

# Topology of data types and computability concepts

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## Precursors

Plotkin's Pisa notes on domains and semantics.

Smyth's topological view of predicate transformers.

Milner-Hennessy logic and powerlocales (Johnstone, Robinson).

Abramsky's logic of observable properties.

Vicker's topology via logic.

& no doubt many others.

## Pioneers

All this goes back to Kleene, Myhill/Shepherdson, Rice/Shapiro, Nerode, Scott, Ershov and others, and is entangled with Stone's representation theorems and their descendants.

See, in particular,

A. Nerode. Some Stone spaces and recursion theory.

Duke Math J. 26 (1959): 397-406.

## Contemporaries

Taylor's Abstract Stone Duality.

Vicker's double power locales.

Vicker and Townsend's double exponentiation.

Simpson's topological domains.

And there are endless connections with topology, domain theory and the theory of computation.

## 1. Introduction - Smyth's dictionary

topological space  $\approx$  datatype  
point  $\approx$  piece of data

(Warning: In reality, it would be more accurate to say that a datatype is a topological algebra. Monads play a major role here (and indeed in topology itself). In these notes we concentrate on the topological component.)

open set  $\approx$  semidecidable property

(observable property - Abramsky  
affirmable property - Vickers)

continuous map  $\approx$  computable function

G- $\delta$  set  $\approx$  specification

dense set  $\approx$  liveness property

compact set  $\approx$  ?

"The notion of compactness is a little harder to motivate: but it will have the significance for us of a "finitarily specifiable" set or, alternatively, of a set of results attainable by a boundedly non-deterministic process"

The main purpose of these notes is to precisely fill this gap. We'll also expand the dictionary with computational interpretations of the notions of sobriety and Hausdorff separation, among others.

## 2. To which extent "continuity" implies "computability"?

The topology of a data type is somehow induced by its computational structure — we shall see several examples.

With this in mind, it is not entirely surprising that

computability implies continuity.

The converse fails. We shall exhibit counterexamples in the due course, but, for the moment, a cardinality argument suffices:

In general, there are continuum many continuous functions but only countably many computable functions.

Nevertheless, one is entitled to ask to which extent the converse holds. One precise answer for a particular data type is given by the Myhill-Shepherdson theorem: every effectively continuous function is computable. By the way, the Rice-Shapiro theorem is about the extent to which openness implies semi-decidability, this time for a different particular data type: every effectively open set is semi-decidable.

But here we briefly explore a different type of answer. For the sake of definiteness, we consider the programming language PCF (for its call-by-name evaluation strategy).

$PCF^+ = PCF$  extended with parallel-or

(this will implement the requirement that finite unions of open sets be open.)

$PCF^{++} = PCF^+$  extended with  $\exists$

(this will implement the requirement that arbitrary unions of open sets be open.)

$\omega\text{-PCF}++ = \text{PCF}++$  extended with infinitely long programs (thought of as infinite trees).

For the sake of brevity, we call the language  $\Omega$ . We get topological spaces as follows:

0. The operational semantics of  $\Omega$  is defined in the same way as for  $\text{PCF}++$ , and so is the notion of observational equivalence (defined by ground contexts).

1. For each type  $\sigma$ , the space  $X_\sigma$  is defined as follows:

a. Its points are the equivalence classes\* of closed terms of type  $\sigma$ .

b. A set  $U \subseteq X_\sigma$  is open iff the function

$$\chi_U : X_\sigma \rightarrow X_{\text{bool}}$$

$$x \mapsto \begin{cases} \text{true} & \text{if } x \in U \\ \perp & \text{if } x \notin U \end{cases}$$

is  $\Omega$ -definable. Here 'true' (normal typeface) is the equivalence class of the term 'true' (typewriter typeface) and  $\perp$  is the equivalence class of divergent programs.

N.B. Notice that we are not invoking the observational preorder in these definitions.

\* If we don't take equivalence classes, we get a non- $T_0$  space, whose  $T_0$ -reflection amounts to taking the equivalence classes, so in principle we could postpone the quotient - but see below.

## Theorem

1. The open sets of  $X_\sigma$  form a topology on  $X_\sigma$ .
2. A function  $X_\sigma \rightarrow X_\tau$  is continuous if and only if it is Omega-definable.

Proof sketch Interpret Omega in the standard Scott model of PCF. Extend Plotkin's proof of Turing-universality of PCF++ to prove that every  $d \in D_\sigma$  is Omega-definable, using the facts that each domain  $D_\sigma$  in the model is countably based and that the finite elements are definable. Conclude that the domain order of  $D_\sigma$  is isomorphic to the partial-order reflection of the observational preorder on closed terms of type  $\sigma$ . Hence the open sets of  $X_\sigma$  are the Scott open sets of its observational partial order. This concludes the proof of (1). Because  $D_\sigma \rightarrow_\tau$  under the Scott topology is homeomorphic to  $X_\sigma \rightarrow_\tau$  and because  $f: X_\sigma \rightarrow X_\tau$  is continuous iff  $f \in X_\sigma \rightarrow_\tau$ , (2) follows.  $\square$

Thus

computable  $\Rightarrow$  continuous.

continuous  $\Rightarrow$  computable by an infinitely long program.

(Cf. Scott's slogan "computable functions on continuous data".)

Exercise Define a notion of computability for infinitely long programs, and show that an Omega program is computable iff it is observationally equivalent to a finite program.  $\square$

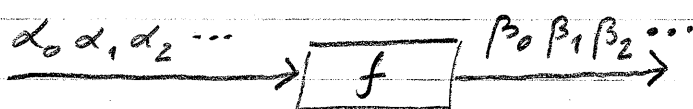
Further reading Scott's paper "The lattice of flow diagrams".  
Find the full reference at MathSciNet.

### 3. Computationally induced topologies

We all know one such kind of topology — the Scott topology, which has already featured in the proof of the theorem of the previous section. The Scott topology is highly non-Hausdorff.

Let's consider Hausdorff spaces for a change.

To begin with, consider computations of functions  $2^w \rightarrow 2^w$  where  $2 = \{0, 1\}$ .



$$\beta = f(\alpha)$$

$$\alpha = \alpha_0 \alpha_1 \alpha_2 \dots$$

$$\beta = \beta_0 \beta_1 \beta_2 \dots$$

The blackbox alternates between reading from the input, performing some internal computations and writing to the output. Bad input-suppliers will provide a finite sequence of digits and then give up — these are ruled out. Bad blackboxes will engage into infinite internal computations at some point, neglecting the output for ever — these are also ruled out of consideration.

#### Example

$$f(\alpha) = \beta \quad \text{where } \beta_i = \bar{\alpha}_i \quad (\text{digit negation}).$$

A functional program (in so-called lazy languages) :

$$f(0 :: \alpha) = 1 :: f(\alpha)$$

$$f(1 :: \alpha) = 0 :: f(\alpha)$$

This is clearly computable.  $\square$

We emphasize that there is no need to restrict to computable inputs.

Counter-example 1  $f: 2^{\omega} \rightarrow 2^{\omega}$  defined by

$$f(0^{k_0} 1 0^{k_1} 1 0^{k_2} 1 \dots 0^{k_n} 1 0^{\omega}) =$$

$$0^{k_0} h_0 0^{k_1} h_1 0^{k_2} h_2 \dots 0^{k_n} h_n 0^{\omega}$$

$$f(0^{k_0} 1 0^{k_1} 1 0^{k_2} 1 \dots 0^{k_n} 1 \dots) =$$

$$0^{k_0} h_0 0^{k_1} h_1 0^{k_2} h_2 \dots 0^{k_n} h_n \dots$$

where  $h_n = \begin{cases} 1 & \text{if } k_n \text{ belongs to the halting set,} \\ 0 & \text{otherwise,} \end{cases}$

is not computable.  $\square$

Counter-example 2  $g: 2^{\omega} \rightarrow 2^{\omega}$  defined by

$$g(\alpha) = \begin{cases} 10^{\omega} & \text{if } \forall i \in \omega. \alpha_i = 0, \\ 0^{\omega} & \text{otherwise,} \end{cases}$$

is not computable either.  $\square$

But the reasons are fundamentally different.

1. The Halting set is undecidable

2. The first digit of output depends on infinitely many digits of input.

A blackbox could compute  $f$  if antiprotonic computers were built in order to compute the Halting set (of Turing machines — that of antiprotonic computers would require a further technological development). However, in order to compute  $g$ , a blackbox has to be in possession of a crystal ball. In any case, based on what went wrong with  $g$ , we can say

If a function  $f: 2^{\omega} \rightarrow 2^{\omega}$  is computable, then finite parts of its output can depend only on finite parts of its input.

More formally, define  $\alpha \equiv_n \beta \iff \forall i < n. \alpha_i = \beta_i$ . Then the condition amounts to

$$\forall \varepsilon \in \omega \exists \delta \in \omega \forall \alpha, \beta \in 2^{\omega}. \alpha \equiv_{\delta} \beta \implies f(\alpha) \equiv_{\varepsilon} f(\beta).$$

We say that  $f$  is of finite character.

Proposition Endow  $2$  with the discrete topology and  $2^{\omega}$  with the product topology. Then  $f: 2^{\omega} \rightarrow 2^{\omega}$  is of finite character iff it is continuous.

Proofs (1) This readily follows from the definition of the product topology.

(2) Define  $d(\alpha, \beta) = \inf \{ 2^{-n} \mid \alpha \equiv_n \beta \}$ . This is a ultrimetric that induces the product topology on  $2^{\omega}$ .  $\square$

This topology is called the Cantor topology, because it makes it homeomorphic to the Cantor third-middle set of  $[0, 1]$  with the relative topology. For us, it has a computational significance:

$$U \subseteq 2^{\omega} \text{ is open iff } \forall \alpha \in U \exists n. \forall \beta. \beta \equiv_n \alpha \implies \beta \in U.$$

If  $\alpha$  passes a test, then it has a finite part such that every  $\beta$  sharing this part also passes the test.



We mentioned that it is not terribly surprising that computable functions are continuous with respect to a computationally induced topology. What is surprising is that these topologies are familiar.

We have ruled out bad input-suppliers and bad blackboxes. Let's now allow them. Then a black box of the kind we are considering is best modelled by a function

$$f: 2^\omega \rightarrow 2^\omega$$

where

$$2^\omega = 2^* \cup 2^\omega.$$

Because outputs, once written out, cannot be retracted,

$$\alpha \text{ is a prefix of } \alpha' \Rightarrow f(\alpha) \text{ is a prefix of } f(\alpha').$$

Again, such functions have to be of finite character — but this time we don't need the relations  $\equiv_n$  to express the condition:

If  $\beta$  is a finite prefix of  $f(\alpha)$ , then there is a finite prefix  $\alpha'$  of  $\alpha$  such that already  $\beta$  is a prefix of  $f(\alpha')$

Exercise This implies the previous (monotonicity) condition.  $\square$

Proposition  $f: 2^\omega \rightarrow 2^\omega$  is of finite character iff it is continuous wrt the Scott topology of the prefix order of  $2^\omega$ .

Proposition The relative Scott topology of  $2^\omega$  on  $2^w$  is the Cantor topology.

Cocollary Suppose a potentially bad function  $g: 2^\omega \rightarrow 2^\omega$  turns out to be good, i.e.

$$\begin{array}{ccc} 2^\omega & \xrightarrow{f} & 2^\omega \\ \downarrow & & \downarrow \\ 2^\omega & \xrightarrow{g} & 2^\omega \end{array}$$

for a (necessarily unique)  $f: 2^\omega \rightarrow 2^\omega$ . If  $g$  is continuous then so is  $f$ .

Aside Computationally, it is clear that whenever we implement a black box  $f: 2^\omega \rightarrow 2^\omega$  we are in reality implementing a black box  $g: 2^\omega \rightarrow 2^\omega$  such that the above diagram commutes. Topologically, we have

Proposition Every continuous  $f: 2^\omega \rightarrow 2^\omega$  extends to at least one continuous function  $g: 2^\omega \rightarrow 2^\omega$  (in the sense of the above diagram).

Proof (For the expert continuous-lattice theorist) The embedding  $2^\omega \hookrightarrow 2^\omega$  is dense, and  $2^\omega$ , being (an algebraic and hence) a continuous Scott domain, is injective over dense topological embeddings.  $\square$

End of aside.

Our second example in this section are the reals. For simplicity, we consider the unit interval  $[0, 1]$ , to begin with, and then  $[-1, 1]$ . There are many approaches. We consider three, of which the first is flawed.

We may compute with reals via their binary expansions (as Turing 1936a did)

$$\mathbb{I} - \mathbb{I} : \mathbb{Z}^\omega \rightarrow [0, 1]$$

$$\alpha \mapsto \sum_{i \geq 0} \alpha_i 2^{-(i+1)}$$

Think of  $\alpha$  as  $0.\alpha_0\alpha_1\alpha_2\dots$

Proposition The quotient topology on  $[0, 1]$  induced by this surjection is the usual, Euclidean topology.

Proof With this topology, the map is continuous. But continuous surjections of compact Hausdorff spaces are always quotient maps.  $\square$

Corollary In a situation

$$\begin{array}{ccc} \mathbb{Z}^\omega & \xrightarrow{f} & \mathbb{Z}^\omega \\ \downarrow & & \downarrow \\ [0, 1] & \xrightarrow{g} & [0, 1] \end{array},$$

If  $f$  is continuous, then so is  $g$ .

Proof This is a well-known, basic property of quotient maps.  $\square$

So, digitally computable functions on  $[0, 1]$  are continuous.

The converse fails badly. We illustrate this using decimal notation.

$$\begin{array}{ccc} (10)^\omega & \xrightarrow{f} & (10)^\omega & (10) = \{0, 1, \dots, 9\} \\ \downarrow & & \downarrow \\ [0, 1] & \xrightarrow{g} & [0, 1] \end{array}$$

Bad news The function  $g(x) = 3x/10$  has no continuous realizer  $f$ .

Proof (Brouwer 1920) The first digit of  $f(3^n 2 \dots)$  has to be 0. That of  $f(3^n 4 \dots)$  has to be 1. For  $f$  to be a realizer of  $g$ , the first digit of  $f(3^w)$  can be either 0 or else 1, because  $\llbracket 10^w \rrbracket = \llbracket 09^w \rrbracket = 0.1 = 3 \cdot \llbracket 3^w \rrbracket / 10$ . But, by continuity, it can be neither.  $\square$

Good news (a variant of Brouwer's solution adopted by Turing 1936b)

Take  $\mathbb{Z} = \{-1, 0, 1\}$ ,  $\llbracket - \rrbracket : \mathbb{Z}^w \rightarrow [-1, 1]$

$$\alpha \mapsto \sum_{i \in \mathbb{N}} \alpha_i 2^{-(i+1)}$$

"signed-digit binary representation".

This is a topological quotient map for the same reasons, and the same corollaries follow, with the bad news overcome:

Proposition For every continuous  $g : [-1, 1] \rightarrow [-1, 1]$  there is at least one continuous realizer  $f : \mathbb{Z}^w \rightarrow \mathbb{Z}^w$ .

Proof omitted. It is easier to show first that every continuous map  $\mathbb{Z}^w \rightarrow [-1, 1]$  lifts to a continuous map  $\mathbb{Z}^w \rightarrow \mathbb{Z}^w$

$$\begin{array}{ccc} \mathbb{Z}^w & \dashrightarrow & \mathbb{Z}^w \\ & \searrow & \downarrow \\ & & [-1, 1] \end{array}$$

(in categorical language,  $\mathbb{Z}^w$  (in the left top corner) is projective over the down quotient.)  $\square$

There are 700 equivalent definitions of computability in the literature for functions on the reals. The above proposition holds with "continuous" replaced by "computable".

Very briefly, we consider the analogue of the situation

$$\begin{array}{ccc} 2^{\omega} & \longrightarrow & 2^{\omega} \\ \downarrow & & \downarrow \\ 2^{\infty} & \longrightarrow & 2^{\infty} \end{array}$$

with the Cantor space  $2^{\omega}$  and the partial Cantor space  $2^{\infty}$  replaced by the unit interval  $[-1, 1]$  and the partial unit interval  $I[-1, 1]$ . For more details, see my "MGS notes".

$I[-1, 1]$  = closed subintervals of  $[-1, 1]$  under Scott topology of the reverse inclusion order.

We have a topological embedding

$$\begin{array}{ccc} [-1, 1] & \hookrightarrow & I[-1, 1] \\ x & \longmapsto & \{x\} = [x, x] \end{array}$$

That is, again a computational topology induces a familiar topology. Hence in a situation

$$\begin{array}{ccc} [-1, 1] & \xrightarrow{f} & [-1, 1] \\ \downarrow & & \downarrow \\ I[-1, 1] & \xrightarrow{g} & I[-1, 1] \end{array}$$

If  $g$  is continuous then so is  $f$ .

Because  $I[-1, 1]$  is a densely injective topological space and  $[-1, 1] \hookrightarrow I[-1, 1]$  is dense, for any given continuous  $f$  there is at least one continuous  $g$  s.t. the above diagram commutes.

Further reading:

- (1) Bauer, Escardo & Simpson. "Comparing functional paradigms for exact real-number computation"

This analyses the situation

$$3^{\omega} \twoheadrightarrow [-1,1] \hookrightarrow \mathbb{I}[-1,1]$$

from topological and computational points of view, in particular regarding higher-type computation.

- (2) Escardo. "Effective and sequential definition by cases on the reals via infinite signed-digit numerals"

This looks at  $3^{\omega} \twoheadrightarrow [-1,1]$  and  $2^{\omega} \twoheadrightarrow [0,1]$

Essentially,  $[-1,1]$  looks (order-)Hausdorff when viewed from  $3^{\omega}$ , as it should (from both topological and computational perspectives - see below). This fails for  $[0,1]$  wrt  $2^{\omega} \twoheadrightarrow [0,1]$ .

#### 4. The Sierpinski space

$S = \{\perp, \top\}$      $\{\perp\}$  closed, but not open.

$\{\top\}$  open, but not closed.

Scott (and Alexandroff) topology of the order  $\begin{matrix} \top \\ | \\ \perp \end{matrix}$ .

$\top$  classifies open subspaces.

$\perp$  classifies closed subspaces.

For  $U \subseteq X$ ,  $\chi_U : X \rightarrow T$

$$x \mapsto \llbracket x \in U \rrbracket = \begin{cases} \top & \text{if } x \in U \\ \perp & \text{if } x \notin U \end{cases}$$

we have

$U$  is open iff  $\chi_U$  is continuous.

Computationally,

$\top = \text{true}$

observable

termination

$\perp = \text{false}$

not observable

non-termination

$U$  is semidecidable iff  $\chi_U$  is computable

But this depends, of course, on a computability structure on the space  $X$ . We shall not be explicit in these notes about this in general, relying on the context.

A space is Hausdorff if any two distinct points can be separated by disjoint neighbourhoods. One quickly learns that this is equivalent to saying that its diagonal  $\{(x, x) \mid x \text{ a point of the space}\}$  is closed in the product topology. The diagonal is closed iff its complement is closed. Thus, a space is Hausdorff iff the map

$$\begin{aligned}(\neq) : X \times X &\longrightarrow S \\ (x, y) &\longmapsto \llbracket x \neq y \rrbracket\end{aligned}$$

is continuous. We can say that  $X$  is computationally Hausdorff (again presuming a computability structure of some sort) if this map is not only continuous but also computable — that is, if inequality is computably semidecidable.

Proposition A space has continuously semidecidable equality iff it is discrete.

Proof Exercise.  $\square$

Soon we shall look at many topological theorems (some basic and some more advanced) from a computational perspective. In order to do this, we need to develop some machinery, which, in particular, will allow us to "see" what compactness means from a computational perspective.



## 5. Exponentiable spaces and $\lambda$ -definable continuous maps

If all spaces were exponentiable (see below), that is, if the category of topological spaces were cartesian closed, which it isn't, then every function which is  $\lambda$ -definable from continuous functions would be automatically continuous. Interestingly, the premise fails but the conclusion holds.

### Lemma (The $\lambda$ -definability lemma)

If a set-theoretical function  $f: X \rightarrow Y$  of (points of) topological spaces  $X$  and  $Y$  is  $\lambda$ -definable from continuous maps, then it is itself continuous.

Proof Will be sketched later. For the moment, the reader is invited to wonder why and how this might be true.  $\square$

This generalizes the well-known fact that compositions of continuous maps produce continuous maps. The force of the lemma is that the definition of  $f$  may involve some "function-spaces" or exponentials which ... don't exist!

Unfortunately, this lemma is not enough for the applications we have in mind — we shall need to form exponentials which don't exist, primarily of the form  $S^X$ . We shall spell out the definition of exponentiability in a moment. If  $S^X$  exists then its points are the continuous maps  $X \rightarrow S$  (this is because our category is well-pointed). We ask the reader to take the following (remarkable in my view) fact on faith for the moment.

Lemma Suppose that  $X$  is an exponentiable topological space and let  $Q \subseteq X$ . The function

$$\forall_Q : S^X \longrightarrow S$$

$$p \longmapsto [\forall x \in Q. p(x) = T]$$

is continuous if and only if  $Q$  is compact.  $\square$

What we shall eventually do is to remove the hypothesis of exponentiability. For the moment, let's  $\lambda$ -prove some basic theorems in topology and extract computational conclusions from this, assuming that  $X$  is exponentiable.

Proposition If  $X$  is Hausdorff and  $Q \subseteq X$  is compact then  $Q$  is closed.

Proof It suffices to  $\lambda$ -define  $\chi_{X \setminus Q} : X \rightarrow S$  from continuous functions. Firstly, notice that  $x \notin Q \iff \forall y \in Q. x \neq y$ . Hence  $\chi_{X \setminus Q}(x) = \forall Q (\lambda y. x \neq y)$ . Because  $X$  is Hausdorff the map  $(\neq)$  is continuous, and because  $Q$  is compact, the map  $\forall_Q$  is continuous.  $\square$

Let's look at this proof in more detail. Using the exponential law, we get

$$\overline{(\neq)} = X \rightarrow S^X \quad \text{from} \quad (\neq) = X \times X \rightarrow S.$$

Composing with  $\forall_Q$ , we get

$$X \xrightarrow{(\neq)} S^X \xrightarrow{\forall_Q} S.$$

This is just the mechanics of the interpretation of the  $\lambda$ -expression.

Now the computational content of the proof. If we can computationally tell points of  $X$  apart and we can computationally quantify over  $\mathcal{Q}$ , then we can computationally semi-decide the complement of  $\mathcal{Q}$ . Programs for the first two tasks give a program for the third. This is what the  $\lambda$ -expression is-

We can say that a space  $X$  (with computability structure) is computationally compact if  $\forall_X$  is not only continuous but also computable. That is, if we can effectively universally quantify over it. We shall see that the Cantor space and the unit interval are computationally compact.

Proposition If  $X$  is compact and  $C \subseteq X$  is closed then  $C$  is compact.

Proof We  $\lambda$ -define  $\forall_C = S^X \rightarrow S$ . Notice that  $\forall_{x \in C}. p(x)$  iff  $\forall x \in X. x \in C \Rightarrow p(x)$  iff  $\forall x \in X. x \notin C \vee p(x)$ . Hence  $\forall_C(p) = \forall_X(\lambda x. \neg_{x \in C}(x) \vee p(x))$ , where  $\neg: S \times S \rightarrow S$  is the evident (continuous) disjunction map.  $\square$

Computationally, this has the drawback that it relies on the so-called "weak parallel-or". In any case, this says that if we can computationally quantify over  $C$  and semi-decide the complement of  $C$ , then we can computationally quantify over  $C$ .

Proposition If  $f: X \rightarrow Y$  is continuous and  $Q \subseteq X$  is compact then  $f(Q)$  is compact.

Proof  $\forall y \in f(Q). p(y)$  iff  $\forall x \in Q. p(f(x))$ .  $\square$

Proposition If  $X$  and  $Y$  are compact then so is  $X \times Y$ .

Proof  $\forall z \in X \times Y. p(z)$  iff  $\forall x \in X \forall y \in Y. p(x, y)$ .  $\square$

(c.f. Fubini)

$$S^{X \times Y} \xrightarrow{\cong} (S^Y)^X \xrightarrow{\forall_Y^X} S^X \xrightarrow{\forall_X} S$$

$\searrow \forall_{X \times Y}$

Proposition If  $Y$  is Hausdorff and  $X$  is exponentiable then  $Y^X$  is Hausdorff.

Proof  $f \neq g$  iff  $\exists x \in X$   $f x \neq g x$  and following lemma.

Lemma If  $X$  is exponentiable then

$$\begin{aligned} \exists_X : S^X &\longrightarrow S \\ p &\longmapsto \llbracket \exists x \in X. p(x) = \top \rrbracket \end{aligned}$$

is continuous.

Proof Cannot be provided yet.  $\square$

This suggests the following "dual" for the above proposition: We start from the ~~proof~~  $f = g$  iff  $\forall x \in X$   $f x = g x$ .

Proposition If  $Y$  is discrete and  $X$  is exponentiable and compact then  $Y^X$  is discrete.

As soon as we know that the cantor space  $2^\omega$  is exponentiable, we conclude that  $2^{2^\omega}$  is discrete. In fact, it is homeomorphic to  $\mathbb{N}$ , because  $2^\omega$  has countably many clopens. In particular, the PCF type  $(\text{nat} \rightarrow \text{bool}) \rightarrow \text{bool}$  has decidable equality for total elements. In fact, the decision procedure is PCF definable (parallel or and  $\exists$  are not needed) — the program and its topological proof of correctness will be given below.

Notice that we didn't need to know what the topology of  $Y^X$  is! Everything is encapsulated in the proofs of continuity of  $\forall$  and  $\exists$ .

"Topology via first-order logic" vs "Topology via [propositional] logic."

(Maybe even higher-order — but I don't have examples so far.)

Now a theorem whose usual topological proofs are not considered obvious at all. Recall that a map is called closed if it maps closed sets to closed sets. The Kuratowski-Mrowka theorem is:

Theorem  $X$  is compact iff, for every space  $Y$ , the projection  $X \times Y \rightarrow Y$  is closed.

Proof The projection is closed iff for every open  $W \subseteq X \times Y$ , the set  $Y \setminus \text{projection}(X \times Y \setminus W)$  is open. But  $y \in Y$  belongs to the latter iff

$$\begin{aligned} & y \notin \text{projection}(X \times Y \setminus W) \\ \text{iff} & \neg \exists x \in X. (x, y) \notin W \\ \text{iff} & \forall x \in X. (x, y) \in W. \end{aligned}$$

Thus, the projection is closed iff for every  $W \subseteq X \times Y$  the set  $\{y \in Y \mid \forall x \in X. (x, y) \in W\}$  is open. This immediately gives  $(\Rightarrow)$ . To prove  $(\Leftarrow)$ , choose  $Y = S^X$  and  $W = \{(x, p) \in X \times S^X \mid p(x) = T\}$ , which is clearly open. Then  $\{p \mid \forall x. p(x) = T\}$  is open, i.e.,  $\bigvee_x : S^X \rightarrow S$  is continuous.  $\square$

Exercise Generalize the above proof to get: A continuous map  $f: X \rightarrow Z$  is proper (i.e. preserves closed sets and reflects compact sets) iff for every continuous map  $g: Y \rightarrow Z$ , the pullback  $f^*g$  is closed:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f^*g} & Y \\ \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

$X \times_Z Y = \{(x, y) \in X \times Y \mid fx = gy\}$  with the relative topology. The map  $f^*g$  is just the projection.

Hint: The theorem is this with  $Z = 1$ .  $\square$

Proposition If  $Q \subseteq X$  is compact and  $V \subseteq Y$  then

$$N(Q, V) = \{ f \in Y^X \mid f(Q) \subseteq V \}$$

is an open subset of  $Y^X$ , assuming that  $X$  is exponentiable.

Proof  $f \in N(Q, V)$  iff  $\forall x \in Q. f(x) \in V$ . Hence

$$\chi_{N(Q, V)}(f) = \bigwedge_Q (\lambda x. \chi_V(f(x))). \quad \square$$

The topology generated by these sets (as a subbase) is known as the compact-open topology. If  $X$  is locally compact then it is exponentiable and the topology of  $Y^X$ , for any space  $Y$ , is the compact-open topology. (There can't be any  $\lambda$ -proof of this.)

We now have to clean up the above development. Firstly, we need to define exponentiability. Secondly, we have to say how we proceed when we don't have it.

If you know the categorical definition, which I am not assuming, I just emphasize that, because our category is well-pointed, the definition given below coincides with the categorical one in our special case.

Definition (a) A topology on the set  $C(X, Y)$  of continuous maps from a topological space  $X$  to a topological space  $Y$  is exponential iff continuity of a function

is equivalent to that of

$$f: A \times X \rightarrow Y$$

$$\bar{f}: A \rightarrow C(X, Y)$$

$$a \mapsto (x \mapsto f(a, x)).$$

In this case,  $C(X, Y)$  with the exponential topology is denoted by  $Y^X$ .

(ii) A space  $X$  is exponentiable iff  $C(X, Y)$  has an exponential topology for every space  $Y$ .  $\square$

The following is useful for the applications:

Lemma If  $A$  and  $X$  are exponentiable then so is  $A \times X$ , and, moreover

$$Y^{A \times X} \cong (Y^X)^A$$

$$f \longmapsto \bar{f}$$

Proof Categorically trivial.  $\square$

For an expository, elementary and brief proof of the following facts, together with credits and references to other proofs, see

Escardó & Heckmann. Topologies on spaces of continuous functions. To appear in the journal *Topology Proceedings*.

Lemma For any  $X$  and  $Y$ ,  $C(X, Y)$  admits at most one exponential topology.

This is categorically, trivially trivial. The following is a peculiarity of our category.

Lemma  $X$  is exponentiable iff the single exponential  $S^X$  exists.

Moreover, in this case, the topology of  $Y^X$  is induced by those of  $S^X$  and  $Y$  via the subbasic neighbourhoods

$$N(H, V) = \{ f \in Y^X \mid f^{-1}(V) \in H \}$$

where  $H \in \mathcal{O} S^X$  and  $V \in \mathcal{O} Y$ .

Exercise Give a  $\lambda$ -proof of the fact that these sets are open if  $Y^X$  exists.  $\square$

Proposition (Alternative subbase) The exponential topology of  $Y^X$  is generated by the sets

$$N(U, V) = \{ f \in Y^X \mid U \ll f^{-1}(V) \}$$

way-below relation

Exercise In the lattice of open sets of a topological space,  $U \ll U'$  iff every open cover of  $U$  has a finite subcover of  $U'$ .  $\square$

Theorem  $X$  is exponentiable iff the lattice  $\mathcal{O}X$  of open sets of  $X$  is continuous in the sense of Dana Scott.

Exercise (Set-theoretically easy) This is the case iff every neighbourhood  $U$  of a point  $x$  of  $X$  contains a smaller neighbourhood  $U'$  of  $x$  such that every open cover of  $U$  has a finite subcover of  $U'$ .  $\square$

Notice that this is a purely topological characterization. In this case,  $X$  is called core-compact.

Proposition A sober space (in particular, a Hausdorff space) is core-compact iff it is locally compact (every neighbourhood of a point contains a compact (not necessarily open) neighbourhood of the point.)

Proposition Locally compact spaces are core compact. If  $X$  is locally compact, the exponential topology of  $Y^X$  coincides with the compact-open topology.

same



Now, recall that we have a bijection of sets

$$p \longmapsto p^{-1}(\tau)$$

$$C(X, S) \cong OX$$

$$X_U \longleftarrow U.$$

Hence any topology on  $C(X, S)$  induces a homeomorphic topology on  $OX$ . We are interested in the exponential topology when  $X$  is exponentiable.

The Sierpinski space has three opens  $\emptyset$ ,  $\{\top\}$  and  $S$ . Now,

$$\begin{aligned} N(U, \emptyset) &= \{p \in S^X \mid U \ll p^{-1}(\emptyset) = \emptyset\} \\ &= \begin{cases} S^X & \text{if } U \ll \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} N(U, S) &= \{p \in S^X \mid U \ll p^{-1}(S) = X\} \\ &= \begin{cases} S^X & \text{if } U \ll X, \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned}$$

$$N(U, \{\top\}) = \{p \in S^X \mid U \ll p^{-1}(\top)\}$$

The last open induces the open  $\uparrow U$  on  $OX$ . Because  $OX$  is a continuous lattice, such opens generate the Scott topology.

$$\text{Now, } \forall Q: S^X \longrightarrow S$$

$$p \longmapsto \llbracket \forall x \in Q. p(\tau) \rrbracket$$

corresponds to

$$OX \longrightarrow S$$

$$U \longmapsto \llbracket \forall x \in Q \ \chi_U(x) = T \rrbracket$$

$$= \llbracket \forall x \in Q \cdot x \in U \rrbracket$$

$$= \llbracket Q \subseteq U \rrbracket$$

The inverse image of  $\{T\}$  along this map is thus

$$\{U \in OX \mid Q \subseteq U\}$$

But this set is Scott open iff every directed open cover of  $Q$  has a singleton subcover iff every open cover has a finite subcover iff  $Q$  is compact. This proves that  $\forall_Q$  is continuous iff  $Q$  is compact, as we claimed above.

This is a good opportunity to talk about the Hofmann-Mislove representation theorem. We say that a <sup>continuous</sup> map  $A: S^X \rightarrow S$  is an abstract universal quantifier

$$A(p \wedge q) = A(p) \wedge A(q)$$

where  $\wedge$  on the lhs is the meet of the lattice  $\mathbb{I}$  and  $\wedge$  on  $S^X$  as in the lhs is defined pointwise.

Lemma Let  $Q, R \subseteq X$  be compact. Then  $\forall_Q = \forall_R$  iff  $\uparrow Q = \uparrow R$ ; (upper closure in specialization order).

Proof Continuous maps preserve the specialization order.  $\square$

Now,  $Q$  is compact iff its saturation is compact, hence

$$A_Q = A_{\uparrow Q}.$$

Theorem (H&M) T.F.A.E. for any space  $X$ .

1.  $X$  is sober

2. Every abstract universal quantifier  $A: S^X \rightarrow S$  is of the form  $\forall_Q$  for a unique compact saturated set  $Q \subseteq X$ .

(Compare with the Riesz representation theorem.)

Proof discussion Abstract universal quantifiers are in bijection with Scott open (continuity) filters (preservation of  $\wedge$ ) of opens. Hence the theorem says that  $X$  is sober iff every Scott open filter is of the form  $\{Q \mid Q \subseteq U\}$  for a unique compact saturated set. The direction ( $\Rightarrow$ ) is that formulated by H&M. The simplest proof I know is due to Kermel and Paseka - The direction ( $\Leftarrow$ ) is folklore. A space is sober iff every completely prime filter is the open neighbourhood filter of a unique point. A filter is completely prime iff it is Scott open and prime (unreachable by finite joins). Hence completely prime filters correspond to abstract universal quantifiers that also preserve  $\vee$ , i.e., which also existentially quantify. This can be the case iff the set  $Q$  is the upper set of a point.  $\square$

We now return to the question of existence of  $S^X$  and what to do when it fails. We first formulate as a lemma something that we have already proved.

Lemma If  $S^X$  exists its topology is the Scott topology of the pointwise (specialization) order on continuous maps  $X \rightarrow S$ .

Definition The pseudo-exponential  $S^X$  is  $C(X, S)$  with the Scott topology of the pointwise order.  $\square$

Because the exponential  $(-1)^{\frac{1}{2}}$  doesn't exist on the reals, we pass to the complex numbers, which form a conservative extension. What we need are "imaginary" or "complex" spaces.

Theorem There exists a category  $\hat{\text{Top}}$  containing  $\text{Top}$  (the category of continuous maps of topological spaces) as a fully embedded subcategory,

$$\begin{array}{ccc} \text{Top} & \hookrightarrow & \hat{\text{Top}} \\ X & \mapsto & \hat{X} \end{array} \quad (f: X \rightarrow Y) \mapsto (\hat{f}: \hat{X} \rightarrow \hat{Y})$$

with the following properties

(i)  $\hat{\text{Top}}$  is cartesian closed

(ii) The embedding  $\text{Top} \hookrightarrow \hat{\text{Top}}$  preserves products and existing exponentials

(iii) It has a left adjoint  $R: \hat{\text{Top}} \rightarrow \text{Top}$  (a reflection)

(iv) For any  $X \in \text{Top}$ , the reflection of  $\hat{X}$  is naturally homeomorphic to the pseudo-exponential  $S^X$ .

(v)  $\hat{\text{Top}}$  is well pointed.

The  $\lambda$ -definability lemma follows, but we get more than that: (iv) tells us how to deal with compactness.

Proof sketch Firstly, the Yoneda embedding into presheaves doesn't work (Russell paradox). One can work, however, with what Rosolini & Streicher call extensional presheaves and previous authors call quasi-topological spaces — provided one is willing to postulate a Grothendieck universe. In ZFC or BNC, one can choose filter spaces (investigated by many authors under various names) or equilogical spaces (in fact Dana Scott and Andrej Bauer are playing the  $\lambda$ -topology game in the category of equilogical spaces). What doesn't seem to be emphasized in the literature is that, for either choice, (UV) holds. But all the needed technical material is there.  $\square$

We now have all ingredients needed to remove all assumptions\* of exponentiability in the above development. Actually, in exactly two cases, it cannot be removed, unless one is happy with a statement about a potentially imaginary exponentials. Can you spot them?

### Addendum

Tychonoff theorem gives

$$X \text{ compact } D \text{ discrete} \Rightarrow X^D \text{ compact}$$

We have the following consequence of the dual Tychonoff theorem

$$D \text{ discrete } X \text{ compact} \Rightarrow D^X \text{ discrete}$$

But I don't know what the dual Tychonoff theorem is.  $\square$

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\* Implicit or explicit

## 6. The lower, upper and double power spaces

Missing for lack of time and energy.

$$(1) \quad LUX \cong ULX \cong S^{S^X} \quad (\text{Johnstone, Vickers})$$

$$(2) \quad \text{For } X \text{ stably compact.} \quad (\text{Escardó \& Vickers indep.})$$

$$LX \cong (S^X)^{\text{op}} \quad UX \cong S^{X^{\text{op}}}$$

Here  $X^{\text{op}}$  is the co-compact dual (c.f. Adám's & Moshier's lectures)

$$(3) \quad (\text{Escardó, based on Vickers}) \quad \text{No assumption on } X.$$

$$\begin{array}{ccc} LX & \hookrightarrow & S^{S^X} \\ C & \longmapsto & \lambda p. \exists x \in C. p(x) = T \end{array}$$

$$\begin{array}{ccc} UX & \hookrightarrow & S^{S^X} \\ Q & \longmapsto & \lambda p. \forall x \in Q. p(x) = T \end{array}$$

$$\underline{\underline{\text{or}}}$$
$$\begin{array}{ccc} LX & \hookrightarrow & S^{S^X} \\ C & \longmapsto & \exists_c \end{array}$$

$$\begin{array}{ccc} UX & \hookrightarrow & S^{S^X} \\ Q & \longmapsto & \forall_Q \end{array}$$

I'll play some interesting games with this.

## 7. Some computer programs

I'll exhibit, run and prove some computer programs using the above ideas, and I'll give some toy and serious applications.

Further reading A.K. Simpson. Lazy functional algorithms for exact real functionals. LNCS 1450, pp. 456 - 469, 1998. (MFCS).

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