Comparing cartesian closed categories of (core) compactly generated spaces

By MARTÍN ESCARDÓ
School of Computer Science
University of Birmingham, UK

JIMMIE LAWSON
Department of Mathematics
Louisiana State University, USA

ALEX SIMPSON
LFCS, School of Informatics
University of Edinburgh, UK

12th February 2004

Abstract

It is well known that, although the category of topological spaces is not cartesian closed, it possesses many cartesian closed full subcategories, e.g.: *(i)* compactly generated Hausdorff spaces; *(ii)* quotients of locally compact Hausdorff spaces, which form a larger category; *(iii)* quotients of locally compact spaces without separation axiom, which form an even larger one; *(iv)* quotients of core compact spaces, which is at least as large as the previous; *(v)* sequential spaces, which are strictly included in (ii); and *(vi)* quotients of countably based spaces, which are strictly included in the category (v).

We give a simple and uniform proof of cartesian closedness for many categories of topological spaces, including (ii)–(v), and implicitly (i), and we also give a self-contained proof that (vi) is cartesian closed. Our main aim, however, is to compare the categories (i)–(vi), and others like them.

When restricted to Hausdorff spaces, (ii)–(iv) collapse to (i), and most non-Hausdorff spaces of interest, such as those which occur in domain theory, are already in (ii). Regarding the cartesian closed structure, finite products coincide in (i)–(vi). Function spaces are characterized as coreflections of both the Isbell and natural topologies. In general, the function spaces differ between the categories, but those of (vi) coincide with those in any of the larger categories (ii)–(v). Finally, the topologies of the spaces in the categories (i)–(iv) are analysed in terms of Lawson duality.

MSC (2000): 54D50, 54D55, 54C35, 06B35.

Keywords: k-space, compactly generated space, function space, Isbell topology, cartesian closed category, quotient space, core-compact space, countably based space, domain theory.

1. *Introduction*

Let X and Y be topological spaces and let $\mathrm{C}(X, Y)$ denote the set of continuous maps from X to Y. Given any continuous map $f\colon A \times X \to Y$, one has a function $\overline{f}\colon A \to \mathrm{C}(X, Y)$ defined by $\overline{f}(a) = (x \mapsto f(a, x))$, called the exponential *transpose* of f. A topology on $\mathrm{C}(X, Y)$ is said to be *exponential* if continuity of a function $f\colon A \times X \to Y$ is equivalent to that of its transpose $\overline{f}\colon A \to \mathrm{C}(X, Y)$. When an exponential topology exists, it is unique, and we denote the resulting space by $[X \Rightarrow Y]$. This is elaborated in Section 2 below. In this case, transposition is a bijection from the set $\mathrm{C}(A \times X, Y)$ to the set $\mathrm{C}(A, [X \Rightarrow Y])$; this bijective equivalence, when it holds, is often referred to as the *exponential law.*

A space X is said to be *exponentiable* if for every space Y there is an exponential topology on $\mathrm{C}(X, Y)$. Among Hausdorff spaces, the exponentiable ones are those that are locally compact. In general, a space is exponentiable if and only if it is *core compact*; that is, every neighbourhood V of a point contains a neighbourhood U of the point such that every open cover of V has a finite subcover of U. Thus, not all topological spaces are exponentiable. In categorical terminology, the category Top of continuous maps of topological spaces fails to be cartesian closed. Furthermore, if X and Y are exponentiable spaces, it is hardly ever the case that the exponential $[X \Rightarrow Y]$ is again exponentiable. In fact, the full subcategory of exponentiable spaces also fails to be cartesian closed.

Nonetheless, Top is known to possess several cartesian closed full subcategories, including:

(i) Compactly generated Hausdorff spaces, also known as k-spaces or Kelley spaces — see e.g. [**33**, **22**, **4**]. These are the Hausdorff spaces such that any subspace whose intersection with every compact subspace is closed is itself closed.

(ii) Compactly generated spaces without separation axiom [**7**, **8**, **34**]. These are characterized as the quotients of locally compact Hausdorff spaces and include the ones of the previous category. We show that they also include most of the non-Hausdorff spaces that occur in domain theory [**15**].

(iii) The quotients of locally compact spaces [**8**, **34**]. (The appropriate notion of local compactness in the absence of the Hausdorff separation axiom is that every point has a base of compact, not necessarily open, neighbourhoods.) This subcategory includes the previous, and Isbell showed that the inclusion is strict, by exhibiting a T_1 example of a space which belongs to this but not to the previous [**20**].

(iv) The quotients of core compact spaces [**8**, **34**]. These of course include the spaces of the previous subcategory, but it is not known whether the inclusion is strict. However, we show that the categories (ii)–(iv) collapse to (i) when restricted to Hausdorff spaces.

(v) The sequential spaces; that is, those whose topologies are determined by sequential convergence. These are strictly included in (ii).

(vi) The quotients of countably based spaces [**27**, **24**]. The spaces here are strictly included in those of the previous category. This category is particularly relevant for computability considerations, and, as the third author has reported recently [**31**], it supports a great deal of programming-language semantics constructions.

(vii) The densely injective topological spaces [**15**]. These are strictly included in (ii). (The injective topological spaces form yet another example, which is properly included in this one, see [**15**].)

One of the contributions of this paper is to provide a simple and uniform proof of cartesian closedness for many subcategories of Top including (ii)–(v) (and thus implicitly (i)). In addition, we give a proof that the category (vi) is cartesian closed. However, the main goal of the paper is to compare the different categories.

It is well-known that, for the category (vii), products and exponentials are calculated as in the category of topological spaces. It follows that products and exponentials are preserved by the inclusions of category (vii) in the categories (ii)–(vi). It is also known that products in the categories (i)–(vi) again coincide with topological products in other special cases, for example when one of the factors is the unit interval (which is relevant for homotopy theory). In general, however, products in the categories (i)–(vi) have topologies finer than the topological product. Nevertheless, although, in principle, products might vary when we move between categories (i)–(vi) (and others like them), we shall prove that, in fact, finite products are always calculated as in the largest category (iv).

Function spaces, in contrast, do vary between the categories. However, we identify two important situations in which they remain the same: (i) For Hausdorff spaces, function spaces are calculated in the same way in each of the categories (i)–(iv). (ii) The function spaces of (vi) are calculated in the same way as in any of its supercategories (ii)–(v). In order to prove these results, we derive useful general characterizations of function spaces in the different categories.

Organization.

In Section 2 we recall basic facts concerning exponentiability in the category of topological spaces. In Section 3 we give a simple proof of the cartesian closedness of certain subcategories of Top, including the categories (ii)–(v) and implicitly (i). Section 4 shows that the Hausdorff objects of (ii)–(iv) coincide and that most non-Hausdorff spaces that occur in domain theory belong to (ii) and hence to (iii) and (iv). Section 5 has a closer look at products and function spaces in the categories (ii)–(v) and implicitly (i). Section 6 contains various characterizations of the objects of (vi) in terms of pseudobases. This is applied to investigate products and function spaces, in Section 7. Section 8 develops characterizations of (some of) the objects of the categories in terms of Lawson duality. We close the paper with a few open problems, in Section 9.

2. *Exponentiable topological spaces*

In this preliminary section we recall known facts about function spaces of topological spaces which are needed for our purposes.

For sets A, X, Y, the natural bijection $f \leftrightarrow \overline{f}$ between the sets of functions $\mathrm{Set}(A \times X, Y)$ and $\mathrm{Set}(A, \mathrm{Set}(X, Y))$ associates with $f\colon A \times X \to Y$ its *transpose* $\overline{f}\colon A \to \mathrm{Set}(X, Y)$ given by $(\overline{f}(a))(x) = f(a, x)$.

A topological space X is *exponentiable* in the category Top of topological spaces if for every space Y there is a topology on the set $\mathrm{C}(X, Y)$ of continuous maps from X to Y such that for any space A the association $f \leftrightarrow \overline{f}$ is a bijection from the set of continuous maps $\mathrm{C}(A \times X, Y)$ to the set of continuous maps $\mathrm{C}(A, [X \Rightarrow Y])$, where $[X \Rightarrow Y]$ denotes $\mathrm{C}(X, Y)$ equipped with the hypothesised topology, which is referred to as an *exponential topology.* We often refer to this bijection as the *exponential law.* To maintain a categorical perspective, we provisionally ignore the fact that these are known to be precisely the core compact spaces.

LEMMA 2·1. *For X exponentiable, the evaluation mapping*

$$E\colon [X \Rightarrow Y] \times X \to Y$$

defined by $E(f, x) = f(x)$ is continuous.

Proof. The transpose of E is the identity map on $[X \Rightarrow Y]$, hence continuous, and thus E is continuous since X is exponentiable. □

LEMMA 2·2. *For X exponentiable, the exponential topology on $\mathrm{C}(X, Y)$ is uniquely determined.*

Proof. Suppose that $[X \Rightarrow Y]$ and $[X \Rightarrow' Y]$ are $\mathrm{C}(X, Y)$ endowed with topologies that satisfy the properties of an exponential. By Lemma 2·1 the evaluation map $E\colon [X \Rightarrow Y] \times X \to Y$ is continuous, and then the identity function on $\mathrm{C}(X, Y)$ from $[X \Rightarrow Y]$ to $[X \Rightarrow' Y]$ is continuous since the latter is an exponential. Reversing the roles of $[X \Rightarrow Y]$ and $[X \Rightarrow' Y]$ gives continuity of the identity in the reverse direction. □

In light of the preceding observation we denote by $[X \Rightarrow Y]$ the set $\mathrm{C}(X, Y)$ of continuous maps endowed with the exponential topology.

LEMMA 2·3. *For X exponentiable and $g \in \mathrm{C}(Y, Z)$, the map*

$$[X \Rightarrow g]\colon [X \Rightarrow Y] \to [X \Rightarrow Z]$$

defined by $[X \Rightarrow g](f) = g \circ f$ is continuous.

Proof. The map $g \circ E\colon [X \Rightarrow Y] \times X \to Z$ is continuous by Lemma 2·1, and thus its transpose, which one sees directly to be $[X \Rightarrow g]$, is continuous. □

It follows from the preceding that every exponentiable X gives rise to a functor $[X \Rightarrow \cdot]\colon \mathrm{Top} \to \mathrm{Top}$ defined by $Y \mapsto [X \Rightarrow Y]$ and $g \mapsto [X \Rightarrow g]$. One observes directly from the fact that $[X \Rightarrow Y]$ is an exponential that this functor is right adjoint to the functor $\cdot \times X$.

LEMMA 2·4. *The product of two exponentiable spaces is exponentiable.*

Proof. Let X_1 and X_2 be exponentiable spaces. Then for all spaces A, one has bijections $\mathrm{C}(A \times (X_1 \times X_2), Y) \cong \mathrm{C}((A \times X_1) \times X_2, Y) \cong \mathrm{C}(A \times X_1, [X_2 \Rightarrow Y]) \cong \mathrm{C}(A, [X_1 \Rightarrow [X_2 \Rightarrow Y]])$. Further, $\mathrm{C}(X_1, [X_2 \Rightarrow Y]) \cong \mathrm{C}(X_1 \times X_2, Y)$. Hence the exponential topology on $\mathrm{C}(X_1, [X_2 \Rightarrow Y])$ induces an exponential topology on $\mathrm{C}(X_1 \times X_2, Y)$. □

Given a family X_i of spaces, we endow the disjoint sum $\sum_i X_i$ with the final topology of the inclusions $X_i \to \sum_i X_i$, that is, the finest topology making the inclusions continuous. As it is well-known, this construction gives the categorical coproduct in Top.

LEMMA 2·5. *The disjoint sum of a family of exponentiable spaces is exponentiable.*

Proof. Similar to the above. Show that if X_i is a family of exponentiable spaces and Y is any space then $\prod_i [X_i \Rightarrow Y]$ induces an exponentiable topology on $\mathrm{C}(\sum_i X_i, Y)$ via the natural bijection $\prod_i \mathrm{C}(X_i, Y) \cong \mathrm{C}(\sum_i X_i, Y)$. □

A useful characterization of exponentiable spaces is the following.

LEMMA 2·6. *A space X is exponentiable if and only if $q \times \mathrm{id}_X : A \times X \to B \times X$ is a quotient map for every quotient map $q\colon A \to B$.*

Proof. We provide the proof only in the direction that will be useful for our purposes at a later point — see Remark 2·7 below for a sketch of the other. Suppose that X is an exponentiable space, $q\colon A \to B$ is a quotient map, and $h\colon B \times X \to Y$ is a function such that

$$A \times X \xrightarrow{q \times \mathrm{id}_X} B \times X \xrightarrow{h} Y$$

is continuous. Since X is exponentiable,

$$A \xrightarrow{q} B \xrightarrow{\overline{h}} [X \Rightarrow Y]$$

is continuous, and thus $\overline{h}$ is continuous, since q is quotient. Again using the exponentiability of X, we conclude that h is continuous. It follows that $q \times \mathrm{id}_X$ is quotient. □

Remark 2·7. A more categorical, and brief, proof of the direction provided would just observe that quotient maps are coequalizers and that left-adjoint functors preserve colimits. The other direction is also easy categorically speaking. It uses the adjoint-functor theorem. The solution-set condition is easy. One needs to check that the product functor $\cdot \times X$ preserves colimits. For this, it is enough to check the preservation of disjoint sums and of coequalizers. The first is easy and the last is our assumption. See e.g. Isbell [**19**].

The definition of exponentiability of a space quantifies over all topological spaces. We finish this section with an intrinsic characterization.

Definition 2·8 (Core compact space). A topological space X is called *core compact* if every open neighbourhood V of a point x of X contains an open neighbourhood U of x with the property that every open cover of V has a finite subcover of U.

For Hausdorff spaces, core compactness coincides with local compactness [**15**].

Theorem 2·9. *A space is exponentiable if and only if it is core compact.*

Proof. In this generality, the theorem is essentially due to Day and Kelly [**9**]. We refer the reader to e.g. Isbell [**19**], Chapter II.4 of [**15**] or the expository paper [**13**]. □

3. $\mathcal{C}$*-generated spaces*

The purpose of this section is to provide a simple proof of a known theorem, stating conditions under which the category of $\mathcal{C}$*-generated* topological spaces is cartesian closed. We begin by recalling basic notions and facts.

Definition 3·1 ($\mathcal{C}$-generated space). Let $\mathcal{C}$ be a fixed collection of spaces, referred to as *generating spaces*. By a *probe* over a space X we mean a continuous map from one of the generating spaces to X. The $\mathcal{C}$*-generated* topology $\mathcal{C}X$ on a space X is the final topology of the probes over X, that is, the finest topology making all probes continuous. We say that a topological space X is $\mathcal{C}$*-generated* if $X = \mathcal{C}X$. The category of continuous maps of $\mathcal{C}$-generated spaces is denoted by $\mathrm{Top}_{\mathcal{C}}$.

Notice that any collection generates the same class of spaces as the collection extended with the one-point space. Hence we shall assume without loss of generality that $\mathcal{C}$ contains at least one non-empty space.

It is easily seen that, when X is $\mathcal{C}$-generated, a function $f\colon X \to Y$ is continuous if and only if $f\colon X \to \mathcal{C}Y$ is continuous. This means that $\mathrm{Top}_{\mathcal{C}}$ is a coreflective subcategory of Top (the so-called *coreflective hull* of $\mathcal{C}$). The coreflection functor maps any continuous

$f\colon X \to Y$ to $f\colon \mathcal{C}X \to \mathcal{C}Y$, and the identity function $\mathcal{C}Y \to Y$ gives the counit of the coreflection. The lemma below summarizes basic properties of such coreflective hulls.

Lemma 3·2.
(i) *Every generating space is* $\mathcal{C}$*-generated.*
(ii) $\mathcal{C}$*-generated spaces are closed under the formation of quotients.*
(iii) $\mathcal{C}$*-generated spaces are closed under the formation of disjoint sums.*
(iv) *Any* $\mathcal{C}$*-generated space is a quotient of a disjoint sum of generating spaces.*
(v) *A space is* $\mathcal{C}$*-generated if and only if it is a colimit in* Top *of generating spaces.*

Proof. (i): Consider the identity probe.

(ii): If $f\colon X \to Y$ is a quotient map and X is $\mathcal{C}$-generated, then the composite probes $f \circ p\colon C \to Y$ suffice to generate the topology of Y, where $p\colon C \to X$ varies over all probes of X.

(iii): Similar to that of (ii).

(iv): Let X be a $\mathcal{C}$-generated space and I be the set of non-open subsets of X. By definition of final topology, for each $i \in I$, there exists a probe $p_i\colon C_i \to X$ with $p_i^{-1}(i)$ non-open. By the choice of probes, a subset V of X is open if and only if $p_i^{-1}(V)$ is open for all $i \in I$, which shows that the topology of X coincides with the final topology of the family $\{p_i\colon C_i \to X\}_{i\in I}$. To ensure that all points of X are covered by probes, we enlarge the family $\{p_i\colon C_i \to X\}_{i\in I}$ slightly, if necessary, by including all constant maps from some non-empty generating space. Since I is a set (rather than just a proper class) we can take the disjoint sum of the spaces C_i. Call it S and, for each $i \in I$, let $j_i\colon C_i \to S$ be the inclusion. By the universal property of sums, there is a unique map $q\colon S \to X$ such that $q \circ j_i = p_i$ for all $i \in I$. Again by the universal property, a function $f\colon X \to Y$ is continuous if and only if $f \circ q\colon S \to X$ is continuous, which shows that q is a quotient map.

(v): The construction of arbitrary small limits in any small cocomplete category (as Top is) can be reduced to coproducts (which in Top are disjoint sums) followed by coequalizers (which in Top are quotient maps). $\square$

We are particularly interested in the following four examples:

Definition 3·3 (Main examples). If $\mathcal{C}$ consists of respectively (i) core compact spaces, (ii) locally compact spaces, (iii) compact Hausdorff spaces, (iv) the one-point compactification of the countable discrete space, then we refer to $\mathcal{C}$-generated spaces as (i) *core compactly generated spaces*, (ii) *locally compactly generated spaces*, (iii) *compactly generated spaces*, (iv) *sequentially generated spaces.*

More examples are given at the end of this section.

Corollary 3·4.
(i) *A space is core compactly generated if and only if it is a quotient of a core compact space.*
(ii) *A space is locally compactly generated if and only if it is a quotient of a locally compact space.*
(iii) *A space is compactly generated if and only if it is a quotient of a locally compact Hausdorff space.*
(iv) *A space is sequentially generated if and only if it is sequential.*

Proof. This follows directly from Lemma 3·2 in light of the following observations. (i) and (ii): The classes of core compact spaces and locally compact spaces are each closed under disjoint sums. (iii): A disjoint sum of compact Hausdorff spaces is locally compact Hausdorff. Conversely a locally compact Hausdorff space is compactly generated. (iv): The final topology for probes on the one-point compactification of the countable discrete space is the topology of sequential convergence. □

DEFINITION 3·5 (Productive class of generating spaces). We say that $\mathcal{C}$ is *productive* if every generating space is exponentiable, and also the topological product of any two generating spaces is $\mathcal{C}$-generated.

Examples (i)–(iv) formulated in Definition 3·3 are all productive. In each case, the exponentiability requirement follows from the explicit characterization of exponentiable spaces as core compact spaces (Theorem 2·9). In cases (i)–(iii), the product requirement holds because the generating spaces are themselves closed under finite topological products. In the case of example (iv), the condition is met because the topological product of any two countably based spaces, in particular of two copies of the generating space, is again countably based, and hence sequential.

We now come to the main theorem concerning $\mathcal{C}$-generated spaces, first proved by Day [**8**].

THEOREM 3·6. *If $\mathcal{C}$ is productive then* $\mathrm{Top}_{\mathcal{C}}$ *is cartesian closed.*

We give a proof which is simpler and more direct than that of [**8**].

DEFINITION 3·7 ($\mathcal{C}$-continuous map). We say that a map $f\colon X \to Y$ of topological spaces is *$\mathcal{C}$-continuous* if the composite $f \circ p\colon C \to Y$ is continuous for every probe $p\colon C \to X$.

Of course, continuous maps are $\mathcal{C}$-continuous. Moreover, for maps defined on $\mathcal{C}$-generated spaces, $\mathcal{C}$-continuity coincides with continuity.

LEMMA 3·8 (The category $\mathrm{Map}_{\mathcal{C}}$).

(i) *The $\mathcal{C}$-continuous maps of arbitrary topological spaces form a category, denoted by* $\mathrm{Map}_{\mathcal{C}}$.

(ii) *The identity $\mathcal{C}X \to X$ is an isomorphism in* $\mathrm{Map}_{\mathcal{C}}$.

(iii) *The assignment that sends a space X to $\mathcal{C}X$ and a $\mathcal{C}$-continuous map to itself is an equivalence of categories $\mathcal{C}\colon$* $\mathrm{Map}_{\mathcal{C}} \to \mathrm{Top}_{\mathcal{C}}$.

Proof. (i): The identity function is continuous, hence $\mathcal{C}$-continuous. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be $\mathcal{C}$-continuous maps. In order to show that $g \circ f\colon X \to Z$ is $\mathcal{C}$-continuous, let $p\colon C \to X$ be a probe. By the $\mathcal{C}$-continuity of f, the composite $f \circ p\colon C \to Y$ is a probe, and by that of g, so is $g \circ (f \circ p) = (g \circ f) \circ p$, which shows that $g \circ f$ is $\mathcal{C}$-continuous.

(ii): Being continuous, the identity $\mathcal{C}X \to X$ is $\mathcal{C}$-continuous. By definition of the final topology, $p\colon C \to \mathcal{C}X$ is continuous for each probe $p\colon C \to X$, which, by definition, means that the identity $X \to \mathcal{C}X$ is $\mathcal{C}$-continuous.

(iii): $\mathrm{Top}_{\mathcal{C}}$ is a full subcategory of $\mathrm{Map}_{\mathcal{C}}$, because a space X is $\mathcal{C}$-generated if and only if continuity of a function defined on X is equivalent to $\mathcal{C}$-continuity, and, as we have just seen, every space is isomorphic in $\mathrm{Map}_{\mathcal{C}}$ to an object of $\mathrm{Top}_{\mathcal{C}}$. □

LEMMA 3·9. *Finite products in the category* $\mathrm{Map}_{\mathcal{C}}$ *exist and can be calculated as in* Top.

Proof. We show that a topological product $X_1 \times X_2$ has the required universal property. The projections $\pi_i \colon X_1 \times X_2 \to X_i$, being continuous, are $\mathcal{C}$-continuous. For each index i, let $f_i \colon A \to X_i$ be $\mathcal{C}$-continuous, and let $f \colon A \to X_1 \times X_2$ be the unique (set-theoretical) function with $f_i = \pi_i f$. In order to show that it is $\mathcal{C}$-continuous, let $p \colon C \to A$ be a probe. We have to show that $f \circ p$ is continuous. By definition of topological product, this is equivalent to showing that $\pi_i \circ f \circ p$ is continuous for each index i. By construction, this is the same as $f_i p$, which is continuous by $\mathcal{C}$-continuity of f_i. $\square$

Remark 3·10. Essentially the same proof shows that *all* limits in $\mathrm{Map}_{\mathcal{C}}$ exist and can be calculated as in Top. We emphasise, however, that the limits, including the finite products, are not uniquely determined as topological spaces up to homeomorphism, but only up to isomorphism in $\mathrm{Map}_{\mathcal{C}}$, where an isomorphism is a $\mathcal{C}$-continuous bijection with $\mathcal{C}$-continuous inverse.

LEMMA 3·11. *If* $f \colon X \times Y \to Z$ *is a* $\mathcal{C}$*-continuous map, then for each* $x \in X$*, the function* $f_x \colon Y \to Z$ *defined by* $f_x(y) = f(x, y)$ *is* $\mathcal{C}$*-continuous.*

Proof. The function $y \mapsto (x, y) \colon Y \to X \times Y$ is continuous and hence $\mathcal{C}$-continuous. Composition with f yields f_x, which is then $\mathcal{C}$-continuous by Lemma 3·8(i). $\square$

For a $\mathcal{C}$-continuous map $f \colon X \times Y \to Z$, we thus have a function $\overline{f} \colon X \to \mathrm{Map}_{\mathcal{C}}(Y, Z)$ defined by $\overline{f}(x) = f_x$, where $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ is the set of $\mathcal{C}$-continuous maps from Y to Z. We show that, when $\mathcal{C}$ is productive and for $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ suitably topologized, the function f is $\mathcal{C}$-continuous if and only if its transpose $\overline{f}$ is, and thus establish the cartesian closedness of the category of $\mathcal{C}$-continuous maps.

In order to define the topology on $\mathrm{Map}_{\mathcal{C}}(Y, Z)$, we require all generating spaces to be exponentiable. The topology is then constructed from the topologies of the exponentials $[C \Rightarrow Z]$ in the category Top, where C ranges over $\mathcal{C}$. Each probe $p \colon C \to Y$ induces a function $T_p \colon \mathrm{Map}_{\mathcal{C}}(Y, Z) \to [C \Rightarrow Z]$ defined by $T_p(g) = g \circ p$. We endow $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ with the initial topology of the family of functions that arise in this way, obtaining a space $\mathrm{Map}_{\mathcal{C}}[Y, Z]$. By definition, this means that, for any space B, a function $h \colon B \to \mathrm{Map}_{\mathcal{C}}[Y, Z]$ is continuous if and only if the composite $T_p \circ h \colon B \to [C \Rightarrow Z]$ is continuous for each probe $p \colon C \to Y$.

LEMMA 3·12. *Suppose* $\mathcal{C}$ *is productive.*

(i) *A function* $h \colon B \to \mathrm{Map}_{\mathcal{C}}[Y, Z]$ *is continuous if and only if for each probe* $p \colon C \to Y$*, the function* $(b, c) \mapsto (h(b))(p(c)) \colon B \times C \to Z$ *is continuous.*

(ii) *The transpose* $\overline{f} \colon X \to \mathrm{Map}_{\mathcal{C}}[Y, Z]$ *of a function* $f \colon X \times Y \to Z$ *is* $\mathcal{C}$*-continuous if and only if for all probes* $p \colon B \to X$ *and* $q \colon C \to Y$*, the map* $f \circ (p \times q) \colon B \times C \to Z$ *is continuous.*

(iii) *A function* $f \colon X \times Y \to Z$ *is* $\mathcal{C}$*-continuous if and only if for all probes* $p \colon B \to X$ *and* $q \colon C \to Y$*, the function* $f \circ (p \times q) \colon B \times C \to Z$ *is continuous.*

(iv) $\mathrm{Map}_{\mathcal{C}}$ *is cartesian closed.*

Proof. (i): The composite $T_p \circ h \colon B \to [C \Rightarrow Z]$ is the transpose of the function $(b, c) \mapsto (h(b))(p(c)) \colon B \times C \to Z$, and, by exponentiability of C in Top, a map $B \to [C \Rightarrow Z]$ is continuous if and only if it is the transpose of a continuous map $B \times C \to Z$.

(ii): By definition, $\overline{f}\colon X \to \mathrm{Map}_{\mathcal{C}}[Y,Z]$ is $\mathcal{C}$-continuous if and only if the map $\overline{f} \circ p\colon B \to \mathrm{Map}_{\mathcal{C}}[Y,Z]$ is continuous for all probes $p\colon B \to X$, and, by (i), this is the case if and only if the function $(b,c) \mapsto ((\overline{f} \circ p)(b))(q(c))\colon B \times C \to Z$ is continuous for all probes $q\colon C \to Y$. The result then follows from the fact that $((\overline{f} \circ p)(b))(q(c)) = f(p(b), q(c)) = f \circ (p \times q)(b,c)$.

(iii)⇒: By productivity, $B \times C$ is $\mathcal{C}$-generated, so $f \circ (p \times q)\colon B \times C \to X \times Z$ is continuous if and only if $f \circ (p \times q) \circ s$ is continuous for each probe $s\colon E \to B \times C$. But $f \circ (p \times q)$ is $\mathcal{C}$-continuous by Lemma 3·8(i), and hence $f \circ (p \times q) \circ s$ is continuous.

(iii)⇐: Let $r\colon D \to X \times Y$ be a probe. By composition with the projections, probes $p\colon D \to X$ and $q\colon D \to Y$ are obtained. By hypothesis, $f \circ (p \times q)\colon D \times D \to Z$ is continuous. And since the diagonal map $\Delta\colon D \to D \times D$ is continuous, so is the map $f \circ (p \times q) \circ \Delta = f \circ r\colon D \to Z$.

(iv): By (ii) and (iii), $\mathcal{C}$-continuity of a map $f\colon X \times Y \to Z$ is equivalent to that of its transpose $\overline{f}\colon X \to \mathrm{Map}_{\mathcal{C}}[Y,Z]$. □

This and Lemma 3·8(iii) conclude our proof of Theorem 3·6. As already stated, this result was first proved by Day [**8**]. Instead of working with our auxiliary category of $\mathcal{C}$-continuous maps of arbitrary topological spaces, Day considered an enlarged category thereof that could readily be shown to be cartesian closed, and he reflected this category onto the category of $\mathcal{C}$-generated spaces. Using fairly elaborate categorical machinery, Day inferred the cartesian closedness of the category of $\mathcal{C}$-generated spaces, via an application of a general categorical reflection theorem. Our proof is considerably simpler because we directly obtain an equivalence rather than just a reflection.

Proofs more closely related to ours involving what we have called probes can be found in Brown [**5**][**7**, Chapter 5] and Vogt [**34**], although they restrict their attention to function spaces that are given by the compact-open topology and its $\mathcal{C}$-coreflection. Barr [**1**] gives a general categorical argument for establishing the *monoidal closure* of categories, which provides another proof of Day's theorem when specialized to categories of topological spaces.

Steenrod [**33**] is responsible for popularizing the category of compactly generated spaces (alternatively k-spaces) in the Hausdorff case, crediting Spanier [**32**], Weingram [**35**] and Brown [**6**], and referencing Kelley's book [**21**]. The first reference in print to k-spaces seems to be Gale's paper [**14**], which attributes the notion and terminology to Hurewicz, who gave seminars on the subject at Princeton as early as 1948-1949.

Examples. It follows from Theorem 3·6 (together with the characterization of exponentiable spaces as core compact spaces) that the four examples introduced in Definition 3·3 are cartesian closed. Other examples are the following.

(1) Because discrete spaces are 1-generated, where 1 is the one-point space, they form a cartesian closed category. This of course amounts to the familiar fact that the category of sets is cartesian closed.

(2) An *Alexandroff space* is a space in which arbitrary intersections of open sets are open. It is an easy exercise to show that a space is Alexandroff if and only if it is $\mathbb{S}$-generated, where $\mathbb{S}$ is the Sierpinski space, that is, the two-point space $\{0,1\}$ with open sets $\mathbb{S}$, $\{1\}$, and $\emptyset$. Hence Alexandroff spaces form a cartesian closed category. But it is well known that Alexandroff spaces form a category isomorphic to that of monotone maps of preordered sets, which is a familiar example of a cartesian closed category.

(3) A *domain* is a continuous directed complete poset under the Scott topology [**15**].

Equivalently, in purely topological terms, a domain is a locally supercompact sober space, where a space is called *supercompact* if every open cover has a singleton subcover. Because domains are closed under disjoint sums, a space is domain-generated if and only if it is a topological quotient of a domain. Because domains are exponentiable in Top and are closed under finite topological products, and hence form a productive class of exponentiable spaces, we conclude that topological quotients of domains form a cartesian closed category.

4. *(Core) compactly generated spaces*

In this section we restrict our attention to the categories of core compactly generated and compactly generated spaces and consider some of their relationships (cf. Definition 3·3). Obviously, the smaller the set $\mathcal{C}$ of generating spaces, the finer the generated topology. In particular the core compactly generated topology is always contained in the compactly generated topology.

Our first observation will be that for Hausdorff spaces the two notions agree. In order to prove this, we introduce the relation of *relative compactness* between arbitrary subsets of a topological space.

DEFINITION 4·1 (The relation $\ll$ on sets). For arbitrary subsets A and B of a topological space X, we say that A is *relatively compact* in B, written $A \ll B$, if for every open cover of B, there exist finitely many elements in the cover that cover A.

The notion of relative compactness, restricted to open sets, plays an important role (as the "way below" relation) in the study of core compact spaces. Indeed, the definition of core compactness says that X is core compact if, for every open neighbourhood V of x, there exists an open neighbourhood U of x with $U \ll V$. Moreover, as is well known, the following interpolation property holds [**15**].

LEMMA 4·2 (Interpolation). *If X is core compact and $U, V \subseteq X$ are open with $U \ll V$ then there exists open $W \subseteq X$ such that $U \ll W \ll V$.*

The extension of the notion of relative compactness to arbitrary subsets generalizes the familiar concept of compact subset. Indeed, a subset A is compact if and only if $A \ll A$. Furthermore, basic topological results about compactness typically have analogues for relative compactness. We give some examples, of which the first three are needed for the purposes of this section and the last two are used in Section 7 below.

LEMMA 4·3. *Let X and Y be topological spaces.*
(i) *If A is closed and $A \ll X$, then A is compact.*
(ii) *If $A \ll B \subseteq X$ and X is Hausdorff, then $\overline{A} \subseteq B$.*
(iii) *If $f\colon X \to Y$ is continuous and $A \ll B \subseteq X$, then $f(A) \ll f(B)$.*
(iv) *If $A' \ll A \subseteq X$ and $B' \ll B \subseteq Y$ then $A' \times B' \ll A \times B$ in $X \times Y$.*
(v) *If $W \subseteq X \times Y$ is open and $S \ll T \subseteq X$, then*

$$\{y \in Y \mid \forall x \in T.\,(x,y) \in W\} \subseteq \operatorname{int}\{y \in Y \mid \forall x \in S.\,(x,y) \in W\}.$$

Assertion (i) generalizes the fact that a closed subset of a compact space is compact, (ii) the statement that a compact subset of a Hausdorff space is closed, (iii) the fact that continuous maps preserve compactness, (iv) the Tychonoff theorem in the finite case, and (v) the fact that if X is compact then the projection $\pi\colon X \times Y \to Y$ is a closed map, which is one half of the Kuratowski-Mrova theorem. Concerning (v), notice that the projection is closed if and only if for every open set $W \subseteq X \times Y$ the set $Y \setminus \pi(X \times Y \setminus W)$ is open, and that this set is $\{y \in Y \mid \forall x \in X.\, (x,y) \in W\}$. Hence the classical case occurs when X is compact and $S = T = X$. The proofs are modifications of the classical proofs and can safely be left to the reader.

Remark 4·4. It is well known and easy to verify that for a Hausdorff space X, the compactly generated topology agrees with the k-topology, defined by declaring a subset to be k-open if its intersection with each compact subspace is relatively open.

THEOREM 4·5. *For a Hausdorff space X the compactly generated topology agrees with the core compactly generated topology (and hence with the locally compactly generated topology). In particular, X is compactly generated if and only if it is core compactly generated.*

Proof. Suppose that U is open in the compactly generated topology, that is, U has the property that its inverse image is open for all probes $i\colon K \to X$, where K is a compact subset of X and i is the inclusion map. (This is equivalent to $U \cap K$ is open in K for all compact subspaces K.) Let $p\colon C \to X$ be a probe, where C is core compact. We need to show $p^{-1}(U)$ is open in C. If $p(C) \cap U = \emptyset$, then this is clearly the case. Otherwise let $x \in p^{-1}(U)$. By core compactness pick V, W open in C such that $x \in V \ll W \ll C$. By Lemma 4·3, $p(V) \ll p(W) \ll p(C) \subseteq X$, thus $\overline{p(V)} \subseteq p(W) \ll X$, and hence the set $A = \overline{p(V)}$ is compact. Therefore $U \cap A$ is open in A. Since $p(V) \subseteq A$, there exists an open set Q in C such that $x \in Q \subseteq V$ and $p(Q) \subseteq U \cap A \subseteq U$. It follows that $p^{-1}(U)$ is open in C, and hence that U is open in the core compactly generated topology. Since the core compactly generated topology is always contained in the compactly generated topology, we are done. ☐

Since the notion of compactly generated and core compactly generated collapse for Hausdorff spaces, it is natural to wonder if they coincide in general. That this is not the case was shown by J. Isbell [**20**], who gave an example of a locally compact T_1 (but clearly non-Hausdorff) space that is not compactly generated.

However, the notions of core compactly generated and compactly generated do coincide for large classes of T_0-spaces. We give some examples, which include most of those investigated in [**15**]. We begin with an elementary, but useful lemma. Recall that the specialization order on the points of a topological space is defined by $x \leq y$ if and only $x \in \overline{\{y\}}$. Since this is equivalent to saying that every neighbourhood of x is a neighbourhood of y, we see that open sets are upper sets in the specialization order.

LEMMA 4·6.

(i) *If the Sierpinski space (see example (2) after Lemma* 3·12*) is $\mathcal{C}$-generated then a subset U of X which is open in the $\mathcal{C}$-generated topology of X is an upper set in the specialization order of X.*

(ii) *The Sierpinski space is $\mathcal{C}$-generated if and only if $\mathcal{C}$ contains a space C in which not every open subset is closed.*

(iii) *The Sierpinski space is sequentially generated and hence compactly generated.*

Proof. (i): Let U be a $\mathcal{C}$-generated open set in X containing x and suppose $x \leq y$ in the specialization order. The map p from $\mathbb{S}$ to X, sending 0 to x and 1 to y, is trivially continuous. Because $\mathbb{S}$ is $\mathcal{C}$-generated, $p^{-1}U$ is open in $\mathbb{S}$. But $0 \in p^{-1}U$, so $1 \in p^{-1}U$, i.e. $y \in U$ as required.

(ii): If $U \subseteq C \in \mathcal{C}$ is not closed, then the map $q\colon C \to \mathbb{S}$ that takes every $c \in U$ to 1 and all other points to 0 is a quotient, hence $\mathbb{S}$ is $\mathcal{C}$-generated. Conversely, the property that all open subsets are closed is preserved by disjoint sums and quotients. Hence, if this property is satisfied by all spaces in $\mathcal{C}$ then it is satisfied by all $\mathcal{C}$-generated spaces.

(iii): Immediate. □

The preceding lemma is quite useful, because in checking whether a certain space or class of spaces is $\mathcal{C}$-generated, we need only check that the $\mathcal{C}$-generated open upper sets are open in the original topology.

Theorem 4·7. *Any directed-complete poset (dcpo) endowed with the Scott topology is compactly generated.*

Proof. Let X be a dcpo under the Scott topology. By the preceding lemma, we need only check that an upper set in X that is open in the compactly generated topology is Scott-open, or equivalently that a lower set that is closed in the compactly generated topology is Scott-closed. Let F be such a lower set and let D be a directed subset of F. If the supremum of D in not again in F, then we may assume that D is a counterexample of least cardinality. By the theorem of Iwomura (see, for example [**23**]) one can write D as a well-ordered (by inclusion) family of of directed sets of lower cardinality. Then the supremum of each of these sets must belong to F, and these suprema form a well-ordered subset C_1 of F. Close up this well-ordered chain under sups to obtain a chain C closed under sups in X that is also well-ordered. (Let A be a nonempty subset of C. Pick $a_1 \in A$, then $a_2 \in A$ with $a_2 < a_1$ and continue this process. Either it terminates at the least element of A or one obtains a strictly decreasing sequence in A. In the latter case, for each a_i, we can choose a $b_i \in C_1$ such that $a_{i+1} < b_i \leq a_i$. But the sequence (b_i) is then an infinite decreasing sequence in C_1, which violates the fact it is well-ordered.)

Now the chain C is a complete chain in its order, and hence is a compact Hausdorff space in its order topology. Since it is sup-closed in X, the embedding map is continuous into X under the Scott topology, and thus a probe. Since the inverse image of F contains C_1, a dense subset of C, we conclude that $C \subseteq F$. But C contains the sup of C_1, which is the supremum of D by construction. □

An important class of locally compact spaces is that of so-called stably compact spaces (see, for example, Chapter VI of [**15**]). One characterization of these is that they are the topological spaces with topology consisting of the open upper sets in a compact pospace (again see *loc. cit.*).

Theorem 4·8. *Stably compact spaces are compactly generated.*

Proof. Let X be a stably compact space. By Lemma 4·6 we need only verify that a compactly generated open set that is an upper set is open. Consider the probe from X with the Hausdorff topology making it a compact pospace to the topology of the open upper sets. If the inverse of an upper set is open for this probe, then clearly the set is an open upper set, and hence open in the stably compact topology. We use here implicitly the fact that the order of specialization agrees with the partial order of the pospace. □

One could envisage a theory of stably compactly generated spaces, but the preceding theorem shows that this theory yields precisely the same spaces as the compactly generated spaces.

5. *Products and function spaces*

In this section we consider in more detail the topology of products and function spaces in $\mathrm{Top}_{\mathcal{C}}$, for productive $\mathcal{C}$. We show that finite products in $\mathrm{Top}_{\mathcal{C}}$ do not vary when we enlarge $\mathcal{C}$. However, as we shall see, this invariance property fails for infinite limits and for function spaces. Nevertheless, we shall obtain various useful characterizations of function spaces in the different categories.

To begin with, let $\mathcal{C}$ be an arbitrary collection of topological spaces. For $\mathcal{C}$-generated X, Y, we write $X \times_{\mathcal{C}} Y$ for the binary product in the category $\mathrm{Top}_{\mathcal{C}}$. Trivially, if the topological product $X \times Y$ is itself $\mathcal{C}$-generated then $X \times_{\mathcal{C}} Y = X \times Y$. Further, we saw in Section 3 that the topological product $X \times Y$ of two spaces X and Y is also the product in $\mathrm{Map}_{\mathcal{C}}$ by Lemma 3·9. Since the functor $\mathcal{C}\colon \mathrm{Map}_{\mathcal{C}} \to \mathrm{Top}_{\mathcal{C}}$ is an equivalence of categories, it follows that the categorical product of two spaces X and Y in $\mathrm{Top}_{\mathcal{C}}$ is given by $X \times_{\mathcal{C}} Y = \mathcal{C}(X \times Y)$.

To proceed further, we assume henceforth that $\mathcal{C}$ is productive. Thus $\mathrm{Top}_{\mathcal{C}}$ is cartesian closed, and we write $[X \Rightarrow_{\mathcal{C}} Y]$ for exponentials in this category. By the proof of cartesian closedness, and the equivalence $\mathcal{C}\colon \mathrm{Map}_{\mathcal{C}} \to \mathrm{Top}_{\mathcal{C}}$, we have $[X \Rightarrow_{\mathcal{C}} Y] = \mathcal{C}(\mathrm{Map}_{\mathcal{C}}[X, Y])$ (n.b. for $\mathcal{C}$-generated spaces, the set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ equals $\mathrm{C}(X, Y)$).

Example 5·1. We show that finite products in $\mathrm{Top}_{\mathcal{C}}$ are, in general, strictly finer than topological products. Assume that the one-point compactification of a countable discrete space is $\mathcal{C}$-generated. Thus $\mathcal{C}$ may be any of our main examples, see Definition 3·3. Clearly, every sequential space is $\mathcal{C}$-generated. Let $\mathbb{N}^\omega$ be the Baire space, i.e. the countable topological product of the (discrete) non-negative integers. This is sequential, hence $\mathcal{C}$-generated. We show that the $\mathrm{Top}_{\mathcal{C}}$ product $[\mathbb{N}^\omega \Rightarrow_{\mathcal{C}} \mathbb{N}] \times_{\mathcal{C}} \mathbb{N}^\omega$ does not have the product topology.

By the cartesian closedness of $\mathrm{Top}_{\mathcal{C}}$, the set

$$P = \{(f, \gamma) \mid f(\gamma) = 0\}$$

is open (indeed clopen) in $[\mathbb{N}^\omega \Rightarrow_{\mathcal{C}} \mathbb{N}] \times_{\mathcal{C}} \mathbb{N}^\omega$. We prove that this set contains no nonempty subsets of the form $U \times V$, where $U \subseteq [\mathbb{N}^\omega \Rightarrow_{\mathcal{C}} \mathbb{N}]$ and $V \subseteq \mathbb{N}^\omega$ are open.

Suppose, for contradiction, that $U \times V \subseteq P$. Writing $\gamma = \gamma_0\gamma_1\gamma_2\dots$ for elements of $\mathbb{N}^\omega$ (i.e. infinite sequences), we can assume that V has the form

$$V = \{\gamma^\omega \in \mathbb{N} \mid \ (\gamma_0, \dots, \gamma_{k-1}) = \alpha\}$$

for some $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^k$, as such sets form a basis for $\mathbb{N}^\omega$. Let $2 = \{0, 1\}$ have the discrete topology. Then 2^ω is Cantor space, which is sequential, hence $\mathcal{C}$-generated. There is a continuous function $h\colon 2^\omega \times \mathbb{N}^\omega \to \mathbb{N}$ defined by

$$h(\delta, \gamma) = \begin{cases} 0 & \text{if all of } \delta_0, \dots, \delta_{\gamma_k} \text{ are } 0 \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $h\colon 2^\omega \times_{\mathcal{C}} \mathbb{N}^\omega \to \mathbb{N}$ is also continuous (in fact $2^\omega \times \mathbb{N}^\omega$ is sequential so is itself the product in $\mathrm{Top}_{\mathcal{C}}$). By the cartesian closedness of $\mathrm{Top}_{\mathcal{C}}$, the exponential transpose $\overline{h}\colon 2^\omega \to [\mathbb{N}^\omega \Rightarrow_{\mathcal{C}} \mathbb{N}]$ is continuous. We claim that $(\overline{h})^{-1}U = \{0^\omega\}$ (where 0^ω is the constant 0 sequence). This is a contradiction, because the singleton set $\{0^\omega\}$ is not open in 2^ω.

To verify the claim, suppose $\overline{h}(\delta) \in U$. Then, as $U \times V \subseteq P$, for all γ with $(\gamma_0, \dots, \gamma_{k-1}) = \alpha$, it holds that $(\overline{h}(\delta), \gamma) \in P$, i.e. $\overline{h}(\delta)(\gamma) = 0$. That is, $h(\delta, \gamma) = 0$ for all γ with $(\gamma_0, \dots, \gamma_{k-1}) = \alpha$. So, by the definition of h, indeed $\delta = 0^\omega$.

A related example to the above appears as Proposition 7.3 of [**17**].

DEFINITION 5·2 (Saturation). The *saturation* of a productive $\mathcal{C}$ is the collection $\bar{\mathcal{C}}$ of $\mathcal{C}$-generated spaces which are exponentiable in Top.

It is clear that saturation is an idempotent inflationary operation. The following is an immediate consequence of Lemmas 2·5 and 3·2.

LEMMA 5·3.

(i) *$\bar{\mathcal{C}}$ is the largest class of exponentiable spaces that generates the same spaces as $\mathcal{C}$. Moreover, for any space X, the coreflections $\mathcal{C}X$ and $\bar{\mathcal{C}}X$ coincide.*

(ii) *$\bar{\mathcal{C}}$ is closed under arbitrary disjoint sums and finite topological products.*

(iii) *A space is $\mathcal{C}$-generated if and only if it is a quotient of a space in $\bar{\mathcal{C}}$.*

THEOREM 5·4.

(i) *If A, B, Y are $\mathcal{C}$-generated and $q\colon A \to B$ is a topological quotient, then the map $q \times \mathrm{id}_A \colon A \times_{\mathcal{C}} Y \to B \times_{\mathcal{C}} Y$ is a topological quotient.*

(ii) *If X and Y are $\mathcal{C}$-generated spaces with Y exponentiable in* Top*, then the topological product $X \times Y$ is $\mathcal{C}$-generated.*

(iii) *If X and Y are $\mathcal{C}$-generated, then $X \times_{\mathcal{C}} Y = X \times_{\mathcal{E}} Y$, where $\mathcal{E}$ denotes the class of all exponentiable spaces.*

Proof.

A quick category-theoretic proof of (i) is to observe that topological quotients between $\mathcal{C}$-generated spaces are just coequalizers in $\mathrm{Top}_{\mathcal{C}}$, and that the functor $(-) \times_{\mathcal{C}} Y$ preserves colimits, because it is a left adjoint by cartesian closedness. Alternatively, one can easily adapt the proof of Lemma 2·6.

In order to prove (ii), we may assume that $\mathcal{C}$ is saturated. By Lemma 5·3(iii) there is a quotient map $q\colon C \to X$ with $C \in \mathcal{C}$. By Lemma 2·6 and the fact that Y is exponentiable, $q \times \mathrm{id}_Y \colon C \times Y \to X \times Y$ is a quotient map. By the assumption of saturation, $Y \in \mathcal{C}$, and hence $C \times Y \in \mathcal{C}$ by Lemma 5·3(ii). The desired conclusion then follows from Lemma 5·3(iii).

Regarding (iii), we may again assume that $\mathcal{C}$ is saturated and hence that there is a quotient map $q\colon C \to X$ with $C \in \mathcal{C}$. By (i), $q \times \mathrm{id}_C \colon C \times_{\mathcal{E}} Y \to X \times_{\mathcal{E}} Y$ is a quotient map. But, by (ii), $C \times_{\mathcal{E}} Y = C \times Y = C \times_{\mathcal{C}} Y$. Hence $X \times_{\mathcal{E}} Y$, being a quotient of a $\mathcal{C}$-generated space, is itself $\mathcal{C}$-generated, and thus the product in the category of $\mathcal{C}$-generated spaces. □

Of course, by instantiating $\mathcal{C}$ appropriately, the above result specializes to all our example categories, as does the following useful consequence.

COROLLARY 5·5. *If $\mathcal{D} \subseteq \mathcal{C}$ is productive then the inclusion functor* $\mathrm{Top}_{\mathcal{D}} \to \mathrm{Top}_{\mathcal{C}}$ *preserves finite products.*

We now give examples showing that the above preservation result extends neither to infinite products nor to exponentials.

Example 5·6. Let $\mathcal{D}$ contain only the one-point space, so the $\mathcal{D}$-generated spaces are the discrete spaces. As in Example 5·1, let $\mathcal{C}$ be such that every sequential space is $\mathcal{C}$-generated. Let $\mathbb{N}$ be the discrete non-negative integers. Then, $\mathbb{N}^\omega$ with the product topology is the Baire space, which is not discrete, but is sequential. Thus the countable power $\mathbb{N}^\omega$ in $\mathrm{Top}_\mathcal{C}$ is the Baire space, and the inclusion functor $\mathrm{Top}_\mathcal{D} \to \mathrm{Top}_\mathcal{C}$ does not preserve countable products.

Further, $\mathbb{N}$ is exponentiable in Top, and the exponential $[\mathbb{N} \Rightarrow \mathbb{N}]$ is homeomorphic to the Baire space, and hence coincides with the exponential $[\mathbb{N} \Rightarrow_\mathcal{C} \mathbb{N}]$ in $\mathrm{Top}_\mathcal{C}$. Because the Baire space is not discrete, the inclusion functor $\mathrm{Top}_\mathcal{D} \to \mathrm{Top}_\mathcal{C}$ does not preserve exponentials. The same example shows that the inclusion $\mathrm{Top}_\mathcal{D} \to \mathrm{Top}$ does not preserve function spaces that exist in Top, i.e. those where the domain of the function space is exponentiable.

Example 5·7. Let $\mathcal{S}$ contain only the one-point compactification of $\mathbb{N}$, so the $\mathcal{S}$-generated spaces are the sequential spaces. Let $\mathcal{C}$ include, at least, all compact Hausdorff spaces, so every compactly-generated space is $\mathcal{C}$-generated. Consider $2 = \{0, 1\}$ with the discrete topology. Let κ be any uncountable cardinal. By Tychonoff's Theorem, the infinite power 2^κ in Top is a compact Hausdorff space, and hence $\mathcal{C}$-generated. Thus 2^κ is the κ-fold power in $\mathrm{Top}_\mathcal{C}$. Consider the set

$$\{\gamma \in 2^\kappa \mid \gamma_i = 1 \text{ for uncountably many } i \in \kappa\}.$$

This is sequentially open, but not open in the product topology. Thus 2^κ is not a sequential space, and so the inclusion functor $\mathrm{Top}_\mathcal{S} \to \mathrm{Top}_\mathcal{C}$ does not preserve uncountable products.

Let X be a discrete space of cardinality κ. Then X is exponentiable and the exponential $[X \Rightarrow 2]$ in Top is homeomorphic to 2^κ, and hence coincides with the exponential $[X \Rightarrow_\mathcal{C} 2]$ in $\mathrm{Top}_\mathcal{C}$. Thus $[X \Rightarrow_\mathcal{C} 2]$ is not sequential, and so the inclusion $\mathrm{Top}_\mathcal{S} \to \mathrm{Top}_\mathcal{C}$ does not preserve exponentials. The same example shows that the inclusion $\mathrm{Top}_\mathcal{S} \to \mathrm{Top}$ does not preserve exponentials between spaces whose exponential exists in Top.

Example 5·8. Let $\mathcal{E}$ be the collection of all exponentiable spaces. Thus a space is $\mathcal{E}$-generated if and only if it is core compactly generated. We show that the inclusion $\mathrm{Top}_\mathcal{E} \to \mathrm{Top}$ does not preserve exponentials between spaces whose exponential exists in Top. Consider the space $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]$, as in Example 5·1. The space 2 is trivially exponentiable, and the exponential $[2 \Rightarrow [\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]]$ in Top is easily seen to be $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]^2$, i.e. $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}] \times [\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]$ with the product topology. We show that this is not core compactly generated. Because $\mathbb{N}$ is a (continuous) retract of $\mathbb{N}^\omega$, it holds that $[\mathbb{N} \Rightarrow_\mathcal{E} \mathbb{N}]$ is a retract of $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]$. But $[\mathbb{N} \Rightarrow_\mathcal{E} \mathbb{N}]$ coincides with the exponential $[\mathbb{N} \Rightarrow \mathbb{N}]$ in Top, which is homeomorphic to $\mathbb{N}^\omega$, because the latter is (sequential hence) core compactly generated. Thus $\mathbb{N}^\omega$ is a retract of $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]$. It follows that $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}] \times \mathbb{N}^\omega$ is a retract of $[\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}] \times [\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]$. But if the latter were core compactly generated then the former would be too (because retracts are quotients), which contradicts Example 5·1. Thus the exponential $[2 \Rightarrow_\mathcal{E} [\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]]$ in $\mathrm{Top}_\mathcal{E}$ differs from the exponential $[2 \Rightarrow [\mathbb{N}^\omega \Rightarrow_\mathcal{E} \mathbb{N}]]$ in Top.

As function spaces vary across the different categories, it is interesting to characterize the topologies that arise in the different cases. To this end, we consider various alternative topologies on the set $\mathrm{C}(X, Y)$ of continuous maps. Although the topologies themselves differ, we shall prove that, for any productive $\mathcal{C}$, the associated $\mathcal{C}$-generated topologies all coincide with the $\mathrm{Top}_{\mathcal{C}}$ exponential $[X \Rightarrow_{\mathcal{C}} Y]$.

The space $\mathrm{Map}_{\mathcal{C}}[X,Y]$ defined in Section 3 is already one example whose associated $\mathcal{C}$-generated topology is $[X \Rightarrow_{\mathcal{C}} Y]$. The definition of the topology on $\mathrm{Map}_{\mathcal{C}}[X,Y]$ makes use of the collection $\mathcal{C}$ of generating spaces. We now consider topologies on $\mathrm{C}(X,Y)$ determined instead by X and Y alone.

DEFINITION 5·9 (Isbell topology). The space $[X \Rightarrow_I Y]$ is the function space $\mathrm{C}(X,Y)$ endowed with the *Isbell topology*, which has subbasic open sets

$$\langle \mathcal{U}, V\rangle = \{f \mid f^{-1}(V) \in \mathcal{U}\},$$

where $V \subseteq Y$ is open and $\mathcal{U}$ is a Scott-open family of open subsets of X.

The familiar *compact-open* topology on $\mathrm{C}(X,Y)$ is given by subbasic opens of the form $\{f \mid f(K) \subseteq V\}$, for compact $K \subseteq X$ and open $V \subseteq Y$. When X is sober (in particular if it is Hausdorff) it is a straightforward consequence of the Hofmann-Mislove Theorem [**15**, Theorem II-1.21] that the Isbell topology coincides with the compact-open topology whenever the Scott topology on the lattice of opens of X has a basis consisting of filters. This latter property holds for all Čech-complete spaces, see [**10**, Theorem 4.1], a class that includes all complete metric spaces. In general, the Isbell topology is finer than the compact-open topology.

DEFINITION 5·10 (Natural topology). The space $[X \Rightarrow_\natural Y]$ is the function space $C(X,Y)$ endowed with the *natural topology*, that is, the finest topology making all transposes $\overline{f}\colon A \to C(X,Y)$ continuous whenever $f\colon A \times X \to Y$ is continuous, as A ranges over all topological spaces.

A topology on $\mathrm{C}(X,Y)$ is often called a *splitting* topology if $\overline{f}$ is continuous whenever f is. Hence the natural topology is the finest splitting topology.

PROPOSITION 5·11. *If X is an exponentiable space, then:*

$$[X \Rightarrow Y] = [X \Rightarrow_I Y] = [X \Rightarrow_\natural Y],$$

and this topology can also be given by subbasic open sets of the form

$$\{f \mid U \ll f^{-1}V\},$$

generated by open sets $U \subseteq X$ and $V \subseteq Y$.

The facts stated in the above proposition can be found in [**11**, **18**, **29**, **30**, **13**]. In these references, it is also shown that the Isbell topology is splitting. Thus the natural topology is always finer than the Isbell topology.

Example 5·12. The natural topology is, in general, strictly finer than the Isbell topology. Let $\mathbb{N}^\omega$ be the Baire space (see Example 5·1). Consider the set $\mathrm{C}(\mathbb{N}^\omega, \mathbb{N})$. We claim that the set

$$F = \{f \mid f(f(0^\omega)\, f(1^\omega)\, f(2^\omega) \ldots) = 0\}$$

(where n^ω is the constant sequence) is open in $[\mathbb{N}^\omega \Rightarrow_\natural \mathbb{N}]$ but not in $[\mathbb{N}^\omega \Rightarrow_I \mathbb{N}]$.

Concerning the Isbell topology, $\mathbb{N}^\omega$ is complete-metrizable, hence, by [**10**, Theorem 4.1], $[\mathbb{N}^\omega \Rightarrow_I \mathbb{N}]$ is the compact-open topology. It is not hard to show that no nonempty finite intersection of subbasic sets of the compact-open topology is contained in F (cf. [**17**, Proposition 7.5]). Thus F is not open in $[\mathbb{N}^\omega \Rightarrow_I \mathbb{N}]$.

Regarding the natural topology, suppose that $g\colon A \times \mathbb{N}^\omega \to \mathbb{N}$ is continuous. Then the function $h\colon A \to \mathbb{N}$ defined by

$$h(a) = g(a,\, g(a,\, 0^\omega)\, g(a,\, 1^\omega)\, g(a,\, 2^\omega)\, \dots)$$

is continuous. So $h^{-1}\{0\}$ is open in A. But $h^{-1}\{0\} = (\overline{g})^{-1}(F)$, where the function $\overline{g}\colon A \to \mathrm{C}(\mathbb{N}^\omega, \mathbb{N})$ is the transpose of g. Thus $(\overline{g})^{-1}(F)$ is open, for every continuous g. It follows that if F is added as an open set to $[\mathbb{N}^\omega \Rightarrow_\natural \mathbb{N}]$ then the resulting topology is still splitting, hence equal to $[\mathbb{N}^\omega \Rightarrow_\natural \mathbb{N}]$. So F is indeed open in $[\mathbb{N}^\omega \Rightarrow_\natural \mathbb{N}]$.

We next develop an alternative characterization of the natural topology as the topology of continuous convergence. A filter Φ on $\mathrm{C}(X, Y)$ is said to *converge continuously* to $f_0 \in \mathrm{C}(X,Y)$ if for any filter $\mathcal{G}$ converging to b in X, the filter generated by the filter base $\{F(G) \mid F \in \Phi,\, G \in \mathcal{G}\}$ converges to $f_0(b)$, where we write $F(G)$ for $\bigcup\{f(G) \mid f \in F\}$. The *topology of continuous convergence* is obtained in the standard fashion whenever one is given a family of convergent filters: a set U is open if whenever any of the given filters converges to a point in U, then U must belong to the filter.

A topology on $\mathrm{C}(X,Y)$ is called *conjoining* if $f\colon A \times X \to Y$ is continuous whenever $\overline{f}\colon A \to \mathrm{C}(X,Y)$ is. This is equivalent to the continuity of the evaluation map $E : \mathrm{C}(X,Y) \times X \to Y$. Notice that a topology on $\mathrm{C}(X,Y)$ is exponential if and only if it is both splitting and conjoining. Hence an exponential topology must agree with the natural topology since the natural topology is the finest splitting topology and since any splitting topology is contained in any conjoining topology — see e.g. [**11**] or [**13**]. We have the following additional observation.

PROPOSITION 5·13. *The natural topology and the topology of continuous convergence on* $\mathrm{C}(X,Y)$ *coincide and are the intersection of all conjoining topologies.*

Proof. We first show that the topology of continuous convergence is splitting, and then that it is the intersection of a collection of conjoining topologies. The result then follows from the fact that any splitting topology is contained in any conjoining topology.

It is splitting: Let $f\colon A \times X \to Y$ be a continuous map. In order to show that its transpose $\overline{f}\colon A \to \mathrm{C}(X,Y)$ is continuous with respect to the topology of continuous convergence, let $\mathcal{F}$ be a filter converging to $a_0 \in A$ in the topology of A. Let $\mathcal{G}$ be a filter converging to $b \in X$. Then the filter generated by the filter base $\{f(F \times G) : F \in \mathcal{F},\, G \in \mathcal{G}\}$ converges to $f(a_0, b)$. Since b was arbitrary, this means that the filter generated by the filter base $\{\{\overline{f}(a) \mid a \in S\} \mid S \in \mathcal{F}\}$ continuously converges to $\overline{f}(a_0)$, and hence converges in the topology of continuous convergence.

It is the intersection of conjoining topologies: Let Φ be a filter converging to $f_0 \in \mathrm{C}(X,Y)$ in the topology of continuous convergence, and consider the topology $\mathcal{O}_{\Phi,f_0}$ on $\mathrm{C}(X,Y)$ induced by the singleton family of convergence relations $\{\Phi \to f_0\}$. It is clear that the topology of continuous convergence is the intersection of all such topologies. Thus, to conclude, it is enough to show that this topology on $\mathrm{C}(X,Y)$ makes the evaluation map $E\colon \mathrm{C}(X,Y) \times X \to Y$ continuous. Notice that the open sets of $\mathcal{O}_{\Phi,f_0}$ are either sets that miss f_0 or sets that (contain f_0 and) belong to the filter. Assume that $E(f,x) = f(x)$ belongs to an open set $V \subseteq Y$. If $f \neq f_0$ then $\{f\}$ is open, and $\{f\} \times f^{-1}(V)$ is an open set carried into V by evaluation. Otherwise, because the neighbourhood filter of x converges to x, the filter generated by the filter base

$\{\{f(x') \mid f \in S, x' \in T\} \mid S \in \Phi, x \in \operatorname{int} T \subseteq X\}$ converges to $f_0(x)$ by definition of continuous convergence. Thus E is indeed continuous. $\square$

Remark 5·14. In the cartesian closed category of filter spaces (also called convergence spaces), see [**17**], the exponential convergence structure is given by continuous convergence of filters. Thus, for topological spaces X, Y, the topology associated to the filter space exponential on $\mathrm{C}(X, Y)$ is the topology of continuous convergence and hence, by Proposition 5·13, the natural topology. This fact is exploited in the *synthetic topology* of [**12**], see, in particular, Lemma 9.1 of *op. cit.*

Having now thoroughly examined the Isbell and natural topologies, we state our main characterization of exponentials in $\mathrm{Top}_{\mathcal{C}}$.

THEOREM 5·15. *For $\mathcal{C}$-generated spaces X, Y, we have*

$$[X \Rightarrow_{\mathcal{C}} Y] = \mathcal{C}([X \Rightarrow_I Y]) = \mathcal{C}([X \Rightarrow_\natural Y]).$$

Before proving the theorem we give an application, where $\mathbb{S}$ is the Sierpinski space (see example (2) after Lemma 3·12).

COROLLARY 5·16. *If every compact Hausdorff space is $\mathcal{C}$-generated then the function space $[X \Rightarrow_{\mathcal{C}} \mathbb{S}]$ has the Scott topology of the pointwise specialization order.*

Proof. It is immediate from the definition of the Isbell topology that the Scott topology and Isbell topology agree on the function space $[X \Rightarrow_{\mathcal{C}} \mathbb{S}]$. But the pointwise specialization order is a complete lattice isomorphic to that of open sets and hence a dcpo. Therefore the result follows from Theorems 5·15 and 4·7. $\square$

To prove Theorem 5·15, we develop an alternative characterization of the topology $\mathrm{Map}_{\mathcal{C}}[X, Y]$ used in the definition of $[X \Rightarrow_{\mathcal{C}} Y]$. The motivation is to obtain a simple description of a subbasis for $\mathrm{Map}_{\mathcal{C}}[X, Y]$, close in spirit to the subbasis for exponential topologies given in Proposition 5·11. This characterization will prove useful in Section 7 below.

DEFINITION 5·17. The relation, $\ll_{\mathcal{C}}$, on subsets of a topological space X is defined by $S \ll_{\mathcal{C}} T$ iff there exist a $\mathcal{C}$-probe $p\colon C \to X$ and open subsets $W \ll V \subseteq C$ such that $S \subseteq p(W)$ and $p(V) \subseteq T$.

Note that it is possible to have $\mathcal{C} \neq \mathcal{D}$ generating the same class of spaces but such that the relations $\ll_{\mathcal{C}}$ and $\ll_{\mathcal{D}}$ differ.

LEMMA 5·18.

(i) *$S \ll_{\mathcal{C}} T$ implies $S \ll T$.*

(ii) *If $S \ll_{\mathcal{C}} T$ and $\mathcal{U}$ is an open cover of T then there exist finitely many $S_1 \ll_{\mathcal{C}} U_1, \ldots, S_k \ll_{\mathcal{C}} U_k$, with each $U_i \in \mathcal{U}$, such that $S \subseteq S_1 \cup \cdots \cup S_k$.*

(iii) *If $S \ll_{\mathcal{C}} T$ then there exists R such that $S \ll_{\mathcal{C}} R \ll_{\mathcal{C}} T$.*

(iv) *If $f\colon X \to Y$ is continuous and $S \ll_{\mathcal{C}} T$ then $f(S) \ll_{\mathcal{C}} f(T)$.*

(v) *If $\mathcal{D} \subseteq \mathcal{C}$ then $S \ll_{\mathcal{D}} T$ implies $S \ll_{\mathcal{C}} T$.*

(vi) *If $S \ll_{\mathcal{C}} T$ in X if and only if $S \ll_{\mathcal{C}} T$ in $\mathcal{C}X$.*

Proof. We just prove statement (ii), as the others are easy consequences of Lemmas 4·2 and 4·3. Suppose that $S \ll_{\mathcal{C}} T$ and $\mathcal{U}$ is an open cover of T. There exists a probe $p\colon C \to X$ with opens $W \ll V \subseteq C$ such that $S \subseteq p(W)$ and $p(V) \subseteq T$. For each

$c \in V$, there exists $U_c \in \mathcal{U}$ such that $x \in p^{-1}U_c$. Hence, by core compactness, there exists open $W_c \subseteq X$ such that $c \in W_c \ll p^{-1}U_c$. Then $\{W_c \mid c \in V\}$ is an open cover of V, hence it has a finite subfamily $W_{c_1}, \ldots, W_{c_k}$ that covers W. For i with $1 \leq i \leq k$, define $S_i = p(W_{c_i})$ and $U_i = U_{c_i}$. This clearly has the required properties. $\square$

DEFINITION 5·19 ($\ll_{\mathcal{C}}$-topology). The space $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ is the function space $\mathrm{C}(X, Y)$ endowed with the *$\ll_{\mathcal{C}}$-topology* which has subbasic open sets

$$[S, V]_{\mathcal{C}} = \{f \mid S \ll_{\mathcal{C}} f^{-1}(V)\},$$

where $S \subseteq X$ is arbitrary and $V \subseteq Y$ is open.

Remark 5·20. Suppose that $\mathcal{C}$ consists of locally compact spaces. Then the $\ll_{\mathcal{C}}$-topology reduces to the compact-open topology, where the compact sets run over the images of compact sets in the domains of probes from members of $\mathcal{C}$.

Proof. Suppose that $f \in [S, V]_{\mathcal{C}}$, i.e., $S \ll_{\mathcal{C}} f^{-1}(V)$. Then there exists $A \in \mathcal{C}$, a probe $p\colon A \to X$, and $T \ll W \subseteq A$ such that $S \subseteq p(T)$ and $p(W) \subseteq f^{-1}(V)$. It follows that $T \ll W \subseteq (fp)^{-1}(V)$. By local compactness we find for each point of $(fp)^{-1}(V)$ a compact neighbourhood containing that point that is contained in the open set $(fp)^{-1}(V)$. Then finitely many of these neighbourhoods cover T. If we denote their finite union by K then we have that K is compact and $T \subseteq K \subseteq (fp)^{-1}(V)$. It follows that $S \subseteq p(K)$, f carries $p(K)$ into V, hence that $p(K) \ll_{\mathcal{C}} f^{-1}(V)$ (since K is compact), and that the subbasic compact-open neighbourhood of $\{g \mid p(K) \subseteq g^{-1}(V)\}$ of f is contained in $[S, V]_{\mathcal{C}}$. Hence the identity map from the compact-open topology to the $\ll_{\mathcal{C}}$-topology is continuous. That it is continuous in the opposite direction is immediate, since subbasic opens in the compact-open topology of the form $\{g \mid p(K) \subseteq g^{-1}(V)\}$ are a special case of $\ll_{\mathcal{C}}$-open sets. $\square$

PROPOSITION 5·21. *For $\mathcal{C}$-generated spaces X, Y, $[X \Rightarrow_{\ll_{\mathcal{C}}} Y] = \mathrm{Map}_{\mathcal{C}}[X, Y]$.*

Proof. To show that $\mathrm{Map}_{\mathcal{C}}[X, Y]$ refines $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$, we establish that $[S, V]_{\mathcal{C}}$ is open in $\mathrm{Map}_{\mathcal{C}}[X, Y]$. Consider $f \in [S, V]_{\mathcal{C}}$. Then there exists a probe $p\colon C \to X$ and open $W \ll W' \subseteq C$ such that $S \subseteq p(W)$ and $p(W') \subseteq f^{-1}V$. The set $G = \{g\colon C \to Y \mid W \ll g^{-1}V\}$ is open in $[C \Rightarrow Y]$ (because $[C \Rightarrow Y]$ has the Isbell topology). So, by the definition of $\mathrm{Map}_{\mathcal{C}}[X, Y]$ from Section 3, we have that $T_p^{-1}G$ is open in $\mathrm{Map}_{\mathcal{C}}[X, Y]$. We show that $f \in T_p^{-1}G \subseteq [S, V]_{\mathcal{C}}$. As $p(W') \subseteq f^{-1}V$, we have $W \ll W' \subseteq (f \circ p)^{-1}V$, so indeed $f \in T_p^{-1}G$. Also, if $h \in T_p^{-1}G$, then $W \ll (h \circ p)^{-1}V$, where $(h \circ p)^{-1}V$ is open in C. Because $S \subseteq p(W)$ and $p((h \circ p)^{-1}V) \subseteq h^{-1}V$, we have $S \ll_{\mathcal{C}} h^{-1}V$, i.e. $h \in [S, V]_{\mathcal{C}}$, as required.

Conversely, to establish that $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ refines $\mathrm{Map}_{\mathcal{C}}[X, Y]$, we show, for any probe $p\colon C \to X$, that the function $T_p\colon [X \Rightarrow_{\ll_{\mathcal{C}}} Y] \to [C \Rightarrow Y]$ is continuous. Accordingly, consider any subbasic open $G = \{g \mid W \ll g^{-1}V\}$ of $[C \Rightarrow Y]$, given by opens $W \subseteq C$ and $V \subseteq Y$, as in Proposition 5·11, and any $f \in T_p^{-1}G$, i.e. $W \ll (f \circ p)^{-1}V$. By interpolation, there exists open $W' \subseteq C$ with $W \ll W' \ll (f \circ p)^{-1}V$. Then $p(W') \ll_{\mathcal{C}} p((f \circ p)^{-1}V) \subseteq f^{-1}V$, so $f \in [p(W'), V]_{\mathcal{C}}$. Also, for any $h \in [p(W'), V]_{\mathcal{C}}$, we have $p(W') \subseteq h^{-1}V$, so $W \ll W' \subseteq p^{-1}(p(W')) \subseteq p^{-1}(h^{-1}V) = (h \circ p)^{-1}V$, and hence $h \in T_p^{-1}G$. We have shown that $f \in [p(W'), V]_{\mathcal{C}} \subseteq T_p^{-1}G$. Thus $T_p^{-1}G$ is indeed open in $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$. $\square$

Proof of Theorem 5·15. Let $p\colon A \to [X \Rightarrow_{\mathcal{C}} Y]$ be a probe. By the cartesian closedness of $\mathcal{C}$-generated spaces, the function

$$(z, x) \mapsto p(z)(x) \colon A \times_{\mathcal{C}} X \longrightarrow V$$

is continuous. Moreover, as A, X are $\mathcal{C}$-generated spaces and A is core compact, $A \times_{\mathcal{C}} X$ is just the topological product $A \times X$. By definition of the natural topology, we have that $p \colon A \to [X \Rightarrow_{\natural} Y]$ is continuous. Since $[X \Rightarrow_{\mathcal{C}} Y]$ is $\mathcal{C}$-generated, it follows that its topology refines the natural topology, which in turn refines the Isbell topology. Taking the $\mathcal{C}$-generated topologies, we conclude that $[X \Rightarrow_{\mathcal{C}} Y] = \mathcal{C}([X \Rightarrow_{\mathcal{C}} Y])$ refines $\mathcal{C}([X \Rightarrow_{\natural} Y])$, which refines $\mathcal{C}([X \Rightarrow_I Y])$.

However, $[X \Rightarrow_I Y]$ refines $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ because, by Lemma 5·18(i),(iii), the family $\{U \subseteq X \mid U \text{ open and } S \ll_{\mathcal{C}} U\}$ is Scott open. Thus, by Proposition 5·21, $[X \Rightarrow_I Y]$ refines $\mathrm{Map}_{\mathcal{C}}[X, Y]$. So $\mathcal{C}([X \Rightarrow_I Y])$ refines $\mathcal{C}(\mathrm{Map}_{\mathcal{C}}[X, Y]) = [X \Rightarrow_{\mathcal{C}} Y]$. □

We close this section with some simple observations about how the standard separation axioms may imposed upon the categories of $\mathcal{C}$-generated spaces. The theorem justifies our comment in Section 1 that our techniques implicitly cover the case of compactly generated Hausdorff spaces.

THEOREM 5·22.

(i) *Limits in* $\mathrm{Top}_{\mathcal{C}}$ *of diagrams of* T_2 *spaces are again* T_2 *spaces.*

(ii) *Disjoint sums in* $\mathrm{Top}_{\mathcal{C}}$ *of* T_2 *spaces are again* T_2 *spaces.*

(iii) *If* X, Y *are* $\mathcal{C}$*-generated spaces and* Y *is* T_2 *then also* $[X \Rightarrow_{\mathcal{C}} Y]$ *is* T_2.

Moreover, analogous statements hold if T_2 *is replaced throughout by* T_1 *or* T_0.

Proof. For (i), we have that any categorical limit in Top of T_2 spaces is again T_2. The limit in $\mathrm{Top}_{\mathcal{C}}$ is the $\mathcal{C}$-generated topology, which is finer, hence again T_2.

Statement (ii) is obvious.

For (iii), we show that $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ is T_2. Suppose $f \neq g \in \mathrm{C}(X, Y)$. Then there exists $x_0 \in X$ with $f(x_0) \neq g(x_0)$, hence there exist disjoint opens $U, V \subseteq Y$ with $f(x_0) \in U$ and $g_0(x_0) \in V$. The sets $F = \{h \mid h(x_0) \in U\}$ and $G = \{h \mid h(x_0) \in V\}$ are easily seen to be disjoint opens in $[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ with $f \in F$ and $g \in G$ as required.

The proofs for the T_1 and T_0 cases are similar. □

Note that the separation axioms are not preserved by quotients, hence the omission of quotients from the statement of the theorem. Note, further, that all spaces appearing in this section, when giving examples of non-preservation of structure, were Hausdorff. Thus the situation with respect to the preservation of products and function spaces is not improved by the imposition of separation axioms.

6. *Quotients of countably based spaces*

In this section we consider the full subcategory QCB of Top consisting of all spaces that are quotients of countably based spaces. Obviously, this is a subcategory of the category of sequential spaces, and hence a subcategory of $\mathcal{C}$-generated spaces, for any $\mathcal{C} \supseteq \mathcal{S}$, where, as above, $\mathcal{S} = \{\mathbb{N}^\infty\}$. It is known that QCB is cartesian closed, and that its products and function spaces are those in the category of sequential spaces [**27**, **24**]. In Section 7, we shall prove that, more generally, products and function spaces in QCB coincide with those of $\mathrm{Top}_{\mathcal{C}}$, for every $\mathcal{C} \supseteq \mathcal{S}$. To achieve this, we first obtain characterizations of quotients of countably based spaces based on various notions of “pseudobase”, loosely related to the original such notion due to Michael [**25**]. For the purposes of this paper, we need to consider three notions of pseudobase.

Henceforth in this section, let $\mathcal{C}$ be productive with $\mathcal{C} \supseteq \mathcal{S}$. It follows that every sequential space is $\mathcal{C}$-generated.

DEFINITION 6·1 (Pseudobases). A collection $\mathcal{B}$ of subsets of a topological space X is called

(i) an S-*pseudobase* if given any convergent sequence $x_n \to x$ and any open set U containing x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, and $x_n \in B$ for all sufficiently large n,

(ii) a $\ll_{\mathcal{C}}$-*pseudobase* if, whenever $S \ll_{\mathcal{C}} U \subseteq X$ with U open, there exists $B \in \mathcal{B}$ such that $S \subseteq B \subseteq U$.

(iii) a $\ll$-*pseudobase* if, whenever $S \ll U \subseteq X$ with U open, there exists $B \in \mathcal{B}$ such that $S \subseteq B \subseteq U$.

The notion of S-pseudobase is originally due to Schröder [**28**].

LEMMA 6·2.

(i) *Any $\ll_{\mathcal{C}}$-pseudobase is an* S-*pseudobase.*

(ii) *Any $\ll$-pseudobase is a $\ll_{\mathcal{C}}$-pseudobase.*

(iii) *Any topological base is an* S-*pseudobase and its closure under finite unions is a $\ll$-pseudobase (hence $\ll_{\mathcal{C}}$-pseudobase).*

(iv) *The closure under finite unions of an* S-*pseudobase is a $\ll_{\mathcal{S}}$-pseudobase.*

LEMMA 6·3. *If $\mathcal{B}$ is an* S-*pseudobase (resp. $\ll$-pseudobase) for X, and $Y \subseteq X$ is any subset, then the family $\{B \cap Y \mid B \in \mathcal{B}\}$ is an* S-*pseudobase (resp. $\ll$-pseudobase) for the relative topology on Y.*

The straightforward proofs of these lemmas are omitted.

Recall that the *saturation* of a subset B of a topological space, denoted by $\uparrow B$, is the intersection of its neighbourhoods, or, equivalently, its upper set in the specialization order. The lower set of B in the specialization order is denoted by $\downarrow B$. If W is open, then $\uparrow B \subseteq W$ if and only if $B \subseteq W$ for any subset B. It follows readily that if a collection $\mathcal{B}$ is any of the above types of pseudobase then the collection $\{\uparrow B \mid B \in \mathcal{B}\}$ is the same type of pseudobase. Because of this, we shall henceforth assume, without loss of generality, that pseudobases always consist of saturated sets. Also, all pseudobases we construct will indeed consist of saturated sets.

The notions of $\ll_{\mathcal{C}}$- and $\ll$-pseudobase share the slightly unfortunate property that topological bases are not necessarily pseudobases (though their closures under finite unions are). This could be remedied by weakening these notions of pseudobase to require merely that there are finitely many members of $\mathcal{B}$, each contained in U, that cover S. However, as the weakened definitions are equivalent modulo closure under finite unions, we have chosen to use the definition given above, because it is sometimes mildly simpler to work with.

As the first result of this section, we show that every quotient of a countably based space has a countable $\ll$-pseudobase.

PROPOSITION 6·4. *Let $\rho\colon X \to Y$ be a quotient map, and assume that X has a countable base $\mathcal{B}$ for the topology. The collection of finite unions of sets of the form $\uparrow\rho(B)$ with $B \in \mathcal{B}$ is a countable $\ll$-pseudobase for Y.*

Proof. Let W be an open subset of Y and let $A \ll W$. Let $\{B_i : i = 1, 2, 3 \ldots\}$ be an enumeration of all $B \in \mathcal{B}$ such that $\rho(B) \subseteq W$. Set $A_n = \bigcup_{i=1}^n \uparrow\rho(B_i)$. We need to show that some A_n contains A. If not, choose for each n some $y_n \in A \setminus A_n$. Then $\overline{\{y_n\}} = \downarrow y_n$ misses $A_n = \uparrow A_n$. The fact that $A \ll W$ implies that the sequence y_n has a cluster point y in W.

Fix some k and consider the set $V_k = \rho^{-1}(W \setminus \bigcup_{n=k}^{\infty} \downarrow y_n) = \rho^{-1}(W) \setminus \bigcup_{n=k}^{\infty} \rho^{-1}(\downarrow y_n)$. For $x \in V_k$, let $\{O_n\}$ be a countable descending base at x. If each O_n meets the sequence of sets $\{\rho^{-1}(\downarrow y_n)\}$ cofinally, then we can pick a sequence $x_{n_j} \to x$ such that $\rho(x_{n_j}) \in \downarrow y_{n_j}$ for all n_j. Pick $B \in \mathcal{B}$ such that $x \in B$ and $\rho(B) \subseteq W$. Then $y_{n_j} \in \uparrow\rho(x_{n_j}) \subseteq \uparrow\rho(B)$ for all sufficiently large j, since $x_{n_j} \to x$. By the construction of the A_n, we have $B \subseteq A_n$ for all sufficiently large indices n. It follows that $y_{n_j} \in A_{n_j}$ for sufficiently large j. This contradicts the construction of the sequence $\{y_n\}$. We conclude that for any V_k and any $x \in V_k$, there exists some open set around x that misses all but finitely many of the $\rho^{-1}(\downarrow y_n)$.

From the definition of V_k, if $x \in V_k$, then $x \notin \rho^{-1}(\downarrow y_n)$ for any $n \geq k$. Since the latter is a closed set and since from the preceding paragraph there is an open set $U \subseteq \rho^{-1}(W)$ around x that hits at most finitely many of them, we can intersect U with the complements of all $\rho^{-1}(\downarrow y_n)$, $n \geq k$, that U hits, and obtain an open set containing x and contained in V_k. It follows that V_k is open, and thus $W \setminus \bigcup_{n=k}^{\infty} \downarrow y_n$ is open, since ρ is a quotient map.

Now it cannot be the case that $y \in W \setminus \bigcup_{n=k}^{\infty} \downarrow y_n$, since $\{y_n\}$ clusters to y and eventually misses the open set $W \setminus \bigcup_{n=k}^{\infty} \downarrow y_n$. Thus it must be the case that $y \in \downarrow y_n$ for some $n \geq k$. Since k was arbitrary we conclude that $y \in \downarrow y_n$ for infinitely many indices n. Let $p \in X$ such that $\rho(p) = y \in W$. There exists $B \in \mathcal{B}$ such that $p \in B$, $\rho(B) \subseteq W$. Then from all large enough integers n, we have $\uparrow\rho(B) \subseteq A_n$. But for infinitely many indices $y_n \in \uparrow y \subseteq \uparrow\rho(B)$. This contradicts $y_n \notin A_n$ for all n. This contradiction completes the proof. □

Proposition 6·5. *Let X be a topological space with a countable $\ll_{\mathcal{C}}$-pseudobase. Then $\mathcal{C}X$ is the sequential topology on X.*

Proof. Since $\mathcal{S} \subseteq \mathcal{C}$, it holds that $\mathcal{C}X$ refines the sequential topology. We must show that each sequential open set U is open in $\mathcal{C}X$. If it is not, then there exists a probe $p\colon Q \to X$, where $Q \in \mathcal{C}$, such that $p^{-1}(U)$ is not open in Q. Hence there exists $y \in Q$ such that $y \in p^{-1}(U)$, but V is not a subset of $p^{-1}(U)$ for every open subset V containing y. That is, $p(V)$ is not contained in U, for every open subset V containing y.

Let $\mathcal{B}$ be a countable $\ll_{\mathcal{C}}$-pseudobase for X. Let $\mathcal{A}$ consist of all $B \in \mathcal{B}$ such that $p(V) \subseteq B$ for some V open containing y. Because Q is core compact, there exists open $V \subseteq Q$ with $y \in V \ll Q$, so $p(V) \ll_{\mathcal{C}} p(Q) \subseteq X$, whence, because $\mathcal{B}$ is a $\ll_{\mathcal{C}}$-pseudobase, there exists $B \in \mathcal{B}$ such that $p(V) \subseteq B$. This shows that $\mathcal{A}$ is non-empty. Let $\{B_n \mid n \geq 1\}$ be an enumeration of $\mathcal{A}$. For each n, we choose $x_n \in \bigcap_{i=1}^n B_n$ such that $x_n \notin U$ (this is possible since B_i contains some $p(V_i)$, and $p(\bigcap_{i=1}^n V_i) \subseteq \bigcap_{i=1}^n B_i$, but $p(\bigcap_{i=1}^n V_i)$ is not contained in U).

Let W be any open set containing $p(y)$. Then $p^{-1}(W)$ is open in Q and contains y. Thus by core compactness there exists some open set V containing y such that $V \ll p^{-1}(W)$. Then $p(V) \ll_{\mathcal{C}} p(p^{-1}(W)) \subseteq W$. Since $\mathcal{B}$ is a $\ll_{\mathcal{C}}$-pseudobase, there exists some $B \in \mathcal{B}$ such that $p(V) \subseteq B \subseteq W$. Then $B \in \mathcal{A}$, i.e., $B = B_j$ for some j, and thus $x_n \in B_j \subseteq W$ for all $n \geq j$. Since W was an arbitrary open set containing $p(y)$, we conclude that

$x_n \to p(y)$. Since U is a sequentially open set containing $p(y)$, we thus obtain that $x_n \in U$ for some n. But this contradicts the choice of x_n. $\square$

COROLLARY 6·6. *The sequential and core compactly generated topologies agree for any space with countable* $\ll$*-pseudobase.*

Proof. Any countable $\ll$-pseudobase on X is also a $\ll_{\mathcal{E}}$-pseudobase, where $\mathcal{E}$ is the collection of all core compact spaces. By Proposition 6·5, $\mathcal{E}X$ is the sequential topology on X. But $\mathcal{E}X$ is the core compactly generated topology on X. $\square$

Proposition 6·8 below, which shows that any sequential space with countable S-pseudobase is a quotient of a countably based space, is due to Schröder, and our proof is adapted from [**28**, **2**]. We first recall, without proof, a standard (and straightforward) topological lemma.

LEMMA 6·7. *Let* $f\colon Y \to X$ *be a continuous function between sequential spaces. Suppose that a sequence* $(x_i) \to x$ *in* X *implies that there exist* (y_i) *and* y *in* Y *such that* $f(y_i) = x_i$, $f(y) = x$ *and* $(y_i) \to y$ *in* Y, *i.e., convergent sequences lift. Then* f *is a topological quotient map.*

PROPOSITION 6·8. *Let* X *be a topological space with countable* S*-pseudobase. Then* X *equipped with the sequential topology is the quotient of a countably based space.*

Proof. We assume that the elements of the S-pseudobase $\mathcal{B}$ are enumerated as B_n for $n \in \mathbb{N}$.

We construct a countably based space R as follows. The elements of R are pairs (x, β), where $x \in X$ and $\beta \in \prod_{n \in \mathbb{N}}\{0,1\}$ satisfy

i) for all $n \in \mathbb{N}$, $x \in B_n$ if $\beta(n) = 1$, and

ii) for all neighbourhoods V of x there exists $n \in \mathbb{N}$ such that $\beta(n) = 1$ and $B_n \subseteq V$.

We topologize $\Omega = \prod_n\{0,1\}$ with the product of the Sierpinski topologies (with $\{1\}$ the only nontrivial open set). The topology of $X \times \Omega$ is the product of the indiscrete topology on X with the topology just defined on Ω, and the topology of R is the subspace topology. The topology on R has an alternative description as the coarsest topology such that the second projection $R \to \Omega$ is continuous.

The topology of R is countably based since that of $X \times \Omega$ is. Let $p\colon R \to X$ be the first projection function, where the image X has its original topology. Let $(x, \beta) \in R$ with $p(x, \beta) = x \in U$, an open subset of X. Then there exists $n \in \mathbb{N}$ such that $\beta(n) = 1$ and $x \in B_n \subseteq U$. Consider the open set V of all (y, γ) such that $\gamma(n) = 1$. Then $y \in B_n$, so $p(y, \gamma) \in U$. Thus p is continuous. It is easily verified from the properties of a pseudobase that p is surjective.

Claim: *For* $p\colon R \to X$, *convergent sequences lift.* Suppose $(x_i) \to x$ in X. Define $y = (x, \beta)$ where $\beta(n) = 1$ if and only if $x \in B_n$ and (x_i) is eventually in B_n. This is a well-defined element of R because $(x_i) \to x$ and $\mathcal{B}$ is an S-pseudobase. Define $y_i = (x_i, \beta_i)$ where $\beta_i(n) = 1$ if and only if $x_i \in B_n$. We just need to show that $(y_i) \to y$. Let $\beta(n) = 1$. We must show that there exists k such that, for all $i \geq k$, $\beta_i(n) = 1$. But this follows from the definition of β and β_i. This completes the claim.

Putting the above together, we have a second-countable space R and continuous $p\colon R \to X$ satisfying the hypotheses of Lemma 6·7. As R is second-countable, hence sequential, we can factor $p\colon R \to X$ as $R \to \mathrm{Seq}(X) \to X$. By Lemma 6·7 the map $R \to \mathrm{Seq}(X)$ is a topological quotient. $\square$

PROPOSITION 6·9. *Let X be a topological space with countable $\ll_{\mathcal{C}}$-pseudobase $\mathcal{B}$. Then $\mathcal{C}X$ is a quotient of a countably based space and the closure of $\mathcal{B}$ under finite unions and intersections is a countable $\ll$-pseudobase for $\mathcal{C}X$.*

Proof. By Proposition 6·5, $\mathcal{C}X$ is the sequential topology on X. Moreover, $\mathcal{B}$ is a countable S-pseudobase on X. Thus the proof of Proposition 6·8 exhibits $\mathcal{C}X$ as a quotient of $R \subseteq X \times \Omega$.

Let J be a finite subset of $\mathbb{N}$. Then the set

$$\{(x, \beta) \mid \beta(n) = 1 \text{ for all } n \in J\}$$

is a basic open set in $X \times \Omega$. Any point $(x, \beta) \in R$ that is also in this basic open set must be in $\bigcap_{n \in J} B_n$ and conversely for any point in $x \in \bigcap_{n \in J} B_n$, one can construct $\gamma \in \Omega$ such that $(x, \gamma) \in R$. It follows that the images of the specified countable basis of R consist of finite intersections of members of $\mathcal{B}$. Thus, by Proposition 6·4, the closure under finite unions of saturates of finite intersections of members of $\mathcal{B}$ is a $\ll$-pseudobase for the topology $\mathcal{C}X$. But the members of $\mathcal{B}$, and hence their intersections, are already saturated. Thus the $\ll$-pseudobase is simply the closure under finite unions and intersections of $\mathcal{B}$. □

The next theorem follows directly from our preceding results. Recall that $\mathcal{C}$ is assumed to be productive with $\mathcal{C} \supseteq \mathcal{S}$, where $\mathcal{S} = \{\mathbb{N}^\infty\}$.

THEOREM 6·10. *The following are equivalent for any space X.*

(i) *X is a quotient of a countably based space.*
(ii) *X is a sequential space with a countable* S*-pseudobase.*
(iii) *X is a $\mathcal{C}$-generated space with a countable $\ll_{\mathcal{C}}$-pseudobase.*
(iv) *X is a core compactly generated space with a countable $\ll$-pseudobase.*

The equivalence of statements (1) and (2) is originally due to Schröder [**27**].

COROLLARY 6·11. *If a core compact space is a quotient of a countably based space, then it is itself countably based.*

Proof. Take any core compact space X with countable $\ll$-pseudobase $\mathcal{B}$. It is easy to see that the collection of interiors of members of $\mathcal{B}$ is a countable base for X. □

Any quotient of a countably based space is core compactly generated, hence a quotient of a core compact space. The remark below shows, however, that it is not always possible to exhibit a QCB space as a quotient of a space that is simultaneously countably based and core compact.

Remark 6·12. The Baire space $\mathbb{N}^\omega$ (the product of countably many copies of a discrete countable space), although itself countably based, is not a quotient of any countably based core compact space.

Proof. Suppose for contradiction there were a quotient map $q \colon X \to \mathbb{N}^\omega$ with X a countably based core compact space. Then X has a countable base $\{U_i\}$ of opens U_i such that $U_i \ll X$. By Lemma 4·3(ii), the closures of the sets $q(U_i)$ form a countable family of compact sets, and it clearly covers $\mathbb{N}^\omega$. By the Baire category theorem, one of them has nonempty interior. But then this interior is a nonempty locally compact open subset of $\mathbb{N}^\omega$, and $\mathbb{N}^\omega$ has no locally compact nonempty opens. □

7. *Countable limits and function spaces in* QCB

In this section, we investigate the behaviour of QCB spaces within any containing category of $\mathcal{C}$-generated spaces. The main result states that whenever a countable (categorical) limit of QCB spaces or a function space between QCB spaces is calculated in the category of $\mathcal{C}$-generated spaces then the resulting space is again a QCB space. Firstly, this provides a self-contained proof that the category QCB has countable limits and is cartesian closed (see [**27**, **24**] for alternative proofs). Secondly, it follows that the inclusion of QCB in $\mathrm{Top}_{\mathcal{C}}$ preserves function spaces and countable limits. Thus, for quotients of countably based spaces, the topologies underlying function spaces and countable limits are, in a strong sense, canonical, for they are always the same irrespective of which ambient category $\mathrm{Top}_{\mathcal{C}}$ they are calculated within. In contrast, Example 5·7 demonstrates that such a situation does not hold for more general spaces.

We briefly review the construction of countable limits in the category $\mathrm{Top}_{\mathcal{C}}$. As is well known, any countable limit can be constructed from a combination of a countable product and an equalizer, so it suffices to describe the construction of these. Given a sequence $(X_i)_{i\geq 0}$ of $\mathcal{C}$-generated spaces, the countable product, $\prod_i^{\mathcal{C}} X_i$, in the category $\mathrm{Top}_{\mathcal{C}}$ is defined by:

$$\prod_i^{\mathcal{C}} X_i \;=\; \mathcal{C}(\prod_i X_i),$$

where the right-hand side is the $\mathcal{C}$-generated topology of the topological product. If X and Y are $\mathcal{C}$-generated spaces and $f, g\colon X \to Y$ are continuous, then the equalizer E in $\mathrm{Top}_{\mathcal{C}}$ is defined by

$$E_{\mathcal{C}} \;=\; \mathcal{C}(\{x \in X \mid f(x) = g(x)\}),$$

where the subset of X on the right is given the subspace topology.

PROPOSITION 7·1. *The following conditions are equivalent for $\mathcal{C}$:*

(i) *Every* QCB*-space is $\mathcal{C}$-generated.*
(ii) *The one point compactification of $\mathbb{N}$ is $\mathcal{C}$-generated.*
(iii) *Every sequential space is $\mathcal{C}$-generated.*

Proof. The one point compactification of $\mathbb{N}$ is countably based, hence in QCB, and every QCB space is sequential. Finally, we have already seen that if the one point compactification of $\mathbb{N}$ is $\mathcal{C}$-generated then so is every sequential space. □

Henceforth in this section, we assume that $\mathcal{C}$ is productive and satisfies any of the equivalent conditions of Proposition 7·1.

THEOREM 7·2.

(i) *If $f, g\colon X \to Y$ are continuous maps between* QCB *spaces then their equalizer in* $\mathrm{Top}_{\mathcal{C}}$ *is also a* QCB *space.*
(ii) *If $(X_i)_{i\geq 0}$ is a sequence of* QCB *spaces, then $\prod_i^{\mathcal{C}} X_i$ is a* QCB *space.*
(iii) *If X and Y are* QCB *spaces, then so is $[X \Rightarrow_{\mathcal{C}} Y]$.*

Before proving the theorem, we state a corollary, which follows immediately, given the fact that $\mathrm{Top}_{\mathcal{C}}$ is cartesian closed and has arbitrary limits.

COROLLARY 7·3. *The category* QCB *is cartesian closed with countable limits. Moreover, the inclusion of* QCB *in* $\mathrm{Top}_{\mathcal{C}}$ *preserves function spaces and countable limits.*

Although the first half of this corollary is already known [**27**, **24**], the result on structure preservation is new, and is one of the main contributions of the present paper.

Remark 7·4. The results above say nothing about categorical colimits. In fact, the existence of countable colimits in QCB is straightforward, they are computed as in Top, and hence trivially preserved by the inclusion of QCB in $\mathrm{Top}_{\mathcal{C}}$.

To prove Theorem 7·2 we assume henceforth that $\mathcal{C}$ is productive and $\mathcal{C} \supseteq \mathcal{S}$, as in the previous section. The latter assumption can be made without loss of generality, because Theorem 7·2 does not depend upon the choice of generating spaces for $\mathrm{Top}_{\mathcal{C}}$.

Proof of Theorem 7·2(i). Let $f, g\colon X \to Y$ be continuous maps between QCB spaces. Define $E = \{x \in X \mid f(x) = g(x)\}$. By Lemma 6·3, a countable $\ll$-pseudobase for X restricts to a countable $\ll$-pseudobase (hence $\ll_{\mathcal{C}}$-pseudobase) for the relative topology on E. Thus, by Proposition 6·9, the equalizer $E_{\mathcal{C}} = \mathcal{C}(E)$ is a QCB space. □

To prove the remaining statements of Theorem 7·2, it is convenient to develop notions of pseudo*sub*base. Given a family $\mathcal{B}$ of subsets of a set X, we write $\cup(\mathcal{B})$ for the closure of $\mathcal{B}$ under finite unions, $\cap(\mathcal{B})$ for the closure of $\mathcal{B}$ under finite intersections, and $\cup\text{-}\cap(\mathcal{B})$ for the closure of B under finite intersections and unions.

DEFINITION 7·5 (Pseudosubbases). A family $\mathcal{B}$ of saturated sets is said to be a $\ll$-*pseudosubbase* (resp. $\ll_{\mathcal{C}}$-*pseudosubbase*) for a topological space X if $\cup\text{-}\cap(\mathcal{B})$ is a $\ll$-pseudobase (resp. $\ll_{\mathcal{C}}$-pseudobase) for X.

The results on pseudobases from Section 6 adapt easily to pseudosubbases. Trivially, any $\ll$-pseudobase is a $\ll_{\mathcal{C}}$-pseudosubbase. By Theorem 6·10, any QCB space has a countable $\ll$-pseudobase. By Proposition 6·9, if a topological space X has a countable $\ll_{\mathcal{C}}$-pseudo(sub)base $\mathcal{B}$, then $\mathcal{C}X$ is a QCB space and $\mathcal{B}$ is a $\ll$-pseudosubbase for $\mathcal{C}X$. Hence a family $\mathcal{B}$ is a $\ll_{\mathcal{C}}$-pseudosubbase for a $\mathcal{C}$-generated space, X, if and only if it is a $\ll$-pseudosubbase for the space. Thus, for $\mathcal{C}$-generated spaces, we shall henceforth talk unambiguously about pseudosubbases, without qualifying them as being of the $\ll_{\mathcal{C}}$ or $\ll$ variety.

By the above remarks, statement (ii) of Theorem 7·2 follows immediately from Proposition 7·6 below, using the construction of countable products in $\mathrm{Top}_{\mathcal{C}}$. Statement (iii) follows from Proposition 7·7, because $[X \Rightarrow_{\mathcal{C}} Y] = \mathcal{C}[X \Rightarrow_{\ll_{\mathcal{C}}} Y]$ by Proposition 5·21.

PROPOSITION 7·6. *Let $(X_i)_{i\geq 0}$ be a sequence of topological spaces, where each X_i has $\ll_{\mathcal{C}}$-pseudosubbase $\mathcal{B}_i$. Then $\prod_i X_i$ has $\ll_{\mathcal{C}}$-pseudosubbase:*

$$\{(B_0 \times \cdots \times B_{n-1} \times \prod_{i\geq n} X_i) \mid n \geq 0,\, B_0 \in \mathcal{B}_0,\, \ldots,\, B_{n-1} \in \mathcal{B}_{n-1}\},$$

which is countable if each $\mathcal{B}_i$ is.

PROPOSITION 7·7. *If X and Y are $\mathcal{C}$-generated spaces, with countable pseudosubbases $\mathcal{A}$ and $\mathcal{B}$ respectively, then the set*

$$\{(\!(A, B)\!) \mid A \in \cap(\mathcal{A}), B \in \cup(\mathcal{B})\},$$

where

$$(\!(A, B)\!) = \{f \in \mathrm{C}(X, Y) \mid f(A) \subseteq B\},$$

is a countable $\ll_{\mathcal{C}}$-pseudosubbase for $X \Rightarrow_{\ll_{\mathcal{C}}} Y$.

The remainder of this section is devoted to the proofs of Propositions 7·6 and 7·7.

LEMMA 7·8. *Suppose that $\mathcal{W}$ is a subbase for a topological space X. Then a family $\mathcal{B}$ of saturated subsets of X is a $\ll_{\mathcal{C}}$-pseudosubbase for X if and only if, for every $S \ll_{\mathcal{C}} W \in \mathcal{W}$, there exists $B \in \cup\text{-}\cap(\mathcal{B})$ such that $S \subseteq B \subseteq W$.*

Proof. The only-if direction is trivial. For the if direction, suppose that $\mathcal{B}$ satisfies the condition stated. We must show it to be a $\ll_{\mathcal{C}}$-pseudosubbase. Suppose then that $S \ll_{\mathcal{C}} U$ where $U \subseteq X$ is open. We have that $U = \bigcup \mathcal{V}$, for some family $\mathcal{V}$ consisting of finite intersections of elements of $\mathcal{W}$. By Lemma 5·18(ii), for some $k \geq 0$ there exist $S_1 \ll_{\mathcal{C}} V_1, \ldots, S_k \ll_{\mathcal{C}} V_k$ with $V_1, \ldots, V_k \in \mathcal{V}$ and $S \subseteq S_1 \cup \cdots \cup S_k$. Then, for each V_i, we have $V_i = W_{i1} \cap \cdots \cap W_{ik_i}$, with each $W_{ij} \in \mathcal{W}$, so $S_i \ll_{\mathcal{C}} W_{i1}, \ldots, S_i \ll_{\mathcal{C}} W_{ik_i}$. By assumption, for each W_{ij}, there exists $B_{ij} \in \cup\text{-}\cap(\mathcal{B})$ with $S_i \subseteq B_{ij} \subseteq W_{ij}$. Thus, defining $B_i = B_{i1} \cap \cdots \cap B_{ik_i}$, we have $S_i \subseteq B_i \subseteq V_i$; and, defining $B = B_1 \cup \cdots \cup B_k$, we have $S \subseteq S_1 \cup \cdots \cup S_k \subseteq B \subseteq V_1 \cup \cdots \cup V_k \subseteq U$. Thus we indeed have $B \in \cup\text{-}\cap(\mathcal{B})$ with $S \subseteq B \subseteq U$. $\square$

Proof of Proposition 7·6. Consider any basic open set $U_0 \times \cdots \times U_{n-1} \times \prod_{i \geq n} X_i$ in the product topology $\prod_i X_i$, given by opens $U_0 \subseteq X_0, \ldots, U_{n-1} \subseteq X_{n-1}$. Suppose that $S \ll_{\mathcal{C}} (U_0 \times \cdots \times U_{n-1} \times \prod_{i \geq n} X_i)$ where, by Lemma 5·18(iv), for each j with $0 \leq j \leq n-1$, we have $\pi_j(S) \ll_{\mathcal{C}} U_j$, where π_j is the projection. Thus, there exists $A_j \in \cup\text{-}\cap(\mathcal{B}_j)$ such that $\pi_j(S) \subseteq A_j \subseteq U_j$. Then it holds that $S \subseteq (A_0 \times \cdots \times A_{n-1} \times \prod_{i \geq n} X_i) \subseteq (U_0 \times \cdots \times U_{n-1} \times \prod_{i \geq n} X_i)$, and $A_0 \times \cdots \times A_{n-1} \times \prod_{i \geq n} X_i$ is in the closure under finite intersections and unions of the candidate $\ll_{\mathcal{C}}$-pseudosubbase. Thus, by Lemma 7·8, this is indeed a $\ll_{\mathcal{C}}$-pseudosubbase. The countability claim is obvious. $\square$

The proof of Proposition 7·7 requires a sequence of lemmas. The first adapts (and uses) Lemma 4·3(iv),(v) to the $\times_{\mathcal{C}}$ product and the $\ll_{\mathcal{C}}$ relation.

LEMMA 7·9. *Let $\mathcal{C}$ be any collection of exponentiable spaces.*

(i) *If X, Y are $\mathcal{C}$-generated spaces, $A' \ll_{\mathcal{C}} A \subseteq X$ and $B' \ll B \subseteq Y$ then $A' \times B' \ll A \times B$ in $X \times_{\mathcal{C}} Y$.*

(ii) *If X, Y are $\mathcal{C}$-generated spaces, $W \subseteq X \times_{\mathcal{C}} Y$ is open and $S \ll_{\mathcal{C}} T \subseteq X$ then*

$$\{y \in Y \mid \forall x \in T.\, (x, y) \in W\} \subseteq \mathrm{int}\{y \in Y \mid \forall x \in S.\, (x, y) \in W\}.$$

Proof. (i): Suppose $A' \ll_{\mathcal{C}} A \subseteq X$ and $B' \ll B \subseteq Y$. Thus there exist $Q \in \mathcal{C}$, continuous $p\colon Q \to X$ and $W' \ll W \subseteq Q$ with $A' \subseteq p(W')$ and $p(W) \subseteq A$. By Lemma 4·3(iv), we have $W' \times B' \ll W \times B \subseteq Q \times Y$. Then, applying the continuous function

$$Q \times Y = Q \times_{\mathcal{C}} Y \xrightarrow{p \times_{\mathcal{C}} \mathsf{id}} X \times_{\mathcal{C}} Y,$$

we obtain that $p(W') \times B' \ll p(W) \times B$ in $X \times_{\mathcal{C}} Y$. But $A' \times B' \subseteq p(W') \times B'$ and $p(W) \times B \subseteq A \times B$, so indeed $A' \times B' \ll A \times B$ in $X \times_{\mathcal{C}} Y$.

(ii): Suppose $S' \ll_{\mathcal{C}} T \subseteq X$ and $B' \ll B \subseteq Y$. Then here exist $Q \in \mathcal{C}$, continuous $p\colon Q \to X$ and $U \ll V \subseteq Q$ with $S \subseteq p(U)$ and $p(V) \subseteq T$. Applying the continuous function

$$Q \times Y = Q \times_{\mathcal{C}} Y \xrightarrow{p \times_{\mathcal{C}} \mathsf{id}} X \times_{\mathcal{C}} Y,$$

we have, by Lemma 4·3(v),

$$\{y \in Y \mid \forall q \in V.\, (p(q), y) \in W\} \subseteq \mathrm{int}\{y \in Y \mid \forall q \in U.\, (p(q), y) \in W\}.$$

But also

$$\{y \in Y \mid \forall x \in T.\,(x,y) \in W\} \subseteq \{y \in Y \mid \forall q \in V.\,(p(q),y) \in W\},$$

as $p(V) \subseteq T$, and

$$\{y \in Y \mid \forall q \in U.\,(p(q),y) \in W\} \subseteq \{y \in Y \mid \forall x \in S.\,(x,y) \in W\},$$

as $S \subseteq p(U)$, which proves the claim. □

Remark 7·10. For $\mathcal{C}$ closed under binary product, a similar argument shows that if $A' \ll_{\mathcal{C}} A \subseteq X$ and $B' \ll_{\mathcal{C}} B \subseteq Y$ then $A' \times B' \ll_{\mathcal{C}} A \times B$ in $X \times_{\mathcal{C}} Y$.

LEMMA 7·11. *Suppose that X, Y are $\mathcal{C}$-generated spaces, that $\mathcal{A}$ is a countable $\ll_{\mathcal{C}}$-pseudobase for X, that $\mathcal{B}$ is a countable $\ll$-pseudobase for Y, and that $F \ll_{\mathcal{C}} [T,V]_{\mathcal{C}}$ in $X \Rightarrow_{\ll_{\mathcal{C}}} Y$, where $V \subseteq Y$ is open. Then, for any $S \ll_{\mathcal{C}} T \subseteq X$, there exist $A \in \cap(\mathcal{A})$ and $B \in \mathcal{B}$ such that $F \subseteq ((A,B)) \subseteq [S,V]_{\mathcal{C}}$.*

Proof. By Lemma 5·18(iii), there exists G with $F \ll_{\mathcal{C}} G \ll_{\mathcal{C}} [T,V]_{\mathcal{C}}$ in $X \Rightarrow_{\ll_{\mathcal{C}}} Y$. By Lemma 5·18(vi), we also have $F \ll_{\mathcal{C}} G \ll_{\mathcal{C}} [T,V]_{\mathcal{C}}$ in $X \Rightarrow_{\mathcal{C}} Y = \mathcal{C}(X \Rightarrow_{\ll_{\mathcal{C}}} Y)$. By the continuity of evaluation

$$(X \Rightarrow_{\mathcal{C}} Y) \times_{\mathcal{C}} X \to Y, \tag{7·1}$$

the set $\{(f,x) \mid f(x) \in V\}$ is an open subset of $(X \Rightarrow_{\mathcal{C}} Y) \times_{\mathcal{C}} X$. Thus, by Lemma 7·9(ii), we obtain the second inclusion of

$$T \subseteq \{x \mid \forall f \in [T,V]_{\mathcal{C}}.\, f(x) \in V\} \subseteq \mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\},$$

where the first inclusion is immediate from the definition of $[T,V]_{\mathcal{C}}$.

Now take any $S \ll_{\mathcal{C}} T$ in X. By Lemma 5·18(iii), $S \ll_{\mathcal{C}} S' \ll_{\mathcal{C}} T$ for some $S' \subseteq X$. Combining with the above inequalities, we obtain

$$S' \ll_{\mathcal{C}} \mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}. \tag{7·2}$$

Let $\{A_i\}_{i \geq 0}$ be an enumeration of the set

$$\{A \in \mathcal{A} \mid S' \subseteq A \subseteq \mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}\},$$

which is nonempty because $\mathcal{A}$ is a $\ll_{\mathcal{C}}$-pseudobase for X. Let $\{B_i\}_{i \geq 0}$ be an enumeration of the set $\{B \in \mathcal{B} \mid B \subseteq V\}$, which is nonempty because $\mathcal{B}$ is a $\ll$-pseudobase for Y. We claim that there exists $k \geq 0$ with $F \subseteq ((A_1 \cap \cdots \cap A_k, B_k))$.

To establish the claim, suppose, for contradiction, that it fails. Then, for each $i \geq 0$, there exists $x_i \in A_1 \cap \cdots \cap A_i$ and $f_i \in F$ such that $f_i(x_i) \notin B_i$. We first show that

$$S' \cup \{x_i \mid i \geq 0\} \ll \mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}. \tag{7·3}$$

For this, suppose that $\mathcal{U}$ is an open cover of $\mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}$. Then, by (7·2) and Lemma 5·18(i),(iii), there exists finite $\mathcal{U}' \subseteq \mathcal{U}$ with $S' \ll_{\mathcal{C}} \bigcup \mathcal{U}'$. As $\mathcal{A}$ is a $\ll_{\mathcal{C}}$-pseudobase for X, there exists A_i such that $S' \subseteq A_i \subseteq \bigcup \mathcal{U}'$. But, for $j \geq i$, we have $x_j \in A_i$. So $\mathcal{U}'$ is a finite cover of $S' \cup \{x_i \mid i \geq j\}$. Thus, by choosing opens $U_1, \ldots, U_{i-1} \in \mathcal{U}$ with $x_1 \in U_1, \ldots, x_{i-1} \in U_{i-1}$, we have that $\mathcal{U}' \cup \{U_1, \ldots, U_{i-1}\}$ is the required finite subcover of $S' \cup \{x_i \mid i \geq 0\}$. This establishes (7·3).

As $F \ll_{\mathcal{C}} G$ in $X \Rightarrow_{\mathcal{C}} Y$, we have, by (7·3) and Lemma 7·9(i),

$$F \times_{\mathcal{C}} (S' \cup \{x_i \mid i \geq 0\}) \ll G \times_{\mathcal{C}} \mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}$$

in $(X \Rightarrow_{\mathcal{C}} Y) \times_{\mathcal{C}} X$. By the continuity of evaluation (7·1), it follows that

$$F(S' \cup \{x_i \mid i \geq 0\}) \ll G(\mathrm{int}\{x \mid \forall f \in G.\, f(x) \in V\}) \subseteq V$$

in Y. Therefore, there exists $B \in \mathcal{B}$ with $F(S' \cup \{x_i \mid i \geq 0\}) \subseteq B \subseteq V$, because $\mathcal{B}$ is a $\ll$-pseudobase for Y. But then $B = B_i$ for some i, and also $f_i(x_i) \in B_i$, contradicting the choice of f_i and x_i. This contradiction establishes that the claim is indeed true.

We have proved that there exists k with $F \subseteq (\!(A_1 \cap \cdots \cap A_k, B_k)\!)$. It remains to verify that $(\!(A_1 \cap \cdots \cap A_k, B_k)\!) \subseteq [S, V]_{\mathcal{C}}$. But, for $f \in (\!(A_1 \cap \cdots \cap A_k, B_k)\!)$, we have $f(A_1 \cap \cdots \cap A_k) \subseteq B_k \subseteq V$. Therefore,

$$S \ll_{\mathcal{C}} S' \subseteq A_1 \cap \cdots \cap A_k \subseteq f^{-1}(V),$$

i.e. $f \in [S, V]_{\mathcal{C}}$ as required. □

Proof of Proposition 7·7. Suppose that X and Y are $\mathcal{C}$-generated spaces, $\mathcal{A}$ is a countable pseudosubbase for X and $\mathcal{B}$ is a countable pseudosubbase for Y. Thus $\cup\text{-}\cap(\mathcal{A})$ is a $\ll$-pseudobase (hence $\ll_{\mathcal{C}}$-pseudobase) for X, and $\cup\text{-}\cap(\mathcal{B})$ is a $\ll$-pseudobase for Y. We must show that the set

$$\mathcal{F} = \{(\!(A, B)\!) \mid A \in \cap(\mathcal{A}), B \in \cup(\mathcal{B})\}$$

is a pseudosubbase of $X \Rightarrow_{\ll_{\mathcal{C}}} Y$.

Suppose that $F \ll_{\mathcal{C}} [S, V]_{\mathcal{C}}$ in $X \Rightarrow_{\ll_{\mathcal{C}}} Y$, where $V \subseteq Y$ is open. By Lemma 7·8, it suffices to show that there exists $G \in \cup\text{-}\cap(\mathcal{F})$ with $F \subseteq G \subseteq [S, V]_{\mathcal{C}}$.

For every $f \in [S, V]_{\mathcal{C}}$, we have $S \ll_{\mathcal{C}} f^{-1}(V)$, so, by Lemma 5·18(iii), there exists T_f such that $S \ll_{\mathcal{C}} T_f \ll_{\mathcal{C}} f^{-1}(V)$. Thus the family $\{[T_f, V]_{\mathcal{C}}\}_{f \in [S,V]_{\mathcal{C}}}$ is an open cover of $[S, V]_{\mathcal{C}}$. As $F \ll_{\mathcal{C}} [S, V]_{\mathcal{C}}$, there exist, by Lemma 5·18(ii), $F_1 \ll_{\mathcal{C}} [T_1, V]_{\mathcal{C}}, \ldots, F_k \ll_{\mathcal{C}} [T_k, V]_{\mathcal{C}}$, for some $T_1, \ldots, T_k \in \{T_f \mid f \in [S, V]_{\mathcal{C}}\}$, with $F \subseteq F_1 \cup \cdots \cup F_k$.

For each i with $1 \leq i \leq k$, we have $F_i \ll_{\mathcal{C}} [T_i, V]_{\mathcal{C}}$ and $S \ll_{\mathcal{C}} T_i$. So, by Lemma 7·11, there exist $A_i \in \cap(\cup\text{-}\cap(\mathcal{A})) = \cup\text{-}\cap(\mathcal{A})$ and $B_i \in \cup(\cup\text{-}\cap(\mathcal{B})) = \cup\text{-}\cap(\mathcal{B})$ such that $F_i \subseteq (\!(A_i, B_i)\!) \subseteq [S, V]_{\mathcal{C}}$. Thus

$$F \subseteq F_1 \cup \cdots \cup F_k \subseteq (\!(A_1, B_1)\!) \cup \cdots \cup (\!(A_k, B_k)\!) \subseteq [S, V]_{\mathcal{C}}.$$

Define $G = (\!(A_1, B_1)\!) \cup \cdots \cup (\!(A_k, B_k)\!)$. It remains only to verify that $G \in \cup\text{-}\cap(\mathcal{F})$.

We show that each $(\!(A_i, B_i)\!) \in \cap(\mathcal{F})$. For this, we have $A_i = A_{i1} \cup \cdots \cup A_{ik_i}$, with each $A_{ij} \in \cap(\mathcal{A})$, and $B_i = B_{i1} \cap \cdots \cap B_{il_i}$, with each $B_{ij} \in \cup(\mathcal{B})$. Thus

$$(\!(A_i, B_i)\!) = \bigcap_{1 \leq j \leq k_i} \bigcap_{1 \leq j' \leq l_i} (\!(A_{ij}, B_{ij'})\!),$$

where each $(\!(A_{ij}, B_{ij'})\!) \in \mathcal{F}$. So indeed $(\!(A_i, B_i)\!) \in \cap(\mathcal{F})$. □

Remark 7·12. The results of this section show that, whenever QCB is contained in $\mathrm{Top}_{\mathcal{C}}$, then the inclusion preserves countable limits and function spaces. This situation is not mimicked by the inclusion QCB → Top. Given the results of the present section, Example 5·1 shows that the latter inclusion fails to preserve finite products, and Example 5·8 shows that it does not preserve exponentials between spaces whose exponential exists in Top.

Remark 7·13. In the case of two countably based spaces X, Y, the exponential $[X \Rightarrow_{\mathcal{C}} Y]$ (for productive $\mathcal{C}$ satisfying any of the conditions of Proposition 7·1) is given by the natural topology $[X \Rightarrow_{\natural} Y]$. This is established by the following chain of reasoning. By the results of the present section, $[X \Rightarrow_{\mathcal{C}} Y]$ coincides with the exponential in QCB. By [**24**, Theorems 2 & 3], the latter exponential is obtained as the topology associated (via quotient) to the exponential in Scott's category of equilogical spaces [**3**]. Finally, by [**12**, Appendix A], this associated topology is indeed the natural topology.

Remark 7·14. In [**26**, §3.3] a proof is given that, for the "type hierarchy" of iterated exponentials over the discrete natural numbers, the sequential topology coincides with the compactly generated topology. This theorem, which is originally due to Hyland, is an immediate consequence of the structure preservation results of the present section. Furthermore, our results establish analogous coincidences for iterated exponentials over arbitrary QCB spaces, and with respect to taking $\mathcal{C}$-generated topologies for any productive $\mathcal{C}$ satisfying the conditions of Proposition 7·1.

8. *Compact generation, a dual Hofmann–Mislove Theorem, and Lawson duality*

The compact saturated subsets of a topological space form a semilattice with respect to the operation of union. In this section we investigate the following problem: can one use this semilattice of subsets to determine internally the core compactly generated topology, as opposed to the external determination via probes?

There are two additional related questions of a more technical nature that we consider. Recall that the Hofmann–Mislove Theorem asserts that a Scott-open filter in the lattice of open sets of a sober space is the open neighbourhood filter of its intersection, which is a compact saturated set (see, for example, [**15**, Theorem II-1.21]). We consider a dual problem: under what conditions is it true that a Scott-open filter in the semilattice of compact saturated sets (*ordered by reverse inclusion*) is the filter of compact saturated subsets of its union, which is an open set? One may view this question as providing an alternative approach to the theory of compactly generated spaces, since in the presence of an affirmative answer the topology of open sets is determined by the Scott-open filters of the semilattice of compact saturated sets. Finally we consider the problem of under what conditions the lattice of open sets of a topological space and its semilattice of compact saturated sets stand in Lawson duality to each other.

We shall have need of the following useful lemma. Recall that a family of subsets is *filtered* if it forms a filter base.

Lemma 8·1. *Let $f\colon X \to Y$ be continuous, where X is a sober space. If L is the intersection of a filtered family $\mathcal{K}$ of compact saturated subsets of X, then $\bigcap\{\uparrow f(K) \mid K \in \mathcal{K}\} = \uparrow f(L)$.*

Proof. Clearly $\uparrow f(L) \subseteq \bigcap\{\uparrow f(K) \mid K \in \mathcal{K}\}$. Conversely suppose that $y \notin \uparrow f(L)$. Then $\downarrow y \cap f(L) = \overline{\{y\}} \cap f(L) = \emptyset$. Thus L misses the closed set $f^{-1}(\downarrow y)$. By the machinery of the Hofmann–Mislove Theorem (Theorem II-1.21 of [**15**]) there exists $K \in \mathcal{K}$ such that $K \subseteq X \setminus f^{-1}(\downarrow y)$ and thus $y \notin \uparrow f(K)$. $\square$

Lemma 8·2. *Let X be a sober space. Then the topology determined by the probes from locally compact sober spaces agrees with the core compactly generated topology.*

Proof. Let $p\colon C \to X$ be a probe from a core compactly generated space C. Since the space X is sober, the map p factors through the sobrification C^s of C, which is a locally compact space (see Exercise V-4.9 and Proposition V-5.10 of [**15**]). Let $p^s\colon C^s \to X$ be the corresponding probe from the sobrification. Then for $W \subseteq X$, $p^{-1}(W)$ if open in C

if and only if $(p^s)^{-1}(W)$ is open in C^s (Proposition V-5.10 of [**15**] again). It thus follows that the core compactly generated topology is determined by all probes from locally compact sober spaces into X. $\square$

The next lemma establishes a connection between core compactly generated spaces and filters of compact saturated sets.

LEMMA 8·3. *If X is a sober space and $\mathcal{K}$ is a Scott-open filter of compact saturated sets, then the set $V = \bigcup \mathcal{K}$ is open in the core compactly generated topology.*

Proof. We must show for a probe $p\colon C \to X$ that $p^{-1}(V)$ is open. By the preceding lemma we may restrict our attention to probes $p\colon C \to X$ where C is locally compact sober.

Let $\mathcal{F}$ be the collection of all compact saturated subsets F of C such that $\uparrow p(F) \in \mathcal{K}$. We claim that $\mathcal{F}$ is a Scott-open filter of compact saturated subsets. Since $\mathcal{K}$ is a filter, it follows directly that $\mathcal{F}$ is also a filter. Suppose that $\mathcal{M}$ is a filtered family of compact saturated subsets of C such that $L = \bigcap M \in \mathcal{F}$. By Lemma 8·1 we have $\uparrow p(L) = \bigcap\{\uparrow p(M) \mid M \in \mathcal{M}\}$. Since $\mathcal{K}$ is Scott-open, there exists $M \in \mathcal{M}$ such that $\uparrow p(M) \in \mathcal{K}$, i.e., $M \in \mathcal{F}$. We conclude that $\mathcal{F}$ is Scott-open.

By the so-called Lawson duality for locally compact sober spaces, we have that the set $U = \bigcup \mathcal{F}$ is open in C [**15**, Theorem IV-2.18]. From the definition of U and V, we have $U \subseteq p^{-1}(V)$. Conversely if $x \in p^{-1}(V)$, then $p(x) \in V = \bigcup \mathcal{K}$. Thus $p(x) \in K$ for some $K \in \mathcal{K}$. Since K is saturated, we have $\uparrow p(x) \subseteq K$, and since $\mathcal{K}$ is a filter we conclude that $\uparrow p(x) \in \mathcal{K}$. Observing that $\uparrow p(x) = \uparrow p(\uparrow x)$, we conclude that $\uparrow x \in \mathcal{F}$. It follows that $x \in U$. Thus $p^{-1}(V) = U$ is open in C. It follows that V is open in the core compactly generated topology. $\square$

We consider the converse direction.

LEMMA 8·4. *Let X be sober and let U be a saturated subset. Then the compact saturated subsets of B form a Scott-open filter with union U if and only if for any filtered family $\mathcal{K}$ of compact saturated sets with intersection contained in U, then some member of $\mathcal{K}$ is contained in U.*

Proof. That the first condition implies the second is immediate from the definition of the filter being Scott-open. Conversely it is also immediate that the compact saturated subsets contained in U form a filter (under reverse inclusion). Since point saturates belong to this family for all points in U, the union is U. The Scott-openness of the filter follows directly from the last condition. $\square$

LEMMA 8·5. *Let X be a sober space.*

(i) *The compact saturated sets contained in a given open set U form a Scott-open filter in the semilattice of compact saturated sets (ordered by reverse inclusion).*

(ii) *Suppose that each compact saturated subset of X is contained in a saturated set for which the relative topology is core compactly generated. Then the compact saturated sets contained in a core compactly generated open set form a Scott-open filter of compact saturated sets.*

Proof. (i) This follows immediately from the equivalence of the preceding lemma and the Hofmann–Mislove machinery for sober spaces [**15**, Theorem II-1.21].

(ii) Let U be open in the core compactly generated topology. Let $\mathcal{K}$ be a descending family of compact saturated sets with intersection $E \subseteq U$. By the preceding lemma, we need only check that $K \subseteq U$ for some $K \in \mathcal{K}$. Pick $K_0 \in \mathcal{K}$ and $A \supseteq K_0$ such that the relative topology on A is core compactly generated. The relative topology on A is sober since A is saturated [**15**, Exercise O-5.16]. We apply part (i) of this lemma to the space A, the open set $U \cap A$, which is a core compactly generated open subset and hence an open subset of A, and the collection $\{K \in \mathcal{K} : K \subseteq K_0\}$ and conclude that $K \subseteq U \cap A \subseteq U$ for some $K \in \mathcal{K}$. Thus the filter in question is Scott-open. □

Lemmas 8·3 and 8·5 yield the following result, which in turn provides an answer for the question beginning this section.

Theorem 8·6. *Let X be a sober space, and suppose that each compact saturated subset of X is contained in a saturated set for which the relative topology is core compactly generated. Then a subset of X is open in the core compactly generated topology if and only if it can be written as a union of a Scott-open filter of compact saturated sets.*

It follows from the preceding results that for a core compactly generated sober space X, the map that assigns to a Scott-open filter of compact saturated sets its union is a surjection from the set of all such Scott-open filters onto the lattice of open sets of X. What is not clear is whether this map is injective. We consider sufficient conditions for this to be the case. In this case one obtains a dual Hofmann–Mislove theorem.

Theorem 8·7. *Let X be a sober space that is core compactly generated (and hence satisfies the hypotheses of the preceding theorem). Suppose that for each compact subset K of X and each open set U containing K, there exists a locally compact space C, a continuous function $p\colon C \to X$, and a compact subset A of C such that $K \subseteq {\uparrow}p(A) \subseteq U$. Then every Scott-open filter in the semilattice of compact saturated sets (ordered by reverse inclusion) is the filter of all compact saturated subsets of its union, which is an open set.*

Proof. Let $\mathcal{K}$ be a Scott-open filter of compact saturated subsets of X. In view of the remarks before the proposition and the earlier results, it remains to show that $\mathcal{K}$ consists of all compact saturated subsets of the open set $U = \bigcup \mathcal{K}$. Thus let L be a compact saturated subset that is contained in U. By hypothesis there exists a locally compact space C, a continuous map $p\colon C \to X$, and a compact subset A of C such that $L \subseteq {\uparrow}p(A) \subseteq U$.

For each $y \in p^{-1}(U)$, we have $p(y) \in U$ and hence $p(y) \in K$ for some $K \in \mathcal{K}$. Thus ${\uparrow}p(y) \subseteq K$, and hence ${\uparrow}p(y) \in \mathcal{K}$, since the latter is a filter. It follows from local compactness that ${\uparrow}y$ is the filtered intersection of all compact saturated neighbourhoods of y. By Lemma 8·1 we conclude that

$$
{\uparrow}p(y) = {\uparrow}p({\uparrow}y) = \bigcap\{{\uparrow}p(B) \mid B \text{ is a compact saturated neighbourhood of } y\}.
$$

Since $\mathcal{K}$ is Scott-open, it follows that ${\uparrow}p(B) \in \mathcal{K}$ for some compact neighbourhood B of y. Thus for every $y \in A$, we can choose a compact neighbourhood B_y of y such that ${\uparrow}p(B_y) \in \mathcal{K}$. Finitely many of these cover A, and hence the finite union of the saturates of their images contains L. It follows that $L \in \mathcal{K}$. □

Corollary 8·8. *Let X be a compactly generated Hausdorff space, or more generally a core compactly generated sober space in which every compact subset is contained in a locally compact subspace. Then every Scott-open filter in the semilattice of compact* *saturated sets (ordered by reverse inclusion) is the filter of all compact saturated subsets of its union, which is an open set.*

Proof. A Hausdorff space is always sober, each compact subset is Hausdorff, hence locally compact, and compactly generated implies core compactly generated. Thus the first set of hypotheses are a special case of the second. The second case follows immediately from the preceding theorem. □

DEFINITION 8·9 (Co-sober space). A compact saturated subspace A of a topological space X is *k-irreducible* if it cannot be written as the union of two proper compact saturated subsets. We call a space *co-sober* if ever k-irreducible compact saturated set is of the saturate of a point.

(This property is loosely connected to the notion of sobriety for the cocompact topology, but we do not pursue this topic here.) The next theorem gives alternative conditions for the dual Hofmann–Mislove theorem to hold.

THEOREM 8·10. *Let X be a sober and co-sober space that is core compactly generated. Then every Scott-open filter in the semilattice of compact saturated sets (ordered by reverse inclusion) is the filter of all compact saturated subsets of its union, which is an open set.*

Proof. As in the proof of Theorem 8·7 we need only show that if $\mathcal{K}$ be a Scott-open filter of compact saturated subsets of X, then $\mathcal{K}$ consists of all compact saturated subsets of the open set $U = \bigcup \mathcal{K}$.

Suppose that there exists some compact saturated set $K \subseteq U$ that is not in $\mathcal{K}$. We consider a maximal chain of compact saturated subsets such that K is a member of the chain and each member of the chain is contained in U but does not belong to $\mathcal{K}$. By the machinery of the Hofmann–Mislove Theorem, it follows that the intersection A is a nonempty compact saturated set, and A does not belong to $\mathcal{K}$ since the latter is Scott-open.

If we could write A as the union of two strictly smaller compact saturated sets, then each of these would belong to $\mathcal{K}$ (by maximality of the chain), and hence A would belong to $\mathcal{K}$ (by the filter property). We conclude that A is k-irreducible. By co-sobriety $A = {\uparrow}x$ for some $x \in U$. But then x is in some member of $\mathcal{K}$, and thus ${\uparrow}x = A \in \mathcal{K}$ since the latter is a filter. This contradiction establishes the proof. □

We recall the basics of Lawson duality (see Section IV-2 of [**15**]). Let P be a directed complete partially ordered set. We define the *Lawson dual* P' to be the collection of Scott-open filters of P ordered by inclusion, which is again a dcpo. If the natural embedding of P into P'' is an order isomorphism, then we say that P and P' are Lawson duals.

Suppose that X is a sober space and $O(X)$ is the lattice of open sets. By the Hofmann-Mislove Theorem the Scott-open filters on $O(X)$ may be identified with $K(X)$, the compact saturated subsets of X ordered by reverse inclusion, by associating to each Scott-open filter its intersection. With respect to this identification, the mapping from $O(X)$ to the double dual sends an open set U to the Scott-open filter of all compact saturated subsets of U. Thus duality holds if and only if every Scott-open filter in $K(X)$ has union an open subset of X and consists of all compact saturated subsets contained in this union, or, more briefly said, $O(X)$ is the Lawson dual of $K(X)$. This is always the case for locally compact sober spaces, as worked out in Section IV-2 of [**15**]. This was further extended in [**16**], but the results there are rather technical. The preceding results of this section give as an immediate corollary a further generalization that is much less technical to derive.

COROLLARY 8·11. *Let X be a sober core compactly generated space. Then $O(X)$ and $K(X)$ are Lawson duals (via the standard constructions of associating compact saturated sets and open sets with Scott-open filters) if either of the following additional conditions are satisfied:*

(i) *X is co-sober;*
(ii) *for each compact subset K of X and each open set U containing K, there exists a locally compact space C, a continuous function $p : C \to X$, and a compact subset A of C such that $K \subseteq \uparrow p(A) \subseteq U$.*

We consider the converse problem: Does Lawson duality imply core-compact generation? We derive a condition that additionally provides an alternative internal construction of the core compactly generated topology.

THEOREM 8·12. *Let X be a sober space with the following property: for every compact saturated subset K of X there exists a probe $p : C \to X$, where C is locally compact and sober, such that for any compact saturated subset L of K, we have $p^{-1}(L)$ is compact and $L = \uparrow p(p^{-1}(L))$. Then the core compact open sets are precisely those sets that can be obtained as the union of a Scott-open filter of compact saturated sets. Hence, if additionally $K(X)$ and $O(X)$ are Lawson duals, then X itself is core compactly generated.*

Proof. Lemma 8·3 yields the implication in one direction. Conversely let W be a set that is open in the core compactly generated topology. The collection of compact saturated subsets contained in W is easily seen to be a filter (under reverse inclusion). To show it is Scott-open, we consider a filtered collection $\mathcal{F}$ of compact saturated sets such that $\bigcap \mathcal{F} \subseteq W$. Pick $K \in \mathcal{F}$ and pick a probe p as in the hypothesis. We may without loss of generality restrict the filtered collection $\mathcal{F}$ to those contained in K. Then $\{p^{-1}(L) \mid L \in \mathcal{F}\}$ is a filtered collection of compact saturated sets in C. Since C is sober, $p^{-1}(W)$ is open in C, and $\bigcap\{p^{-1}(L) \mid L \in \mathcal{F}\} = p^{-1}(\bigcap \mathcal{F}) \subseteq p^{-1}(W)$, we conclude that $p^{-1}(L) \subseteq p^{-1}(W)$ for some $L \in \mathcal{F}$. It follows that $L \subseteq \uparrow p(p^{-1}(L)) \subseteq p(p^{-1}(W)) \subseteq W$.

If $K(X)$ and $O(X)$ are Lawson duals, then the union of any Scott-open filter of compact saturated sets must be an open set. But we have just seen that such unions yield all core compactly generated open sets. Hence the two topologies agree. □

COROLLARY 8·13. *A Hausdorff space X is compactly generated if and only if $K(X)$ and $O(X)$ are in Lawson duality.*

Proof. A Hausdorff space is sober and easily seen to satisfy the condition of the previous theorem. Hence it is core compactly generated if $K(X)$ and $O(X)$ are Lawson duals. The converse follows from Corollary 8·11 since condition (ii) of that corollary is satisfied in a Hausdorff space. □

9. *Open problems*

We have previously mentioned Isbell's example [**20**] of a space that is locally compact but not generated by the locally compact Hausdorff spaces.

Problem 9·1. Is there an example of a core compactly generated space that is not generated by the locally compact spaces?

It is known a Hausdorff space has the same compact sets as its compactly generated coreflection.

Problem 9·2. When do X and $\mathcal{C}X$ have the same compact sets?

We have seen that for any two $\mathcal{C}$-generated (and hence $\mathcal{E}$-generated) spaces X and Y, the products $X \times_{\mathcal{C}} Y$ and $X \times_{\mathcal{E}} Y$ coincide, where $\mathcal{E}$ is the class of all exponentiable spaces. Thus, there is a single construction that produces the products of all categories of $\mathcal{C}$-generated spaces.

Problem 9·3. Is there an intrinsic characterization of the $\mathcal{C}$-product?

In Example 5·7, it is shown that the inclusion of sequential spaces in core compactly generated spaces does not preserve uncountable products. On the other hand, Theorem 7·2 shows that the inclusion of QCB in core compactly generated spaces does preserve countable limits.

Problem 9·4. Does the inclusion of sequential spaces in core compactly generated spaces preserve countable products? Does it preserve equalizers?

The last three problems relate to the concerns of Section 8.

Problem 9·5. Is the core compactly generated topology of a sober space again sober? Dually, is the sobrification of a core compactly generated space again core compactly generated?

Problem 9·6. In a sober space X, is every core compactly generated open set the union of a Scott-open filter of compact saturated sets?

Problem 9·7. Is a sober space co-sober?

REFERENCES

[**1**] M. Barr, *Building closed categories*, Cahiers Topologie Géom. Différentielle **19** (1978), no. 2, 115–129.

[**2**] A. Bauer, *A relationship between equilogical spaces and type two effectivity*, MLQ Math. Log. Q. **48** (2002), no. suppl. 1, 1–15, Dagstuhl Seminar on Computability and Complexity in Analysis, 2001.

[**3**] A. Bauer, L. Birkedal, and D.S. Scott, *Equilogical spaces*, Theoret. Comput. Sci. (to appear).

[**4**] F. Borceux, *Handbook of categorical algebra. 2*, Cambridge University Press, Cambridge, 1994, Categories and structures.

[**5**] R. Brown, *Some problems of algebraic topology: A study of function spaces, function complexes and fd-complexes*, Ph.D. thesis, New College, Oxford University, December 1961.

[**6**] R. Brown, *Function spaces and product topologies*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 238–250.

[**7**] ———, *Topology*, second ed., Ellis Horwood Ltd., Chichester, 1988, A geometric account of general topology, homotopy types and the fundamental groupoid.

[**8**] B. J. Day, *A reflection theorem for closed categories*, J. Pure Appl. Algebra **2** (1972), no. 1, 1–11.

[**9**] B. J. Day and G. M. Kelly, *On topological quotient maps preserved by pullbacks or products*, Proc. Cambridge Philos. Soc. **67** (1970), 553–558.

[**10**] S. Dolecki, G.H. Greco, and A. Lechicki, *When do the upper kuratowski topology (homeomorphically, scott topology) and the co-compact topology coincide?*, Trans. Amer. Math. Soc. **347** (1995), 2869–2884.

[**11**] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[**12**] M.H. Escardó, *Topology of data types and computability concepts*, Electronic Notes in Theoretical Computer Science (submitted).
[**13**] M.H. Escardó and R. Heckmann, *Topologies on spaces of continuous functions*, Topology Proceedings **26** (2001–2002), no. 2, 545–564.
[**14**] D. Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc. **1** (1950), 303–308.
[**15**] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott, *Continuous lattices and domains*, Cambridge Press, 2003.
[**16**] K.H. Hofmann and Jimmie D. Lawson, *On the order-theoretical foundation of a theory of quasicompactly generated spaces without separation axiom*, J. Austral. Math. Soc. Ser. A **36** (1984), no. 2, 194–212. MR 85g:06002
[**17**] J.M.E Hyland, *Filter spaces and continuous functionals*, Annals of Mathematical Logic **16** (1977), 101–143.
[**18**] J. Isbell, *Function spaces and adjoints*, Mathematica Scandinavica **36** (1975), 317–339.
[**19**] ———, *General function spaces, products and continuous lattices*, Mathematical Proceedings of the Cambridge Philosophical Society **100** (1986), 193–205.
[**20**] ———, *A distinguishing example in k-spaces*, Proc. Amer. Math. Soc. **100** (1987), 593–594.
[**21**] J.L. Kelley, *General topology*, D. van Nostrand, New York, 1955.
[**22**] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971.
[**23**] G. Markowsky, *Chain-complete posets and directed sets with applications*, Algebra Universalis **6** (1976), 53–68.
[**24**] M. Menni and A. Simpson, *Topological and limit-space subcategories of countably-based equilogical spaces*, Math. Structures Comput. Sci. **12** (2002), no. 6, 739–770.
[**25**] E. Michael, *$\aleph_0$-spaces*, J. Math. Mech. **15** (1966), 983–1002.
[**26**] D. Normann, *Recursion on the countable functionals*, Springer LNM 811, 1980.
[**27**] M. Schröder, *Admissible representations for continuous computations*, Ph.D. thesis, Universität Hagen, 2002.
[**28**] ———, *Extended admissibility*, Theoret. Comput. Sci. **284** (2002), no. 2, 519–538.
[**29**] F. Schwarz, *Powers and exponential objects in initially structured categories and applications to categories of limit spaces*, Quaest. Math. **6** (1983), 227–254.
[**30**] F. Schwarz and S. Weck, *Scott topology, Isbell topology and continuous convergence*, Continuous lattices and their applications (Bremen 1982) (New York) (R.-E. Hoffmann and K.H. Hofmann, eds.), Lecture notes in pure and applied mathematics, vol. 101, Marcel Dekker, 1985, pp. 251–273.
[**31**] A. Simpson, *Towards a convenient category of topological domains*, Proceedings of 13th ALGI Workshop, RIMS, Kyoto University, 2003.
[**32**] E. Spanier, *Infinite symmetric products, function spaces, and duality*, Ann. of Math. (2) **69** (1959), 142–198[erratum, 733].
[**33**] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.
[**34**] R.M. Vogt, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. (Basel) **22** (1971), 545–555.
[**35**] S. Weingram, *Bundles, realizations and k-spaces*, Ph.D. thesis, Princeton University, 1961.