witness. Of course:

Lemma 9. *If X has some totally separated, sharp apartness relation, then X is totally separated.*

Proof Assume that $\forall p \in X \to 2(p(x) = p(y))$. The contra-positive of total separatedness of $\sharp$ gives the conclusion $\neg(x \sharp y)$, which sharpness transforms into $x = y$. □

The step that relates choice to total separatedness is this:

Lemma 10. *If the axiom of choice* $\mathrm{AC}(X, 2)$ *holds, then any apartness relation on X is totally separated.*

Proof Assume that $x \sharp y$ and define $A(z, 0) \iff z \sharp y$ and $A(z, 1) \iff z \sharp x$. Then, by co-transitivity, for every $z \in X$ there is $t \in 2$ such that $A(z, t)$. By $\mathrm{AC}(X, 2)$, there is $p \in X \to 2$ such that $A(z, p(z))$ for all z, which then satisfies $p(x) = 0$ and $p(y) = 1$, as required. □

Lemma 11. *Any set X has an apartness relation given by*

$$x \sharp_2 y \iff \exists p \in X \to 2(p(x) \neq p(y)),$$

which is totally separated by construction.

Proof Irreflexivity and symmetry are immediate. To prove co-transitivity, consider $p \in X \to 2$ such that $p(x) \neq p(y)$, and let $z \in Z$. By decidability of equality on 2, either $p(z) = p(y)$ or $p(z) = p(x)$. In the first case $z \sharp_2 x$, and in the second case $z \sharp_2 y$, and hence $z \sharp_2 x$ or $z \sharp_2 y$, as required. □

Remark 2. If a set T comes with a sharp apartness relation $\sharp$, then much of the development of this section routinely adapts to the apartness relation $\sharp_T$ on X defined by $x \sharp_T y \iff \exists p \in X \to T(p(x) \sharp p(y))$, with T in the place of 2.

Lemma 10 can be read, in view of Lemma 11, as saying that $x \sharp y \implies x \sharp_2 y$ for any apartness relation $\sharp$, and hence:

Lemma 12. *The relation $\sharp_2$ is the finest apartness relation if* $\mathrm{AC}(X, 2)$ *holds.*

The apartness relation $\sharp_2$ does not need to be sharp. For example, if X is connected, then $\sharp_2$ is empty.

Lemma 13. *The apartness relation $\sharp_2$ on X is sharp if and only if X is totally separated.*

Proof ($\Leftarrow$): Because $\neg(x \sharp_2 y)$ amounts to $\forall p \in X \to 2(p(x) = p(y))$, which total separatedness of X transforms into $x = y$. ($\Rightarrow$): Lemma 9. □

Hausdorff sets. It is natural to call a set X *Hausdorff* if it has some sharp apartness relation. By the above lemma, any totally separated set X is Hausdorff, with sharp apartness relation $\sharp_2$. Putting the above together:

Theorem 3. *If* $\mathrm{AC}(X, 2)$ *holds,* X *is Hausdorff if and only if it is totally separated.*

Moreover, as we have seen, in this case, any sharp apartness relation is totally separated, and $\sharp_2$ is the finest apartness relation, and is sharp.

By the above discussion, if X is connected and Hausdorff and has two distinct points, then $\mathrm{AC}(X, 2)$ fails. The reals are not Dubuc-Penon compact in the models considered by Dubuc and Penon, but are boolean DP-compact in the same models because they are searchable, as discussed above, as these models validate connectedness of $\mathbb{R}$. Thus, this is an example that distinguishes, in the absence of choice, Dubuc–Penon compactness from boolean Dubuc–Penon compactness.

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