# Notes on compactness

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## 1 Introduction

By definition, the intersection of finitely many open sets of any topological space is open. Nachbin [6] observed that, more generally, the intersection of "compactly many" open sets is open (see Section 2 for a precise formulation of this fact). Of course, this is to be expected, because compact sets are intuitively understood as those sets that, in some mysterious sense, behave as finite sets. Moreover, Nachbin applied this to obtain elegant proofs of various facts concerning compact sets in topology and elsewhere.

A simple calculation (performed in Section 2) shows that Nachbin's observation amounts to the well known fact that if a space $X$ is compact, then the projection map $Z \times X \to Z$ is closed for every space $Z$.

It is also well known that the converse holds: if a space $X$ has the property that the projection $Z \times X \to Y$ is closed for every space $Z$, then $X$ is compact. We reformulate this as a converse of Nachbin's observation, and apply this to obtain further elegant proofs of (old and new) theorems concerning compact sets.

We provide a new proof of (a reformulation of) the fact that a space $X$ is compact if and only if the projection map $Z \times X \to Z$ is closed for every space $Z$. This is generalized in various ways, to obtain results about proper maps, relative compactness, and compactly generated spaces.

## 2 Some characterizations of the notion of compactness

2.1 DEFINITION. Let $Z$ be a topological space and $\{V_i \mid i \in I\}$ be a family of open sets of $Z$. If the index set $I$ comes endowed with a topology, such a family will be called *continuously indexed* if whenever $z \in V_i$, there are neighbourhoods $T$ of $z$ and $U$ of $i$ such that $t \in V_u$ for all $t \in T$ and $u \in U$. This amounts to saying that the graph $\{(z,i) \in Z \times I \mid z \in V_i\}$ of the family is open in the product topology.

2.2 *A space $X$ is compact if and only if the set $\bigcap_{x \in X} V_x$ is open for any continuously indexed family $\{V_x \mid x \in X\}$ of open sets of any space $Z$.*

That is, not only are the open sets of any space closed under the formation of compact intersections, in addition to the postulated finite intersections, but also this characterizes the notion of compactness. In the "synthetic" formulation of compactness developed in [2, Chapter 7], we used continuous universal quantification functionals, for which function spaces were required (see Section 3 below). A related formulation that avoids the function-space machinery is the following:

2.3 *A space $X$ is compact if and only if for any space $Z$ and any open set $W \subseteq Z \times X$, the set $\{z \in Z \mid \forall x \in X.(z,x) \in W\}$ is open.*

This set can be written, more geometrically, as $\{z \in Z \mid \{z\} \times X \subseteq W\}$, but the given logical formulation emphasizes the connection with [2]. To prove 2.2 and 2.3, first observe that any open set $W \subseteq Z \times X$ gives rise to the continuously indexed family $\{V_x \mid x \in X\}$ of open sets of $Z$ defined by $z \in V_x$ iff $(z,x) \in W$, and that this construction is a bijection from open sets of $Z \times X$ to continuously $X$-indexed families of open sets of $Z$. Moreover, $z \in \bigcap_{x \in X} V_x$ iff $\forall x \in X.(z,x) \in W$. Next, consider the closed set $F \overset{\text{def}}{=} (Z \times X) \setminus W$ and the projection $\pi \colon Z \times X \to Z$. Then $z \in \pi(F)$ iff $\exists x \in X.(z,x) \in F$. By the De Morgan law for existential and universal quantifiers, $Z \setminus \pi(F) = \{z \in Z \mid \forall x \in X.(z,x) \in W\}$. It follows that:

2.4 LEMMA. *The following are equivalent for spaces $X$ and $Z$.*

1. *The open sets of $Z$ are closed under continuously $X$-indexed intersections.*

2. *For any open set $W \subseteq Z \times X$, the set $\{z \in Z \mid \forall x \in X.(z,x) \in W\}$ is open.*

3. *The projection $Z \times X \to Z$ is a closed map.*

This concludes the proof of 2.2 and 2.3, because it is well known that compactness of $X$ is equivalent to closedness of the projection $Z \times X \to Z$ for every $Z$. A self-contained proof of a generalization of 2.3, which doesn't rely on previous knowledge of the closed-projection characterization of compactness, will be given in Section 4.

# 3  A further characterization

We apply the formulation of compactness given by 2.3 to derive the "synthetic" formulation based on function spaces [2, Chapter 7]. No previous knowledge on function-space topologies is required.

3.1 DEFINITION.  For given spaces $S$ and $X$, we denote by $S^X$ the set of continuous maps $X \to S$ endowed with a topology such that[1]

1. the evaluation map $e \colon S^X \times X \to S$ defined by $e(f, x) = f(x)$ is continuous,

2. for any space $Z$, if $f \colon Z \times X \to S$ is continuous then so is its exponential transpose $\bar{f} \colon Z \to S^X$ defined by $\bar{f}(z) = (x \mapsto f(z, x))$.

Such an exponential topology doesn't always exist, but when it does it is easily seen to be unique. Criteria for existence and explicit constructions can be found in e.g. [3] or [2, Chapter 8], or in the extensive set of references contained therein, but they are not necessary for our purposes.

3.2 DEFINITION.  Let $\mathbb{S}$ be the Sierpinski space with an isolated point $\top$ (true) and a limit point $\bot$ (false). That is, the open sets are $\emptyset$, $\{\top\}$ and $\{\bot, \top\}$, but not $\{\bot\}$.

Then a map $p \colon X \to \mathbb{S}$ is continuous iff $p^{-1}(\top)$ is open, and a set $U \subseteq X$ is open iff its characteristic map $\chi_U \colon X \to \mathbb{S}$ is continuous. Previous proofs of the following theorem were based on the fact that if the exponential $\mathbb{S}^X$ exists, then its topology is the Scott topology. The present proof doesn't require this knowlegde, relying only on 2.3 and the universal property of exponentials given by Definition 3.1.

3.3  *If the exponential $\mathbb{S}^X$ exists, then the following are equivalent:*

1. *$X$ is compact.*

2. *The universal-quantification functional $A \colon \mathbb{S}^X \to \mathbb{S}$ defined by*

$$A(p) = \top \iff \forall x \in X.p(x) = \top$$

   *is continuous.*

PROOF  ($\Downarrow$): Because the evaluation map $e \colon \mathbb{S}^X \times X \to \mathbb{S}$ is continuous, the set $W \stackrel{\text{def}}{=} e^{-1}(\top)$ is open, and hence $\{p \in \mathbb{S}^X \mid \forall x \in X.(p, x) \in W\} = A^{-1}(\top)$ is open by compactness of $X$, and therefore $A$ is continuous.

($\Uparrow$): Let $Z$ be any space and $W \subseteq Z \times X$ be an open set. Because the transpose $w \colon Z \to \mathbb{S}^X$ of $\chi_W \colon Z \times X \to \mathbb{S}$ is continuous, so is $A \circ w \colon Z \to \mathbb{S}$, and hence $V \stackrel{\text{def}}{=} (A \circ w)^{-1}(\top)$ is open. But $z \in V$ iff $A(w(z)) = \top$ iff $\forall x \in X.w(z)(x) = \top$ iff $\forall x \in X.(z, x) \in W$. This shows that $\{z \in Z \mid \forall x \in X.(z, x) \in W\}$ is open, and hence that $X$ is compact. $\qquad\square$

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[1] Because the category of continuous maps of topological spaces is well pointed, this coincides with the categorical notion of exponential.

# 4 Generalization of Section 2

A proof that compactness of $X$ implies closedness of the projection $Z \times X \to Z$ for every space $Z$, which amounts to the implication 2.3($\Rightarrow$), is relatively easy. We now formulate and prove a generalization of this implication for families of compact subsets of the space $X$.

4.1 DEFINITION. We say that a family $\{Q_y \mid y \in Y\}$ of compact subsets of $X$ is continuously indexed[2] by a topological space $Y$ if for every neighbourhood $U$ of $Q_y$, there is a neighbourhood $T$ of $y$ such that $Q_t \subseteq U$ for all $t \in T$. This amounts to saying that the set $\{y \in Y \mid Q_y \subseteq U\}$ is open for every open set $U \subseteq X$.

The implication 2.3($\Rightarrow$) is a special case of the following, considering the space $Y$ with just one point $y$ and the trivial family $Q_y = X$.

4.2 LEMMA. *Let $\{Q_y \mid y \in Y\}$ be a continuously indexed family of compact sets of a space $X$, let $Z$ be any space, and $W \subseteq Z \times X$ be an open set. Then the set*

$$\{(z, y) \in Z \times Y \mid \forall_{x \in Q_y}.(z, x) \in W\}$$

*is open.*

Equivalently,

$$V_y \overset{\text{def}}{=} \{z \in Z \mid \{z\} \times Q_y \subseteq W\}$$

is a continuously indexed family of open sets of $Z$.

PROOF    To show that the set $M \overset{\text{def}}{=} \{(z, y) \in Z \times Y \mid \{z\} \times Q_y \subseteq W\}$ is open, we construct, for any pair $(z, y) \in M$, open sets $V$ and $T$ with $(z, y) \in V \times T \subseteq M$. So assume that $\{z\} \times Q_y \subseteq W$. For any $x \in Q_y$, we have that $(z, x) \in W$ and hence there are open sets $U_x$ and $V_x$ with $(z, x) \in V_x \times U_x \subseteq W$ by definition of the product topology. Then $Q_y \subseteq \bigcup\{U_x \mid x \in Q_y\}$, and, by compactness of $Q_y$, there is a finite set $I \subseteq Q_y$ such that already $Q_y \subseteq \bigcup\{U_i \mid i \in I\}$. Let $V \overset{\text{def}}{=} \bigcap_{i \in I} V_i$. Then $V$ is an open neighbourhood of $z$. By hypothesis, there is an open neighbourhood $T$ of $y$ such that $Q_t \subseteq \bigcup\{U_i \mid i \in I\}$ for all $t \in T$. To show that $V \times T \subseteq M$, let $(v, t) \in V \times T$. For any $x \in Q_t$, there is $i \in I$ such that $x \in U_i$, and hence $(v, x) \in V \times U_i \subseteq V_i \times U_i \subseteq W$, which shows that $\{v\} \times Q_t \subseteq \bigcup_{i \in I} V \times U_i \subseteq W$, and therefore that $(v, t) \in M$, as required.    □

That closedness of the projection $Z \times X \to Z$ for every space $Z$ implies compactness of $X$, which amounts to the implication 2.3($\Leftarrow$), is less trivial. Typical proofs apply the characterization of compactness via cluster points of filters (see e.g. the proof of [1, Lemma 10.2.1, page 101]). We offer a proof of a slight generalization of 2.3($\Leftarrow$) that is closely related to, and inspired by those of [3, Lemma 4.4] and [2, Theorem 9.5].

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[2]This is equivalent to continuity of the map $y \mapsto Q_y$ when the collection of compact sets is endowed with the upper Vietoris topology. For a family $V_x$ of open sets of $Z$, however, there isn't a topology on the collection of open sets of $Z$ such that continuity of the family is equivalent to continuity of the map $x \mapsto V_x$, unless $Z$ is an exponentiable space — see e.g. [3, Corollary 4.6].

This argument will be reused later to prove a more general fact about relative compactness (Section 6).

Recall that a collection $\mathcal{C}$ of open sets is called directed if for any finite set $\mathcal{S} \subseteq \mathcal{C}$ there is $U \in \mathcal{C}$ with $\bigcup \mathcal{S} \subseteq U$. Any collection of open sets can be made directed by adding the finite unions of its members. Hence a set $Q$ is compact if and only if every directed open cover of $Q$ has a member that covers $Q$.

**4.3** *Let $Q$ be a subset of a space $X$. If for every space $Z$ and any open set $W \subseteq Z \times X$, the set $\{z \in Z \mid \{z\} \times Q \in W\}$ is open, then $Q$ is compact.*

PROOF    Let $\mathcal{C}$ be a directed open cover of $Q$.

We first construct a space $Z$ from $X$ and $\mathcal{C}$: its points are the open sets of $X$, and $V \subseteq Z$ is open iff (1) $U \in V$ and $U \subseteq U' \in Z$ together imply $U' \in V$, and (2) if $\bigcup \mathcal{C} \in V$ then $U \in V$ for some $U \in \mathcal{C}$. Such open sets are readily seen to form a topology[3], using the fact that $\mathcal{C}$ is directed, and if $U$ is an open subset of a member of $\mathcal{C}$ then $\uparrow U \stackrel{\text{def}}{=} \{U' \in Z \mid U \subseteq U'\}$ is clearly open.

Next, we take $W \stackrel{\text{def}}{=} \{(U,x) \in Z \times X \mid x \in U\}$. To show that $W$ is open, let $(U,x) \in W$ and consider two cases. (1) $x \in \bigcup \mathcal{C}$: Then $x \in U'$ for some $U' \in \mathcal{C}$ with $x \in U'$, and hence $(U,x) \in \uparrow(U \cap U') \times (U \cap U') \subseteq W$. (2) $x \notin \bigcup \mathcal{C}$: Then $U \not\subseteq \bigcup \mathcal{C}$ and hence $\uparrow U$ is open, and $(U,x) \in \uparrow U \times U \subseteq W$.

Finally, by the hypothesis, the set $V \stackrel{\text{def}}{=} \{U \in Z \mid \{U\} \times Q \subseteq W\}$ is open, and clearly $U \in V$ iff $Q \subseteq U$. Hence $\bigcup \mathcal{C} \in V$ and so some member of $\mathcal{C}$ is in $V$, that is, covers $Q$, by construction of the topology of $Z$, as required.    □

For future reference, we summarize part of the above development as follows:

**4.4 LEMMA.**  *The following are equivalent for any subset $Q$ of any topological space $X$.*

1. *$Q$ is compact.*

2. *For every space $Z$, the set $\{z \in Z \mid \forall q \in Q.(z,q) \in W\}$ is open whenever the set $W \subseteq Z \times X$ is open.*

3. *For every space $Z$, the set $\{z \in Z \mid \{z\} \times Q \subseteq W\}$ is open whenever the set $W \subseteq Z \times X$ is open.*

# 5   Sample "synthetic" proofs of old theorems

We redevelop the synthetic proofs of [2, Chapter 9] almost literally, but without invoking the function-space machinery or the lambda-calculus.

**5.1** *If $X$ is Hausdorff and $Q \subseteq X$ is compact, then $Q$ is closed in $X$.*

PROOF    Because $X$ is Hausdorff, the complement $W$ of the diagonal is open. Hence $X \setminus Q = \{x \in X \mid \forall q \in Q.x \neq q\} = \{x \in X \mid \forall q \in Q.(x,q) \in W\}$ is open by Lemma 4.4, and so $Q$ is closed.    □

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[3]This is like the Scott topology, but defined with respect to one particular directed set, rather than all directed sets. One cannot use the Scott topology for this proof, as, in general, it doesn't give rise to openness of the set $W$ constructed in the proof — see e.g. [3, Corollary 4.6].

**5.2** *If $X$ is compact and $F \subseteq X$ is closed then $F$ is compact.*

PROOF    We use Lemma 4.4. Let $Z$ be any space and $W \subseteq Z \times X$ be open. We have to show that $V \stackrel{\text{def}}{=} \{z \in Z \mid \forall x \in F.(z,x) \in W\}$ is open. But $z \in V$ iff $\forall x \in X.x \in F \implies (z,x) \in W$ iff $\forall x \in X.x \notin F \vee (z,x) \in W$. Hence $V = \{z \in Z \mid \forall x \in X.(z,x) \in W'\}$ where $W' = (Z \times (X \setminus F)) \cup W$, and $V$ is open by compactness of $X$, openness of $W'$ and 2.3($\Rightarrow$). $\square$

**5.3** *If $f \colon X \to Y$ is continuous and the set $Q \subseteq X$ is compact, then so is $f(Q)$.*

PROOF    For any space $Z$ and any open set $W \subseteq Z \times Y$, we have that $\{z \in Z \mid \forall y \in f(Q).(z,y) \in W\} = \{z \in Z \mid \forall x \in Q.(z, f(x)) \in W\}$, which is open by compactness of $Q$, because the set $W'$ defined by $(z,x) \in W'$ iff $(z, f(x)) \in W$ is open by continuity of $f$. $\square$

**5.4** *If $X$ and $Y$ are compact spaces then so is $X \times Y$.*

PROOF    We show that $V \stackrel{\text{def}}{=} \{z \in Z \mid \forall (x,y) \in X \times Y.(z,x,y) \in W\}$ is open for any space $Z$ and any open set $W \subseteq Z \times X \times Y$. By compactness of $Y$, the set $W' \stackrel{\text{def}}{=} \{(z,x) \in Z \times X \mid \forall y \in Y.(z,x,y) \in W\}$ is open, and, by compactness of $X$, the set $\{z \in Z \mid \forall x \in X.(z,x) \in W'\} = V$ is open, as required. $\square$

Although we don't need the function-space machinery to develop the core of topology, we still can use the function-space-free synthetic approach to prove theorems about function spaces, as we have done in Section 3. Moreover, the abstract definition of function space as an exponential again suffices.

**5.5** *If $Y$ is Hausdorff, then so is the exponential $Y^X$ if it exists.*

PROOF    The codiagonal of $Y^X$ is $\{(f,g) \in Y^X \times Y^X \mid \exists x \in X.f(x) \neq g(x)\} = \bigcup_{x \in X} \{(f,g) \in Y^X \times Y^X \mid f(x) \neq g(x)\}$, which is a union of open sets, because $W \subseteq Y^X \times Y^X$ defined by $(f,g) \in W$ iff $f(x) \neq g(x)$ is open, using openness of the codiagonal of $Y$ and continuity of the evaluation map $Y^X \times X \to Y$. $\square$

For the proof of the following dual proposition, recall that a space is discrete iff its diagonal is open.

**5.6** *If $X$ is compact and $Y$ is discrete, then the exponential $Y^X$ is discrete if it exists.*

PROOF    The diagonal of $Y^X$ is $\{(f,g) \in Y^X \times Y^X \mid \forall x \in X.f(x) = g(x)\}$, which is open by compactness of $X$, because the set $W \subseteq Y^X \times Y^X \times X$ defined by $(f,g,x) \in W$ iff $f(x) = g(x)$ is open, using openness of the diagonal of $Y$ and continuity of the evaluation map. $\square$

As discussed above, these last two propositions don't require an intrinsic description of the topology of $Y^X$. A partial description is given by the following:

**5.7** *If the exponential $Y^X$ exists, and if $Q \subseteq X$ is compact and $V \subseteq Y$ is open, then the set $N(Q,V) \stackrel{\text{def}}{=} \{f \in Y^X \mid f(Q) \subseteq V\}$ is open.*

PROOF    $f \in N(Q,V)$ iff $\forall q \in Q.f(q) \in V$. The result then follows from the fact that $W \subseteq Y^X \times X$ defined by $(f,x) \in W$ iff $f(x) \in V$ is open, using continuity of the evaluation map. $\square$

Recall that a continuous map $f \colon X \to Y$ is called proper if the product map

$$\mathrm{id}_Z \times f \colon Z \times X \to Z \times Y$$

is closed for every space $Z$, where $\mathrm{id}_Z \colon Z \to Z$ is the identity map [1].

5.8 *The following are equivalent for any continuous map $f \colon X \to Y$.*

  1. *$f$ is proper.*

  2. *For every space $Z$ and every open set $W \subseteq Z \times X$, the set*

  $$\{(z,y) \in Z \times Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$$

  *is open.*

  3. *$f$ is closed and the set $f^{-1}(Q)$ is compact for every compact set $Q \subseteq Y$.*

  4. *$f$ is closed and the set $f^{-1}\{y\}$ is compact for every point $y \in Y$.*

  5. *$\{f^{-1}\{y\} \mid y \in Y\}$ is a continuously indexed family of compact sets of $X$.*

We first refomulate closedness in terms of open sets. By taking complements, a continuous map $g \colon A \to B$ is closed iff for every open set $U \subseteq A$, the set $B \setminus g(A \setminus U)$ is open. But an easy calculation shows that this set is $\{b \in B \mid g^{-1}\{b\} \subseteq U\}$. This proves:

5.9 LEMMA. *A continuous map $g \colon A \to B$ is closed if and only if for every open set $U \subseteq A$, the set $\{b \in B \mid g^{-1}\{b\} \subseteq U\}$ is open.*

PROOF    of 5.8.

  (1) $\Leftrightarrow$ (2): Calculate that $(\mathrm{id}_Z \times f)^{-1}\{(z,y)\} = \{z\} \times f^{-1}\{y\}$ and then apply Lemma 5.9 to $g = \mathrm{id}_Z \times f$.

  (1,2) $\Rightarrow$ (3): Considering the case in which $Z$ is the one-point space, we see that any proper map is closed. To show that $f^{-1}(Q)$ is compact, let $Z$ be any space and $W \subseteq Z \times X$ be an open set. Then the set $T \overset{\mathrm{def}}{=} \{(z,y) \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$ is open by hypothesis, and hence the set $U \overset{\mathrm{def}}{=} \{z \in Z \mid \{z\} \times Q \subseteq T\}$ is open by Lemma 4.4. But $z \in U$ iff $(z,y) \in T$ for all $y \in Q$, iff $\{z\} \times f^{-1}\{y\} \subseteq W$ for all $y \in Q$, iff $\{z\} \times f^{-1}(Q) \subseteq W$. Because $Z$ and $W$ are arbitrary, a second application of Lemma 4.4 shows that $f^{-1}(Q)$ is compact, as required.

  (3) $\Rightarrow$ (4): Singletons are compact.

  (4) $\Rightarrow$ (5): By Lemma 5.9 applied to $g = f$, the set $\{y \in Y \mid f^{-1}\{y\} \subseteq U\}$ is open for every $y \in Y$ and every open set $U \subseteq X$.

  (5) $\Rightarrow$ (2): This follows directly from Lemma 4.2.  □

The above characterizations (3) and (4) of propriety are of course well known. The development of synthetic proofs was left as an exercise in [2]. Characterization (2) is clearly just a reformulation of the definition using the language of open sets. Formulation (5) seems to be new.

# 6   Relative compactness

For some topological questions regarding local compactness and function spaces, it is fruitful to consider the domain-theoretic way-below relation on open sets [5]. Again in a context pertaining to function spaces, Escardó, Lawson and Simpson [4] found it profitable to generalize this to arbitrary subsets of topological spaces.

For subsets $S$ and $T$ of a topological space $X$, we define

$$S \ll T \quad \Longleftrightarrow \quad \text{every cover of } T \text{ by open sets of } X$$
$$\text{has a finite subcollection that covers } S.$$

In this case one says that $S$ is *way below* $T$, or *compact relative* to $T$. Then it is immediate that a set is compact iff it is compact relative to itself. We also define

$$S \Subset T \quad \Longleftrightarrow \quad S \subseteq T^\circ.$$

The following was formulated as [4, Lemma 4.2]:

6.1  *Let $X$ and $Y$ be topological spaces.*

1. *If $F \ll X$ is closed, then $F$ is compact.*

2. *If $X$ is Hausdorff and $S \ll T$ holds in $X$, then $\overline{S} \subseteq T$.*

3. *If $f \colon X \to Y$ is continuous and $S \ll T$ in $X$, then $f(S) \ll f(T)$ holds in $Y$.*

4. *If $S \ll T$ in $X$ and $A \ll B$ in $Y$, then $S \times A \ll T \times B$ holds in $X \times Y$.*

5. *If $W \subseteq Y \times X$ is open and $S \ll T$ holds in $X$, then*

$$\{y \in Y \mid \{y\} \times T \subseteq W\} \Subset \{y \in Y \mid \{y\} \times S \subseteq W\}.$$

Assertion (1) generalizes the fact that a closed subset of a compact space is compact, (2) the statement that a compact subset of a Hausdorff space is closed, (3) the fact that continuous maps preserve compactness, and (4) the Tychonoff theorem in the finite case. In this section we prove (5) and a converse, generalizing 2.3 and the development of Section 4, and use this to derive (1)–(4), generalizing the development of Section 5.

6.2 THEOREM. *The following are equivalent for any two subsets $S$ and $T$ of a topological space $X$.*

1. *$S \ll T$.*

2. *For every space $Z$ and every open set $W \subseteq Z \times X$,*

$$\{z \in Z \mid \{z\} \times T \subseteq W\} \Subset \{z \in Z \mid \{z\} \times S \subseteq W\}.$$

3. *For every space $Z$, every $z \in Z$ and every open set $W \subseteq Z \times X$,*

$$\{z\} \times T \subseteq W \implies V \times S \subseteq W \text{ for some neighbourhood } V \text{ of } z.$$

4. *For every space $Z$ and all $M, N \subseteq Z \times X$,*

$$M \Subset N \implies \{z \in Z \mid \{z\} \times T \subseteq M\} \Subset \{z \in Z \mid \{z\} \times S \subseteq N\}.$$

PROOF  $(2) \Leftrightarrow (3)$: By definition of interior. $(2) \Leftrightarrow (4)$: Consider $W = N^\circ$ in one direction and $M = N = W$ in the other.

$(1) \Rightarrow (3)$: Assume that $\{z\} \times T \subseteq W$. Then for any $t \in T$, we have that $(z, t) \in W$ and hence there are open sets $U_t$ and $V_t$ with $(z, t) \in V_t \times U_t \subseteq W$. Because $T \subseteq \bigcup_{t \in T} U_t$ and $S \ll T$, there is a finite set $I \subseteq T$ such that $S \subseteq \bigcup_{i \in I} U_i$. Then $V \overset{\text{def}}{=} \bigcap_{i \in I} V_i$ is open and $z \in V$. To show that $V \times S \subseteq W$, let $(v, s) \in V \times S$. Because $s \in S \subseteq \bigcup_{i \in I} U_i$, there is $j \in I$ such that $s \in U_j$, and because $V = \bigcap_{i \in I} V_i$ we have that $v \in V_j$. Hence $(v, s) \in V_j \times U_j \subseteq W$, as required.

$(3) \Rightarrow (1)$. To show that $S \ll T$, let $\mathcal{C}$ be a directed open cover of $T$. We have to conclude that $S \subseteq U$ for some $U \in \mathcal{C}$. We first construct a space $Z$ from $X$ and $\mathcal{C}$, and an open set $W \subseteq Z \times X$ as in the proof of 4.3. Because $T \subseteq \bigcup \mathcal{C}$, we have that $\{\bigcup \mathcal{C}\} \times T \subseteq W$. Hence, by the hypothesis, $V \times S \subseteq W$ for some neighbourhood $V$ of $\bigcup \mathcal{C}$, which may be assumed to be open. By construction of the topology of $Z$, we have that $U \in V$ for some $U \in \mathcal{C}$. To show that $S \subseteq U$, concluding the proof, let $s \in S$. Then $(U, s) \in V \times S \subseteq W$, and hence $s \in U$, as required.  $\square$

Notice that 2.3 follows directly from Theorem 6.2$(1 \Leftrightarrow 2)$, because a set is open iff it is contained in its interior. Observe also that the implication $(1) \Rightarrow (3)$ amounts to saying that if the relation $\{y\} \times T \subseteq W$ holds, and if we make $T$ significantly smaller by passing to a set way below, then we can make $\{y\}$ significantly bigger by passing to a whole neighbourhood so that the relation will still hold. We now apply Theorem 6.2 to generalize some of the proofs of Section 5.

6.3  *If $X$ is Hausdorff and $S \ll T$, then $\overline{S} \subseteq T$.*

PROOF  Because the complement $W \subseteq X \times X$ of the diagonal is open as $X$ is Hausdorff, Theorem 6.2$(1 \Rightarrow 2)$ shows that $X \setminus T = \{x \in X \mid \{x\} \times T \subseteq W\} \Subset \{x \in X \mid \{x\} \times S \subseteq W\} = X \setminus S$, and hence $\overline{S} \subseteq T$.  $\square$

**6.4** *If $F$ is closed in $X$ and $F \ll X$, then $F$ is compact.*

PROOF    Let $Z$ be any space and $W \subseteq Z \times X$ be open. Then $W' = (Z \times (X \setminus F)) \cup W$ is also open, and Theorem 6.2($2 \Rightarrow 1$) gives $M \stackrel{\text{def}}{=} \{z \in Z \mid \{z\} \times X \subseteq W'\} \Subset N \stackrel{\text{def}}{=} \{z \in Z \mid \{z\} \times F \subseteq W'\}$. But one readily checks that $M$ and $N$ are equal to $\{z \in Z \mid \forall x \in F.(z,x) \in W\}$, and hence, being contained in its own interior, this set is open. Because the space $Z$ and the open set $W \subseteq Z \times X$ are arbitrary, the desired result follows from Lemma 4.4.    $\square$

**6.5** *If $f\colon X \to Y$ is continuous and $S \ll T$ in $X$, then $f(S) \ll f(T)$ holds in $Y$.*

PROOF    Let $Z$ be a space, $W \subseteq Z \times Y$ be open, and assume that $\{z\} \times f(T) \subseteq W$. Then $W' \stackrel{\text{def}}{=} (\operatorname{id}_Z \times f)^{-1}(W) = \{(z,x) \in Z \times X \mid (z, f(x)) \in W\}$ is also open by continuity of $f$, and $\{z\} \times T \subseteq W'$. By Theorem 6.2($1 \Rightarrow 3$), there is a neighbourhood $V$ of $z$ with $V \times S \subseteq W'$. Hence $V \times f(S) \subseteq W$. Because the space $Z$, the open set $W \subseteq Z \times Y$ and the point $z \in Z$ are arbitrary, Theorem 6.2($3 \Rightarrow 1$) shows that $f(S) \ll f(T)$, as required.    $\square$

**6.6** *If $S \ll T$ in $X$ and $A \ll B$ in $Y$, then $S \times A \ll T \times B$ holds in $X \times Y$.*

PROOF    Let $Z$ be a space and let $M, N \subseteq Y \times X \times Y$ with $M \Subset N$. Then, by two successive applications of Theorem 6.2($1 \Rightarrow 4$), we first have that

$$M' \stackrel{\text{def}}{=} \{(z, x_0) \in Z \times X \mid \{(z,x)\} \times B \subseteq M\}$$
$$\Subset N' \stackrel{\text{def}}{=} \{(z, x) \in Z \times X \mid \{(z,x)\} \times A \subseteq N\}$$

and then that $M'' \stackrel{\text{def}}{=} \{z \in Z \mid \{z\} \times T \subseteq M'\} \Subset N'' \stackrel{\text{def}}{=} \{z \in Z \mid \{z\} \times S \subseteq N'\}$. But one readily checks that $M'' = \{z \in Z \mid \{z\} \times T \times B \subseteq M\}$ and $N'' = \{z \in Z \mid \{z\} \times S \times A \subseteq N\}$. Because the space $Y$ and the sets $M, N \subseteq Y \times X \times Y$ are arbitrary, the result follows from Theorem 6.2($4 \Rightarrow 1$).    $\square$

# 7   Compactly generated spaces

In this section we assume familiarity with the notions and results developed in [4] and with domain theory [5].

Let $\mathcal{E}$ be the class of all spaces that are exponentiable in the category of topological spaces, and $\mathcal{C} \subseteq \mathcal{E}$ be any productive class of spaces. If $\mathcal{C}$ consists of the compact Hausdorff spaces, then the $\mathcal{C}$-generated spaces (or $\mathcal{C}$-spaces for short) are known as the *compactly generated spaces*.

The categorical product in the category of $\mathcal{C}$-spaces is given by the $\mathcal{C}$-coreflection of the topological product: $X \times_{\mathcal{C}} Y = \mathcal{C}(X \times Y)$. Recall that the $\mathcal{C}$-coreflection $\mathcal{C}X$ of a topological space $X$ is obtained by keeping the same points and suitably refining the given topology of $X$. By [4, Theorem 5.4], we know that $X \times_{\mathcal{C}} Y = X \times_{\mathcal{E}} Y$ for all $\mathcal{C}$-spaces $X$ and $Y$. That is, the $\mathcal{C}$-product doesn't depend on $\mathcal{C}$, even though the $\mathcal{C}$-coreflection does. We were thus led to ask whether there is an instrinsic characterization of the $\mathcal{C}$-product [4, Problem 9.3]. We now develop and answer to

this question, formulated as Theorem 7.2 below. We know that the Sierpinski space is a $\mathcal{C}$-generated space if and only if the generating class $\mathcal{C}$ includes a space in which not every open set is closed [4, Lemma 4.6(ii)]. In particular, the Sierpinski space is $\mathcal{E}$-generated.

7.1 LEMMA. *Assume that the Sierpinski space is $\mathcal{C}$-generated. For a $\mathcal{C}$-generated space $X$, let*

$$\mathcal{O}_\mathcal{C}\, X$$

*be the lattice of open sets of $X$ endowed with the topology that makes the bijection $U \mapsto \chi_U \colon \mathcal{O}_\mathcal{C}\, X \to \mathbb{S}^X$ into a homeomorphism, where the exponential is calculated in the category $\mathrm{Top}_\mathcal{C}$ of $\mathcal{C}$-spaces.*

1. *The topology of $\mathcal{O}_\mathcal{C}\, X$ is finer than the Scott topology.*

2. *The topology of $\mathcal{O}_\mathcal{C}\, X$ coincides with the Scott topology if $\mathcal{C}$ generates all compact Hausdorff spaces.*

3. *A set $W \subseteq Y \times_\mathcal{C} X$ is open if and only if its transpose $w : Y \to \mathcal{O}_\mathcal{C}\, X$ defined by $w(y) = \{x \in X \mid (y, x) \in W\}$ is continuous.*

4. *The set $\{(U, x) \in \mathcal{O}_\mathcal{C}\, X \times_\mathcal{C} X \mid x \in U\}$ is open in the $\mathcal{C}$-product.*

PROOF    (1): [4, Theorem 5.15]. (2): [4, Corollary 5.16]. (3): By definition of exponential transpose. (4): Its transpose is the identity of $\mathcal{O}_\mathcal{C}\, X$.    □

If $\mathcal{C}$ doesn't generate all compact Hausdorff spaces, the second item doesn't necessarily hold. For example, if $\mathcal{C}$ is a singleton consisting of the one-point compactification of the discrete natural numbers (known the "generic convergent sequence"), then a space is $\mathcal{C}$-generated if and only if it is sequential, and for a sequential space $X$ we have that $\mathcal{U} \subseteq \mathcal{O}_\mathcal{C}\, X$ is open if and only if it is upwards closed and inaccessible by unions of countable directed sets. If $X$ is a Lindelöf space, as is the case if $X$ is a QCB space, this does coincide with the Scott topology, but, in general, this is strictly finer than the Scott topology. The following holds without any assumption on $\mathcal{C}$ other than that it is contained in $\mathcal{E}$ and that it is productive.

7.2 THEOREM. *If $X$ and $Y$ are $\mathcal{C}$-spaces, then the following are equivalent for any set $W \subseteq Y \times X$.*

1. *$W$ is open in $Y \times_\mathcal{C} X$.*

2. (a) *For each $y \in Y$, the set $U_y \stackrel{\text{def}}{=} \{x \in X \mid (y, x) \in W\}$ is open, and*

   (b) *for each Scott open set $\mathcal{U} \subseteq \mathcal{O}\, X$, the set $V_\mathcal{U} \stackrel{\text{def}}{=} \{y \in Y \mid U_y \in \mathcal{U}\}$ is open.*

3. (a) *For each $x \in X$, the set $V_x \stackrel{\text{def}}{=} \{y \in Y \mid (y, x) \in W\}$ is open, and*

   (b) *for each Scott open set $\mathcal{V} \subseteq \mathcal{O}\, Y$, the set $U_\mathcal{V} \stackrel{\text{def}}{=} \{x \in X \mid V_x \in \mathcal{V}\}$ is open.*

Proof    We prove (1) $\Leftrightarrow$ (2). A proof of (1) $\Leftrightarrow$ (3) is obtained via the canonical homeomorphism $X \times_{\mathcal{C}} Y \cong Y \times_{\mathcal{C}} X$.

(1) $\Rightarrow$ (2): As we have already discussed, if $W$ is open in the $\mathcal{C}$-product, then it is also open in the $\mathcal{E}$-product. Because the Sierpinski space $\mathbb{S}$ is an $\mathcal{E}$-space, its transpose $w \colon Y \to \mathcal{O}_{\mathcal{E}} X$ defined in the previous lemma is continuous. Then one readily checks that $w(y) = U_y$ and $w^{-1}(\mathcal{U}) = V_{\mathcal{U}}$, which shows that $U_y$ and $V_{\mathcal{U}}$ are open, as required.

(2) $\Rightarrow$ (1): By the hypothesis, the map $w \colon Y \to \mathcal{O}_{\mathcal{E}} X$ given by $w = U_y$ is well defined and continuous. But one readily checks that this is the transpose of $W$ defined in Lemma 7.1, and hence $W$ is open in the $\mathcal{E}$-product, and therefore in the $\mathcal{C}$-product, as required. $\square$

We now return to the subject of compactness. We henceforth assume that the Sierpinski space is $\mathcal{C}$-generated.

7.3 Theorem. *The following are equivalent for any subset $Q$ of a $\mathcal{C}$-space $X$.*

1. *The set $\{U \in \mathcal{O}_{\mathcal{C}} X \mid Q \subseteq U\}$ is open.*

2. *For every $\mathcal{C}$-space $Y$, and every open set $W \subseteq Y \times_{\mathcal{C}} X$, the set*

$$\{y \in Y \mid \{y\} \times Q \subseteq W\}$$

   *is open.*

3. *The universal-quantification functional $A_Q \colon \mathbb{S}^X \to \mathbb{S}$ defined by*

$$A_Q(p) = \top \iff \forall x \in Q.p(x) = \top$$

   *is continuous, where the exponential is calculated in $\mathrm{Top}_{\mathcal{C}}$.*

Proof    (1) $\Rightarrow$ (2): One readily checks that the set $\{y \in Y \mid \{y\} \times Q \subseteq W\}$ is the same as $V_{\mathcal{U}}$ in Theorem 7.2(2) for the choice $\mathcal{U} = \{U \in \mathcal{O}_{\mathcal{C}} X \mid Q \subseteq U\}$.

(2) $\Rightarrow$ (3) Because the evaluation map $e \colon \mathbb{S}^X \times_{\mathcal{C}} X \to \mathbb{S}$ is continuous, the set $W \stackrel{\text{def}}{=} e^{-1}(\top)$ is open, and hence the set $\{p \in \mathbb{S}^X \mid \{p\} \times Q \subseteq W\} = \{p \in \mathbb{S}^X \mid \forall x \in Q.p(x) = \top\} = A^{-1}(\top)$ is open by the hypothesis, and therefore $A_Q$ is continuous.

(3) $\Rightarrow$ (1): The set $\{U \in \mathcal{O}_{\mathcal{C}} X \mid Q \subseteq U\}$ is the inverse image of $\{\top\}$ for the composite $A_Q \circ (U \mapsto \chi_U) \colon \mathcal{O}_C X \to \mathbb{S}^X \to \mathbb{S}$. $\square$

7.4 Definition. When these equivalent conditions hold, we say that $Q$ is *$\mathcal{C}$-compact.*

For example, it follows from the above observations that if the class $\mathcal{C}$ is a singleton consisting of the generic convergent sequence, then a $\mathcal{C}$-generated space (i.e. a sequential space) is $\mathcal{C}$-compact if and only if every countable open cover has a finite subcover. However, for compactly generated spaces, the same notion of compactness is obtained, as shown by the next proposition. We first formulate an immediate consequence of the above theorem.

7.5 Corollary. *A $\mathcal{C}$-space $X$ is $\mathcal{C}$-compact if and only if the projection $Y \times_{\mathcal{C}} X \to Y$ is closed for every $\mathcal{C}$-space $Y$.*

Proof    Use the De Morgan laws as in Section 2. $\square$

7.6 PROPOSITION. *Any compact set is $\mathcal{C}$-compact. If the class $\mathcal{C}$ generates all compact Hausdorff spaces, the converse holds.*

PROOF   If $Q$ is compact subset of a $\mathcal{C}$-space $X$, then $\{U \in \mathcal{O}X \mid Q \subseteq U\}$ is Scott open by definition of the Scott topology, and hence open in $\mathcal{O}_{\mathcal{C}}X$ by Lemma 7.1. Conversely, if $Q$ is $\mathcal{C}$-compact and the hypothesis holds, then $\{U \in \mathcal{O}X \mid Q \subseteq U\}$ is Scott open by Lemma 7.1, and hence compact by definition of the Scott topology.   $\square$

Thus, even though $Y \times_{\mathcal{C}} X$ has a greater (and somewhat mysterious) supply of open sets than $Y \times X$, it is still the case that if $Q$ is compact then for every open set $W \subseteq Y \times_{\mathcal{C}} X$, the set $\{y \in Y \mid \{y\} \times Q \subseteq W\}$ is open.

7.7 DEFINITION. We say that a $\mathcal{C}$-space $X$ is $\mathcal{C}$-*Hausdorff* if its diagonal is closed in $X \times_{\mathcal{C}} X$, and that it is $\mathcal{C}$-*discrete* if its diagonal is open in $X \times_{\mathcal{C}} X$.

If a $\mathcal{C}$-space is Hausdorff (resp. discrete) then it is $\mathcal{C}$-Hausdorff (resp. -discrete), because the $\mathcal{C}$-product has a topology finer than the topological product. There must be $\mathcal{C}$-Hausdorff spaces which are not Hausdorff, but I doubt that this holds for discreteness.

We have developed enough ideas and techniques to routinely develop proofs of the following, and hence we omit them:

7.8 PROPOSITION. *Let $X$ and $Y$ be $\mathcal{C}$-spaces.*

1. *If $X$ and $Y$ are $\mathcal{C}$-compact, then so is $X \times_{\mathcal{C}} Y$.*

   This potentially fails if one replaces $\mathcal{C}$-compactness by topological compactness, because the $\mathcal{C}$-product has a topology finner than the topological product.

2. *If $f : X \to Y$ is continuous and $Q \subseteq X$ is $\mathcal{C}$-compact, then so is $f(Q)$.*

3. *If $X$ is $\mathcal{C}$-Hausdorff and $Q \subseteq X$ is $\mathcal{C}$-compact, then $Q$ is closed.*

   Notice that this is stronger than the statement that a compact subspace of a Hausdorff $\mathcal{C}$-space is closed, as it has weaker hypotheses.

4. *If $F \subseteq X$ is closed and $X$ is $\mathcal{C}$-compact, then so is $F$.*

5. *If $Y$ is $\mathcal{C}$-Hausdorff, then so is the exponential $Y^X$.*

6. *If $X$ is $\mathcal{C}$-compact and $Y$ is $\mathcal{C}$-discrete, then the exponential $Y^X$ is $\mathcal{C}$-discrete.*

7. *If $Q \subseteq X$ is $\mathcal{C}$-compact and $V \subseteq Y$ is open, then $\{f \in Y^X \mid f(Q) \subseteq V\}$ is open.*

Nb. We can define the $\mathcal{C}$-*Isbell topology* on the set of continuous maps $X \to Y$ as the usual Isbell topology, replacing Scott openness by openness in $\mathcal{O}_{\mathcal{C}}X$. It is easy to see that the exponential topology is finer than the $\mathcal{C}$-Isbell topology.

We now develop another application of Theorem 7.2. It is well-known that the (full and faithful) functor $\Sigma : \mathrm{DCPO} \to \mathrm{Top}$ from the category of dcpos to topological spaces, that endows a dcpo with its Scott topology and acts identically on maps, fails to preserve finite products [5]. By [4, Theorem 4.7], we know that dcpos under the

Scott topology are compactly generated. Thus, if every compactly generated space is a $\mathcal{C}$-space then $\Sigma$ factors through the category $\mathrm{Top}_\mathcal{C}$ of $\mathcal{C}$-spaces. This is the case, for instance, if $\mathcal{C} = \mathcal{E}$ or $\mathcal{C}$ consists of all compact Hausdorff spaces or of all locally compact spaces.

7.9 THEOREM. *If $\mathcal{C} \subseteq \mathcal{E}$ generates all compact Hausdorff spaces, then the functor $\Sigma\colon \mathrm{DCPO} \to \mathrm{Top}_\mathcal{C}$ preserves finite products.*

PROOF    Let $D$ and $E$ be dcpos. By Theorem 7.2($1 \Leftrightarrow 2$), it is enough to show that $W \subseteq D \times E$ is Scott open iff (a) for each $d \in D$ the set $V_d \stackrel{\mathrm{def}}{=} \{e \in D \mid (d, e) \in W\}$ is Scott open, and (b) for each Scott open set $\mathcal{V}$ of Scott open sets of $E$, the set $U_\mathcal{V} \stackrel{\mathrm{def}}{=} \{d \in D \mid V_d \in \mathcal{V}\}$ is Scott open. We omit the somewhat long, but routine verification that this is the case.

$\square$

Here is a another argument that side-steps Theorem 7.2 but uses the same ingredients as its proof:

PROOF    Let $D$ and $E$ be two dcpos. Write $\mathcal{A}(A, B)$ to denote the hom-set of a pair $A, B$ of objects of a category $A$, and $\mathcal{A}[A, B]$ to denote the exponential $B^A$ if it exists. Then, regarding $\mathbb{S}$ both as a ($\mathcal{C}$-)space and a dcpo by an abuse of notation, we calculate, using obvious canonical isomorphisms:

$$
\begin{aligned}
\mathcal{O}(\Sigma D \times_\mathcal{C} \Sigma E) &\cong \mathrm{Top}_\mathcal{C}(\Sigma D \times_\mathcal{C} \Sigma E, \mathbb{S}) \\
&\cong \mathrm{Top}_\mathcal{C}(\Sigma D, \mathrm{Top}_\mathcal{C}[\Sigma E, \mathbb{S}]) \\
&\cong \mathrm{Top}_\mathcal{C}(\Sigma D, \Sigma \mathrm{DCPO}[E, \mathbb{S}]) \quad \text{by [4, Corollary 5.16]} \\
&\cong \mathrm{DCPO}(D, \mathrm{DCPO}[E, \mathbb{S}]) \\
&\cong \mathrm{DCPO}(D \times_{\mathrm{DCPO}} E, \mathbb{S}) \\
&\cong \mathcal{O}\, \Sigma(D \times_{\mathrm{DCPO}} E).
\end{aligned}
$$

Moreover, the composition of all the canonical isomorphisms is easily seen to be the identity, because the transpositions are calculated as in the category of sets, and hence $\mathcal{O}(\Sigma D \times_\mathcal{C} \Sigma E) = \mathcal{O}\, \Sigma(D \times_{\mathrm{DCPO}} E)$. Because both products are set-theoretical products with appropriate structure, we conclude that $\Sigma D \times_\mathcal{C} \Sigma E = \Sigma(D \times_{\mathrm{DCPO}} E)$, as required. $\square$

As a further corollary we obtain the known fact that the restriction of the functor $\Sigma\colon \mathrm{DCPO} \to \mathrm{Top}$ to continuous dcpos preserves finite products. The reason is that continuous dcpos are core-compact in the Scott topology, and hence are in the class $\mathcal{E}$, and that $X \times_\mathcal{E} Y = X \times Y$ if one of the factors is in $\mathcal{E}$. Moreover, this argument establishes, more generally, the following fact, which is also known [5]:

7.10 COROLLARY. *The restriction of the functor $\Sigma\colon \mathrm{DCPO} \to \mathrm{Top}$ to dcpos that are core-compact in their Scott topology preserves finite products.*

# 8 To do

For the moment, this set of notes is still evolving, in slow motion, because I have many other things to do. The following are notes for myself, so that I don't forget what I intend to include in the future.

Beyond topology and related work: (1) Locale theory. (2) Dubuc and Penon's notion of compactness for "sets" in a topos. (3) Connections with Taylor's ASD. (4) Bauer and Scott's work on equilogical spaces. (5) Computational spaces. When this is done, rename these notes to "Compactness in topology and beyond", and write a more substantial introduction.

Further work: Define a "calculus of open relations", to make the synthetic proofs more transparent, but yet avoiding the lambda-calculus. Any relation defined in this calculus is automatically open. The terms of the calculus are built from continuous maps and automatically denote continuous maps of their free variables. The formulas are built from terms starting from open relations, using conjunction, disjunction, existential quantification over any space (warning: for locales in an arbitrary topos this will not the case), universal quantification over compact spaces, equality for discrete spaces, and apartness for Hausdorff spaces. Rewrite the sample synthetic proofs using this. The calculus has no implication or negation, because relations built in terms of them are not open in general. (Keep the more cumbersome proofs to illustrate the gain?) In any case, when this is done we can say precisely what a synthetic proof is: it is one written in this calculus.

Further synthetic proofs: Include the synthetic proof from [2] that if $X$ is compact and exponentiable (and non-empty!), then the maximum-value functional $\mathbb{R}^X \to \mathbb{R}$ is continuous. Is that's not much trouble, also consider (Riemann) integration. By the way, is there a synthetic proof that any continuous map from a compact metric space to any metric space is uniformly continuous? (There are restricted form of this in the unpublished work of the author with Bauer, and in the published work of the author with Ho Weng Kin.) Ah, about E. Hewitt too.

Further synthetic proofs: A continuous $f \colon X \to Y$ is proper iff it is closed and the right adjoint $f_* \colon \mathcal{O}X \to \mathcal{O}Y$ of the frame map $f^* \colon \mathcal{O}Y \to \mathcal{O}X$ preserves directed joins. Whether or not $f$ is proper, observe that $f_*(U) = \{y \in Y \mid f^{-1}\{y\} \subseteq U\}^\circ$. A continuous map $f \colon X \to Y$ is proper iff the interior operator can be removed from this formula. Proper maps reflect the way-below relation between arbitrary sets.

Consider stably (semi)open maps too?

Further synthetic proofs: We call a continuous map $f \colon X \to Y$ is perfect it is stably semiclosed. Here a continuous map is semiclosed if the lower set, in the specialization order, of the image of any closed set is closed. Then a continuous map $f \colon X \to Y$ is perfect iff $\{f^{-1}(upy) \mid y \in Y\}$ is a continuously indexed family of compact sets, and perfect maps reflect compact saturated sets. A continuous map is perfect iff $f_*(U) = \{y \in Y \mid f^{-1}\{\uparrow y\} \subseteq U\}^\circ$. The map $f$ is perfect iff it is semiclosed and $f_*$ preserves directed joins. Cf. work of Hofmann and Lawson. Perfect maps reflect the way-below relation between *open* sets, and, more generally between saturated sets.

Further work: Consider proper and perfect maps in the compactly generated case. (And in the sequential case, etc.)

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# Contents

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