Computability of continuous solutions of higher-type equations

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Abstract. Given a continuous functional $f: X \to Y$ and $y \in Y$, we wish to compute $x \in X$ such that $f(x) = y$, if such an x exists. We show that if x is unique and X and Y are subspaces of Kleene–Kreisel spaces of continuous functionals with X exhaustible, then x is computable uniformly in f, y and the exhaustion functional $\forall_X : 2^X \to 2$. We also establish a version of the above for computational metric spaces X and Y, where is X computationally complete and has an exhaustible set of Kleene–Kreisel representatives. Examples of interest include functionals defined on compact spaces X of analytic functions.

Keywords: Higher-type computability, Kleene–Kreisel spaces of continuous functionals, exhaustible set.

1 Introduction

Given a continuous functional $f: X \to Y$ and $y \in Y$, we consider the equation $f(x) = y$ with the unknown $x \in X$. We show that if X and Y are subspaces of Kleene–Kreisel spaces [1] with X *exhaustible* [2], the solution is computable uniformly in f, y and the exhaustion functional $\forall_X : 2^X \to 2$, provided there is a unique solution (Section 3). Here exhaustibility plays the role of a computational counter-part of the topological notion of compactness (Section 2). Moreover, under the same assumptions for X and Y, it is uniformly semi-decidable whether a solution $x \in X$ fails to exist.

Recall that the Kleene–Kreisel spaces are obtained from the discrete space $\mathbb{N}$ by iterating finite products and function spaces in a suitable cartesian closed category (e.g. filter spaces, limit spaces, k-spaces, equilogical spaces or QCB spaces). For computability background, see [1] or [2]. We exploit the fact that, by cartesian closedness, computable functionals are closed under λ-definability.

The computation of unique solutions of equations of the form $g(x) = h(x)$ with $g, h: X \to Y$ is easily reduced to the previous case, because there are (abelian) computable group structures on the ground types that can be lifted componentwise to product types and pointwise to function types, and hence $x \in X$ is a solution of such an equation iff it is a solution of the equation $f(x) = 0$, where $f(x) = h(x) - g(x)$. And, by cartesian closedness, the case in which g and h computably depend on parameters, and in which the solution computably depends on the same parameters, is covered. Moreover, because the Kleene–Kreisel spaces are closed under finite products and countable powers, this includes the solution of finite and countably infinite systems of equations with functionals of finitely many or countably infinitely many variables.

We also consider a generalization to computational metric spaces that applies to computational analysis, where f can be a functional and x a function (Section 4). And we develop examples of sets of analytic functions that are exhaustible and can play the role of the space X (Section 5).

Organization. (2) Exhaustible subspaces of Kleene–Kreisel spaces. (3) Equations over Kleene–Kreisel spaces. (4) Equations over metric spaces. (5) Exhaustible spaces of analytic functions.

2 Exhaustible Subspaces of Kleene–Kreisel Spaces

In previous work we investigated exhaustible sets of total elements of effectively given domains and their connections with Kleene–Kreisel spaces of continuous functionals [2]. Here we work directly with exhaustible subspaces of Kleene–Kreisel spaces, where in this section we translate notions and results for them from that work. Denote by Y^X the space of continuous functionals from X to Y.

Definition 2.1. Let $2 = \{0, 1\}$ be discrete.

1. A space K is called *exhaustible* if the universal quantification functional
$$\forall_K : 2^K \to 2$$
defined by $\forall_K(p) = 1$ iff $p(x) = 1$ for all $x \in K$ is computable.
2. It is called *searchable* if there is a computable selection functional
$$\varepsilon_K : 2^K \to K$$
such that for all $p \in 2^K$, if there is $x \in K$ with $p(x) = 1$ then $p(\varepsilon_K(p)) = 1$.
3. A set $F \subseteq X$ is *decidable* if its characteristic map $X \to 2$ is computable. □

Equivalently, K is exhaustible iff the functional $\exists_K : 2^K \to 2$ defined by $\exists_K(p) = 1$ iff $p(x) = 1$ for some $x \in K$ is computable. If K is searchable, then it is exhaustible, because $\exists_K(p) = p(\varepsilon_K(p))$. The empty space is exhaustible, but not searchable, because there is no map $2^\emptyset \to \emptyset$.

The following results are directly adapted to our setting from [2].

Lemma 2.1.

1. *The Cantor space* $2^{\mathbb{N}}$ *is searchable.*
2. *Any exhaustible subspace of a Kleene–Kreisel space is compact, and moreover, if it is non-empty, it is searchable, a computable retract, and a computable image of the Cantor space.*
3. *Searchable spaces are closed under computable images, intersections with decidable sets, and finite products.*
4. *A product of countably many searchable subspaces of a common Kleene–Kreisel space is searchable uniformly in the sequence of quantifiers.*

Thus, exhaustibility is a computational counter-part of the topological notion of compactness, at least for subspaces of Kleene–Kreisel spaces.

3 Equations over Kleene–Kreisel Spaces

We emphasize that in this paper, including Section 4, the terminology *uniform* is used in the sense of recursion theory, rather than metric topology.

Theorem 3.1. *If $f\colon X \to Y$ is a continuous map of subspaces of Kleene–Kreisel spaces with X exhaustible, and $y \in Y$, then, uniformly in $\forall_X$, f, and y:*

1. *It is semi-decidable whether the equation $f(x) = y$ fails to have a solution.*
2. *If $f(x) = y$ has a unique solution $x \in X$, then it is computable.*

Hence if $f\colon X \to Y$ is a computable bijection then it has a computable inverse, uniformly in $\forall_X$ and f.

The conclusion is a computational counter-part of the topological theorem that any continuous bijection from a compact Hausdorff space to a Hausdorff space is a homeomorphism.

Remark 3.1. The uniqueness assumption in the second part is essential. In fact, consider e.g. $X = 2$ (which is trivially exhaustible) and $Y = \mathbb{N}^{\mathbb{N}}$. Then a map $f\colon X \to Y$ amounts to two functions $f_0, f_1\colon \mathbb{N} \to \mathbb{N}$. Hence computing a solution to the above equation amounts to finding $i \in 2$ such that $f_i = y$ holds, that is, $f_i(n) = y(n)$ for all $n \in \mathbb{N}$. In other words, under the assumption that $f_0 = y$ or $f_1 = y$, we want to find i such that $f_i = y$. If the only data supplied to the desired algorithm are f_0, f_1, y, this is not possible, because no finite amount of information about the data can determine that one particular disjunct holds. When specialized to this example, the proof of the theorem relies on the additional information that only one of the disjuncts holds. □

The following will be applied to semi-decide absence of solutions:

Lemma 3.1. *Let X be an exhaustible subspace of a Kleene–Kreisel space and $K_n \subseteq X$ be a sequence of sets that are decidable uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

emptiness of $\bigcap_n K_n$ is semi-decidable.

Proof. Because X is compact by exhaustibility, K_n is also compact as it is closed. Because X is Hausdorff, $\bigcap_n K_n = \emptyset$ iff there is n such that $K_n = \emptyset$. But emptiness of this set is decidable uniformly in n by the algorithm $\forall x \in X.x \notin K_n$. Hence a semi-decision procedure is given by $\exists n.\forall x \in X.x \notin K_n$. □

As a preparation for a lemma that will be applied to compute unique solutions, notice that if a singleton $\{u\} \subseteq \mathbb{N}^Z$ is exhaustible, then the function u is computable, because $u(z) = \mu m.\forall v \in \{u\}.v(z) = m$. Moreover, u is computable uniformly in $\forall_{\{u\}}$, in the sense that there is a computable functional

$$U\colon S \to \mathbb{N}^Z \quad \text{with} \quad S = \{\phi \in 2^{2^{\mathbb{N}^Z}} \mid \phi = \forall_{\{v\}} \text{ for some } v \in \mathbb{N}^Z\},$$

such that $u = U\left(\forall_{\{u\}}\right)$, namely $U(\phi)(z) = \mu m.\phi(\lambda u.u(z) = m)$. Lemma 3.2 below generalizes this, using an argument from [2] that was originally used to prove that non-empty exhaustible subsets of Kleene–Kreisel spaces are computable images of the Cantor space and hence searchable. Here we find additional applications and further useful generalizations.

Lemma 3.2. *Let X be an exhaustible subspace of a Kleene–Kreisel space and $K_n \subseteq X$ be a sequence of sets that are exhaustible uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

if $\bigcap_n K_n$ is a singleton $\{x\}$, then x is computable.

Proof. By Lemma 2.1, X is a computable retract of its Kleene–Kreisel superspace. Because any Kleene–Kreisel space is a computable retract of a Kleene–Kreisel space of the form $\mathbb{N}^Z$, and because retractions compose, there are computable maps $s\colon X \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to X$ with $r \circ s = \mathrm{id}_X$. It suffices to show that the function $u = s(x) \in \mathbb{N}^Z$ is computable, because $x = r(u)$. The sets $L_n = s(K_n) \subseteq \mathbb{N}^Z$, being computable images of exhaustible sets, are themselves exhaustible. For any $z \in Z$, the set $U_z = \{v \in \mathbb{N}^Z \mid v(z) = u(z)\}$ is clopen and $\bigcap_n L_n = \{u\} \subseteq U_z$. Because $\mathbb{N}^Z$ is Hausdorff, because $L_n \supseteq L_{n+1}$, because each L_n is compact and because U_z is open, there is n such that $L_n \subseteq U_z$. That is, $v \in L_n$ implies $v(z) = u(z)$. Therefore, for every $z \in Z$ there is n such that $v(z) = w(z)$ for all $v, w \in L_n$. Now, the map $n(z) = \mu n.\forall v, w \in L_n.v(z) = w(z)$ is computable by the exhaustibility of L_n. But $u \in L_{n(z)}$ for any $z \in Z$ and therefore u is computable by exhaustibility as $u(z) = \mu m.\forall v \in L_{n(z)}.v(z) = m$, as required. □

To build sets K_n suitable for applying these two lemmas, we use:

Lemma 3.3. *For every computable retract of a Kleene–Kreisel space, there is a family $(=_n)$ of equivalence relations that are decidable uniformly in n and satisfy*

$$x = x' \iff \forall n.\, x =_n x',$$
$$x =_{n+1} x' \implies x =_n x'.$$

Proof. Let X be a Kleene–Kreisel space and $s\colon X \to \mathbb{N}^Z$ and $r\colon \mathbb{N}^Z \to X$ be computable maps with Z a Kleene–Kreisel space and $r \circ s = \mathrm{id}_X$. By the density theorem, there is a computable dense sequence $\delta_n \in Z$. Then the definition

$$x =_n x' \iff \forall i < n.s(x)(\delta_i) = s(x')(\delta_i)$$

clearly produces an equivalence relation that is decidable uniformly in n and satisfies $x =_{n+1} x' \implies x =_n x'$. Moreover, $x = x'$ iff $s(x) = s(x')$, because s is injective, iff $s(x)(\delta_n) = s(x')(\delta_n)$ for every n, by density, iff $x =_n x'$ for every n, by definition. □

Proof (of Theorem 3.1). The set $K_n = \{x \in X \mid f(x) =_n y\}$, being a decidable subset of an exhaustible space, is exhaustible. Therefore the result follows from Lemmas 3.1 and 3.2, because $x \in \bigcap_n K_n$ iff $f(x) =_n y$ for every n iff $f(x) = y$ by Lemma 3.3. □

Algorithms 3.2. In summary, the algorithm for semi-deciding non-existence of solutions is

$$\exists n. \forall x \in X. f(x) \neq_n y,$$

and that for computing the solution x_0 as a function of $\forall_X$, f, and y is:

$$\begin{aligned} \forall x \in K_n . p(x) &= \forall x \in X. f(x) =_n y \implies p(x), \\ \forall v \in L_n . q(v) &= \forall x \in K_n . q(s(x)), \\ n(z) &= \mu n. \forall v, w \in L_n . v(z) = w(z), \\ u(z) &= \mu m. \forall v \in L_{n(z)} . v(z) = m, \\ x_0 &= r(u). \end{aligned}$$

Here $r \colon \mathbb{N}^Z \to X$ is a computable retraction with section $s \colon X \to \mathbb{N}^Z$, where Z is a Kleene–Kreisel space, as constructed in the proof of Lemma 3.2. □

Of course, even in the absence of uniqueness, *approximate* solutions with precision n are trivially computable with the algorithm

$$\varepsilon_X(\lambda x. f(x) =_n y),$$

using the fact that non-empty exhaustible subsets of Kleene–Kreisel spaces are searchable. But the above unique-solution algorithm uses the quantification functional $\forall_X$ rather than the selection functional ε_X. In the next section we compute solutions as limits of approximate solutions.

4 Equations over Metric Spaces

For the purposes of this and the following section, we can work with computational spaces in the sense of TTE [3] using Baire-space representations, or equivalently, using partial equivalence relations on representatives living in effectively given domains [4]. Our development applies to both, and we can more generally assume for the former that representatives form subspaces of arbitrary Kleene–Kreisel spaces rather than just the Baire space $\mathbb{N}^{\mathbb{N}}$. We first formulate the main result of this section and then supply the missing notions in Definition 4.2:

Theorem 4.1. *Let X and Y be computational metric spaces with X computationally complete and having an exhaustible Kleene–Kreisel representation.*

If $f \colon X \to Y$ is continuous and $y \in Y$, then, uniformly in f, y and the exhaustibility condition:

1. *It is semi-decidable whether the equation $f(x) = y$ fails to have a solution.*
2. *If $f(x) = y$ has a unique solution $x \in X$, then it is computable.*

Hence any computable bijection $f \colon X \to Y$ has a computable inverse, uniformly in f and the exhaustibility condition.

Given that exhaustibility is a computational counter-part of the topological notion of compactness, and that compact metric spaces are complete, it is natural to conjecture that, at least under suitable computational conditions, the assumption of computational completeness in the above theorem is superfluous. We leave this as an open question. In connection with this, notice that this theorem is analogous to a well-known result in constructive mathematics [5], with the assumptions reformulated in our higher-type computational setting.

There is a technical difficulty in the proof of the theorem: at the intensional level, where computations take place, solutions are unique only up to equivalence of representatives. In order to overcome this, we work with pseudo-metric spaces at the intensional level and with a notion of decidable closeness for them. Recall that a *pseudo-metric* on a set X is a function $d\colon X \times X \to [0, \infty)$ such that

$$d(x,x) = 0, \quad d(x,y) = d(y,x), \quad d(x,z) \leq d(x,y) + d(y,z).$$

Then d is a *metric* if it additionally satisfies $d(x,y) = 0 \implies x = y$. If d is only a pseudo-metric, then $(\sim)$ defined by

$$x \sim y \iff d(x,y) = 0$$

is an equivalence relation, referred to as *pseudo-metric equivalence*. A pseudo-metric topology is Hausdorff iff it is T_0 iff the pseudo-metric is a metric. Moreover, two points are equivalent iff they have the same neighbourhoods. Hence any sequence has at most one limit up to equivalence.

A *computational metric space* is a computational pseudo-metric space in which the pseudo-metric is actually a metric, and hence we formulate the following definitions in the generality of pseudo-metric spaces.

Definition 4.2. We work with any standard (admissible) representation of the Hausdorff space $[0, \infty)$.

1. A *computational pseudo-metric space* is a computational space X endowed with a computable pseudo-metric, denoted by $d = d_X : X \times X \to [0, \infty)$.
2. A *fast Cauchy sequence* in a computational pseudo-metric space X is a sequence $x_n \in X$ with $d(x_n, x_{n+1}) < 2^{-n}$. The subspace of $X^{\mathbb{N}}$ consisting of fast Cauchy sequences is denoted by $\mathrm{Cauchy}(X)$.
3. A computational pseudo-metric space X is called *computationally complete* if every sequence $x_n \in \mathrm{Cauchy}(X)$ has a limit uniformly in x_n.
4. A computational pseudo-metric space X has *decidable closeness* if there is a family of relations $\sim_n$ on X that are decidable uniformly in n and satisfy:
   (a) $x \sim_n y \implies d(x,y) < 2^{-n}$,
   (b) $x \sim y \implies \forall_n . x \sim_n y$.
   (c) $x \sim_{n+1} y \implies x \sim_n y$,
   (d) $x \sim_n y \iff y \sim_n x$,
   (e) $x \sim_{n+1} y \sim_{n+1} z \implies x \sim_n z$.
   The last condition is a counter-part of the triangle inequality. It follows from the first condition that if $x \sim_n y$ for every n, then $x \sim y$. Write

   $$[x] = \{y \in X \mid x \sim y\}, \qquad [x]_n = \{y \in X \mid x \sim_n y\}.$$

   Then the equivalence class $[x]$ is the closed ball of radius 0 centered at x. □

For instance, the spaces $\mathbb{R}$ and $[0, \infty)$ are computationally complete metric spaces under the Euclidean metric, but don't have decidable closeness.

Remark 4.1. In the above definition, we don't require the representation topology of X to agree with the pseudo-metric topology generated by open balls. But notice that the metric topology is always coarser than the representation topology, because, by continuity of the metric, open balls are open in the representation topology. Hence the representation topology of any computational metric space is Hausdorff. Moreover, if X has an exhaustible Kleene–Kreisel space of representatives and the metric topology is compact, then both topologies agree, because no compact Hausdorff topology can be properly refined to another compact Hausdorff topology. □

We are ready to prove the theorem.

Lemma 4.1. *For every computational metric space X there is a canonical computable pseudo-metric $d = d_{\ulcorner X \urcorner}$ on the representing space $\ulcorner X \urcorner$ such that:*

1. *The representation map $\rho = \rho_X : \ulcorner X \urcorner \to X$ is an isometry:*
$$d(t, u) = d(\rho(t), \rho(u)).$$
In particular:
(a) $t \sim u \iff d(t, u) = 0 \iff \rho(t) = \rho(u)$.
(b) *If $f : X \to Y$ is a computable map of metric spaces, then any representative $\ulcorner f \urcorner : \ulcorner X \urcorner \to \ulcorner Y \urcorner$ preserves the relation $(\sim)$.*
2. *If X is computationally complete, then so is $\ulcorner X \urcorner$.*
3. *The representing space $\ulcorner X \urcorner$ has decidable closeness.*

Proof. Construct $d_{\ulcorner X \urcorner} : \ulcorner X \urcorner \times \ulcorner X \urcorner \to [0, \infty)$ as the composition of a computable representative $\ulcorner d_X \urcorner : \ulcorner X \urcorner \times \ulcorner X \urcorner \to \ulcorner [0, \infty) \urcorner$ of $d_X : X \times X \to [0, \infty)$ with the representation map $\rho_{[0,\infty)} : \ulcorner [0, \infty) \urcorner \to [0, \infty)$. A limit operator for $\ulcorner X \urcorner$ from a limit operator for X is constructed in a similar manner. For given $t, u \in \ulcorner X \urcorner$, let q_n be the n-th term of the sequence $\ulcorner d_X \urcorner(t, u) \in \ulcorner [0, \infty) \urcorner \subseteq \text{Cauchy}(\mathbb{Q})$, and define $t \sim_n u$ to mean that $[-2^{-n}, 2^{-n}] \subseteq [q_n - 2^{-n+1}, q_n + 2^{-n+1}]$. □

Lemma 4.2. *Let Z be a subspace of a Kleene–Kreisel space with complete computational pseudo-metric structure and decidable closeness, and $K_n \subseteq Z$ be a sequence of sets that are exhaustible uniformly in n and satisfy $K_n \supseteq K_{n+1}$. Then, uniformly in the data:*

if $\bigcap_n K_n$ is an equivalence class, then it has a computable member.

Proof. Let $z \in \bigcap_n K_n$. For any m, we have $\bigcap_n K_n = [z] \subseteq [z]_{m+1}$, and hence there is n such that $K_n \subseteq [z]_{m+1}$, because the sets K_n are compact, because $K_n \supseteq K_{n+1}$, because Z is Hausdorff and because $[z]_{m+1}$ is open. Hence for every $u \in K_n$ we have $u \sim_{m+1} z$, and so for all $u, v \in K_n$ we have $u \sim_m v$. By the exhaustibility of K_n and the decidability of $(\sim_n)$, the function $n(m) = \mu n. \forall u, v \in K_n. u \sim_m v$ is computable. By the searchability of K_n, there is a computable sequence $u_m \in K_{n(m)}$. Because $n(m) \leq n(m+1)$, we have that $K_{n(m)} \supseteq K_{n(m+1)}$ and hence $u_m \sim_m u_{m+1}$ and so $d(u_m, u_{m+1}) < 2^{-m}$ and u_m is a Cauchy sequence. By completeness, u_m converges to a computable point u_∞. Because $z \in K_{n(m)}$, we have $u_m \sim_m z$ for every m, and hence $d(u_m, z) < 2^{-m}$. And because $d(u_\infty, u_m) < 2^{-m+1}$, the triangle inequality gives $d(u_\infty, z) < 2^{-m} + 2^{-m+1}$ for every m and hence $d(u_\infty, z) = 0$ and therefore $u_\infty \in \bigcap_n K_n$. □

The proof of the following is essentially the same as that of Theorem 3.1, but uses Lemma 4.2 rather than Lemma 3.2, and Lemma 4.1 instead of Lemma 3.3.

Lemma 4.3. *Let Z and W be subspaces of Kleene–Kreisel spaces with computational pseudo-metric structure and decidable closeness, and assume that Z is computationally complete and exhaustible.*

If $g\colon Z \to W$ is a computable map that preserves pseudo-metric equivalence and $w \in W$ is computable, then, uniformly in $\forall_Z$, g, and w:

1. *It is semi-decidable whether the equivalence $g(z) \sim w$ fails to have a solution $z \in Z$.*
2. *If $g(z) \sim w$ has a unique solution $z \in Z$ up to equivalence, then some solution is computable.*

Proof. The set $K_n = \{z \in Z \mid g(z) \sim_n w\}$, being a decidable subset of an exhaustible space, is exhaustible. Therefore the result follows from Lemmas 3.1 and 4.2, because $z \in \bigcap_n K_n$ iff $g(z) \sim_n w$ for every n iff $g(z) = w$. □

Algorithm 4.3. The solution $z = u_\infty$ is then computed from $\forall_Z$, g and w as follows, where we have expanded $\forall_{K_n}$ as a quantification over Z:

$$\begin{aligned} n(m) &= \mu n. \forall u, v \in Z.\, g(u) \sim_n w \wedge g(v) \sim_n w \implies u \sim_m v, \\ u_\infty &= \lim_m \varepsilon_K(\lambda z. g(z) \sim_{n(m)} w). \end{aligned}$$

Thus, although there are common ingredients with Theorem 3.1, the resulting algorithm is different from 3.2, because it relies on the limit operator and approximate solutions. □

But, for Theorem 4.1, approximate solutions are computable uniformly in $\ulcorner f \urcorner$ and $\ulcorner y \urcorner$ only, as different approximate solutions are obtained for different representatives of f and y:

Proof (of Theorem 4.1.). Let $f\colon X \to Y$ and $y \in Y$ be computable. Now apply Lemma 4.3 with $Z = \ulcorner X \urcorner$, $W = \ulcorner Y \urcorner$, $g = \ulcorner f \urcorner$, $w = \ulcorner y \urcorner$, using Lemma 4.1 to fulfil the necessary hypotheses. If $f(x) = y$ has a unique solution x, then $g(z) \sim w$ has a unique solution z up to equivalence, and $x = \rho(z)$ for any solution z, and hence x is computable. Because g preserves $(\sim)$ by Lemma 4.1, if $g(z) \sim w$ has a solution z, then $x = \rho(z)$ is a solution of $f(x) = y$. This shows that $f(x) = y$ has a solution iff $g(z) = w$ has a solution, and we can reduce the semi-decision of absence of solutions of $f(x) = y$ to absence of solutions of $g(z) = w$. □

5 Exhaustible Spaces of Analytic Functions

For any $\epsilon \in (0,1)$, any $x \in [-\epsilon, \epsilon]$, any $b > 0$, and any sequence $a \in [-b, b]^{\mathbb{N}}$, the Taylor series $\sum_n a_n x^n$ converges to a number in the interval $[-b/(1+\epsilon), b/(1-\epsilon)]$. The following is proved by a standard computational analysis argument:

Lemma 5.1. *Any analytic function $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$ of the form $f(x) = \sum_n a_n x^n$ is computable uniformly in any given $\epsilon \in (0,1)$, $b > 0$ and $a \in [-b,b]^{\mathbb{N}}$.*

Definition 5.1. Denote by $A = A(\epsilon, b) \subseteq \mathbb{R}^{[-\epsilon,\epsilon]}$ the subspace of such analytic functions and by $T = T_{\epsilon,b} \colon [-b,b]^{\mathbb{N}} \to A(\epsilon, b)$ the functional that implements the uniformity condition, so that $f = T(a)$. □

The following results also hold uniformly in ϵ and b, but we omit explicit indications for the sake of brevity. The results are uniform in the exhaustibility assumptions too. Because $[-b,b]^{\mathbb{N}}$ is compact and T is continuous, the space A is compact as well. Moreover:

Theorem 5.2. *The space A has an exhaustible set of Kleene–Kreisel representatives.*

Proof. The space $[-b,b]^{\mathbb{N}}$ has an exhaustible space of representatives, e.g. using signed-digit binary representation. Because exhaustible spaces are preserved by computable images, the image of any representative $\ulcorner T \urcorner \colon \ulcorner [-b,b]^{\mathbb{N}} \urcorner \to \ulcorner A \urcorner$ of T gives an exhaustible set of representatives of A contained in the set $\ulcorner A \urcorner$ of all representatives of A. □

Hence the solution of a functional equation with a unique analytic unknown in A can be computed using Theorem 4.1.

Lemma 5.2. *For any non-empty space X with an exhaustible set of Kleene–Kreisel representatives, the maximum- and minimum-value functionals*

$$\max_X, \min_X : \mathbb{R}^X \to \mathbb{R}$$

are computable.

Of course, any $f \in \mathbb{R}^X$ attains its maximum value because it is continuous and because spaces with exhaustible sets of representatives are compact.

Proof. We discuss max only. By e.g. the algorithm given in [6], this is the case for $X = 2^{\mathbb{N}}$. Because the representing space $\ulcorner X \urcorner$, being a non-empty exhaustible subspace of a Kleene–Kreisel space, is a computable image of the Cantor space, the space X itself is a computable image of the cantor space, say with $q \colon 2^{\mathbb{N}} \to X$. Then the algorithm $\max_X(f) = \max_{2^N}(f \circ q)$ gives the required conclusion. □

Corollary 5.1. *If K is a subspace of a metric space X and K has an exhaustible set of Kleene–Kreisel representatives, then K is* computably located *in X, in the sense that the distance function $d_K \colon X \to \mathbb{R}$ defined by*

$$d_K(x) = \min\{d(x,y) \mid y \in K\}$$

is computable.

Corollary 5.2. *For any metric space X with an exhaustible set of Kleene–Kreisel representatives, the* max*-metric* $d(f, g) = \max\{d(f(x), g(x)) \mid x \in X\}$ *on* $\mathbb{R}^X$ *is computable.*

Corollary 5.3. *For* $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$*, it is semi-decidable whether* $f \notin A$*.*

Proof. Because A is computationally located in $\mathbb{R}^{[-\epsilon,\epsilon]}$ as it has an exhaustible set of representatives, and because $f \notin A \iff d_A(f) \neq 0$. □

Another proof, which doesn't rely on the exhaustibility of a set of representatives of A, uses Theorem 4.1: $f \notin A$ iff the equation $T(a) = f$ doesn't have a solution $a \in [-b, b]^{\mathbb{N}}$. But this alternative proof relies on a complete metric on $[-b, b]^{\mathbb{N}}$. For simplicity, we consider a standard construction for 1-bounded metric spaces. Because we apply this to metric spaces with exhaustible sets of representatives, this is no loss of generality as the diameter of such a space is computable as $\max(\lambda x. \max(\lambda y.d(x, y)))$ and hence the metric can be computably rescaled to become 1-bounded.

Lemma 5.3. *For any computational* 1*-bounded metric space X, the metric on* $X^{\mathbb{N}}$ *defined by* $d(x, y) = \sum_n 2^{-n-1} d(x_n, y_n)$ *is computable and* 1*-bounded, and it is computationally complete if X is.*

Proof. Use the fact that the map $[0, 1]^{\mathbb{N}} \to [0, 1]$ that sends a sequence $a \in [0, 1]^{\mathbb{N}}$ to the number $\sum_n 2^{-n-1} a_n$ is computable. Regarding completeness, it is well known that a sequence in the space $X^{\mathbb{N}}$ is Cauchy iff it is componentwise Cauchy in X, and in this case its limit is calculated componentwise. □

Corollary 5.4. *The Taylor coefficients of any* $f \in A$ *can be computed from* f*.*

Proof. Because $[-b, b]^{\mathbb{N}}$ has an exhaustible set of representatives, the function T has a computable inverse by Theorem 4.1 and Lemma 5.3. □

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