Monoidal Categories, Higher Categories

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Midlands Graduate School in the Foundations of Computing Science
University of Birmingham
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Examples will be drawn from sets, relations, and Hilbert spaces, giving insight into applications to classical, nondeterministic, and quantum computation.

Part I

Monoidal categories

I. Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
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Monoidal category theory adds the idea of *parallelism*:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of P and Q to construct a proof of the conjunction (P and Q).

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- What should the relationship be between $A \otimes B$ and $B \otimes A$?

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- and a family of *right unitor* natural isomorphisms

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This data must satisfy the *triangle* and *pentagon* equations, for all objects A, B, C and D:

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

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Exercise. Use the triangle and pentagon equations to prove $\lambda_I = \rho_I$.

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Other tensor products exist, but this one plays a canonical role in our interpretation of classical reality.

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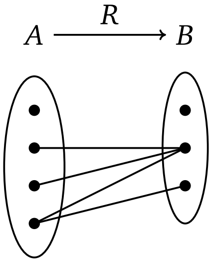

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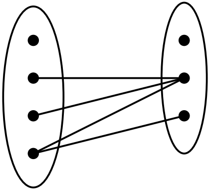

(1)

This is nondeterministic, because an element of A can be related to more than one element of B, or to none.

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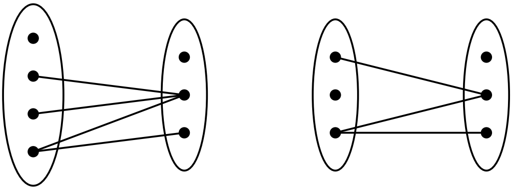
Suppose we have a pair of head-to-tail relations:

$$A \xrightarrow{R} B \qquad\qquad B \xrightarrow{S} C$$



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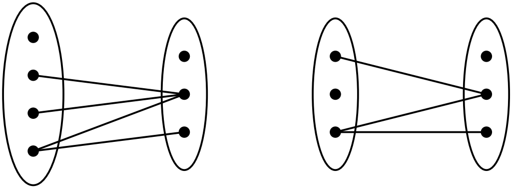
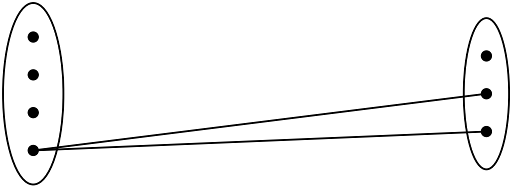


Then our interpretation gives a natural notion of composition:

$$A \xrightarrow{S \circ R} C$$



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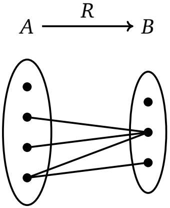
We can write relations as (0,1)-valued matrices:



$$\leftrightsquigarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Composition of relations is then ordinary matrix multiplication, with logical disjunction (OR) and conjunction (AND) for $+$ and $\times$.

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Define the category **FRel** to be the restriction of **Rel** to finite sets.

While **Set** is a setting for classical physics, and **Hilb** is a setting for quantum physics, **Rel** is somewhere in the middle.

It seems like **Rel** should be a lot like **Set**, but we will discover it behaves a lot more like **Hilb**.

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The Cartesian product is *not* a categorical product in **Rel**, so although this monoidal structure looks like that of **Set**, it is more similar to the structure on **Hilb**.

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Theorem 9. *Any morphisms* $A \xrightarrow{f} B$*,* $B \xrightarrow{g} C$*,* $D \xrightarrow{h} E$ *and* $E \xrightarrow{j} F$ *in a monoidal category satisfy the interchange law:*

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Proof. This holds because of properties of the category $\mathbf{C} \times \mathbf{C}$, and from the fact that $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor:

$$\begin{aligned}
(g \circ f) \otimes (j \circ h) &\equiv \otimes(g \circ f, j \circ h) \\
&= \otimes\big((g,j) \circ (f,h)\big) && \text{(composition in } \mathbf{C} \times \mathbf{C}\text{)} \\
&= \big(\otimes(g,j)\big) \circ \big(\otimes(f,h)\big) && \text{(functoriality of } \otimes\text{)} \\
&= (g \otimes j) \circ (f \otimes h)
\end{aligned}$$

Remember the functoriality property: $F(g \circ f) = F(g) \circ F(f)$.

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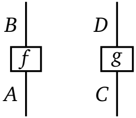
For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ like this:



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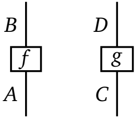

The idea is that f and g represent distinct processes taking place at the same time.

Inputs are drawn at the bottom, and outputs are drawn at the top; in this sense, "time" runs upwards.

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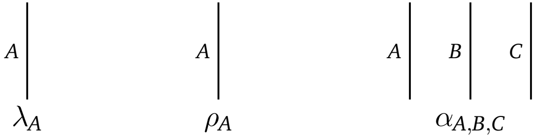

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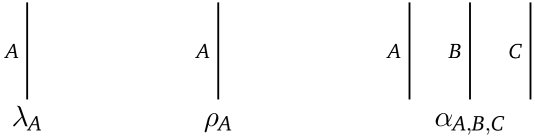

The coherence of α, λ and ρ is essential for the graphical calculus to function. Since there can only be a single morphism built from their components of any given type, it *doesn't matter* that their graphical calculus encodes no information.

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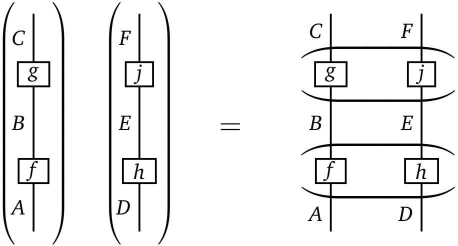

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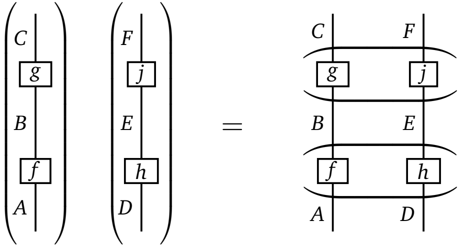

Graphically it's trivial.

The apparent complexity of the theory of monoidal categories—α, λ, ρ, coherence, interchange—was in fact complexity of the *geometry of the plane*. So when we use a geometrical notation, the complexity vanishes.

I. Monoidal categories

Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
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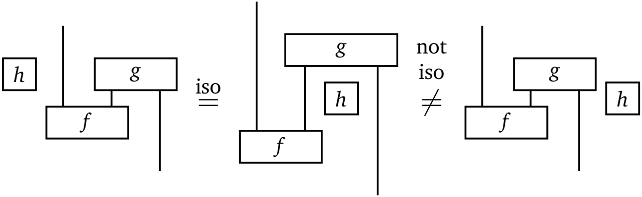

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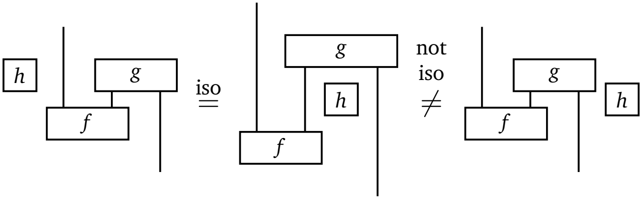

We will allow heights of the diagrams to change, and allow input and output wires to slide horizontally along the boundary, although they must never change order.

I. Monoidal categories

We can now state the correctness theorem.

Theorem 10. *A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.*

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Let f and g be morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

- $P(f, g)$ = 'under the axioms of a monoidal category, $f = g$'
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Completeness is the reverse assertion, that for all such f and g, $Q(f, g) \Rightarrow P(f, g)$. It is hard to prove; one must show that planar isotopy is generated by a finite set of moves, each being implied by the monoidal axioms.

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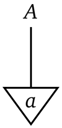
We draw a state $I \xrightarrow{a} A$ like this:



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Example 12. Let's examine the states in our example categories.

- In **Hilb**, states of a Hilbert space H are linear functions $\mathbb{C} \to H$, which correspond to *elements* of H by considering the image of $1 \in \mathbb{C}$.

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- In **Rel**, states of a set A are relations $\{\bullet\} \xrightarrow{R} A$, which correspond to *subsets* by considering all elements related to $\bullet$.

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The dual notion of state is effect.

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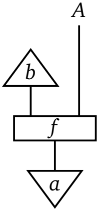
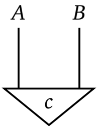
We can use states, effects and other morphisms to build up interesting diagrams, which give 'histories' for a family of systems:



We can interpret an effect as a *property observation* of a system. Overall this composite gives a state of A.

I. Monoidal categories

A morphism $I \xrightarrow{c} A \otimes B$ is a *joint state* of A and B. We depict it graphically in the following way.



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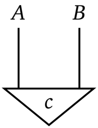
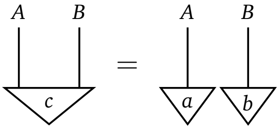


Definition 14. A joint state $I \xrightarrow{c} A \otimes B$ is a *product state* when it is of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:



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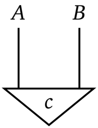

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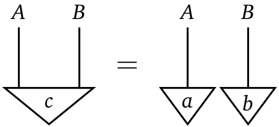

Definition 15. A joint state is *entangled* when it is not a product state.

I. Monoidal categories

Example 16. Let's investigate joint states, product states, and entangled states in our example categories.

- In **Hilb**:
  - **joint states** of H and K are elements of $H \otimes K$;
  - **product states** are factorizable states;
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- In **Rel**:
  - **joint states** of A and B are subsets of $A \times B$;
  - **product states** are subsets $U \subseteq A \times B$ such that, for some $V \subseteq A$ and $W \subseteq B$, $(v, w) \in U$ if and only if $v \in V$, $w \in W$;
  - **entangled states** are subsets that aren't of this form.

I. Monoidal categories

In many theories, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to *braided* and *symmetric* monoidal categories.

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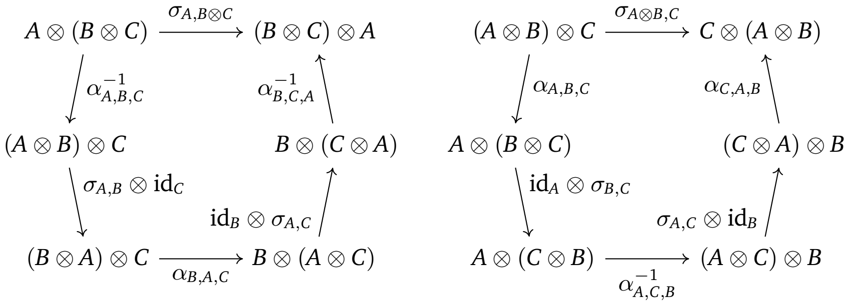
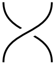


I. Monoidal categories

We include the braiding in our graphical notation like this:



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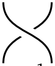

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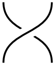

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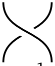

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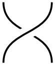

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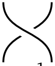

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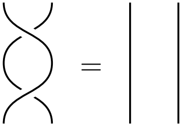
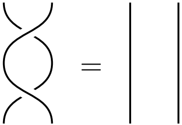
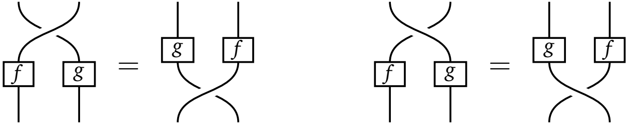
Invertibility takes the following graphical form:





I. Monoidal categories

Naturality has the following graphical representation:



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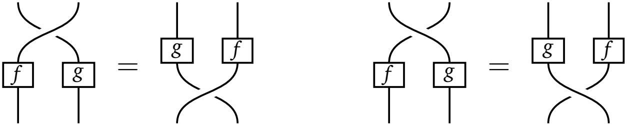
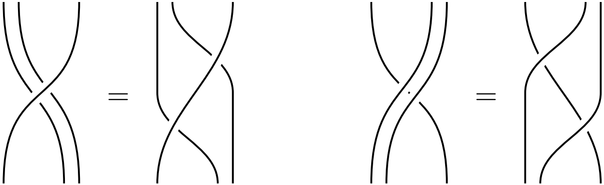


The hexagon equations look like this:



So braiding with a tensor product of two objects is the same as braiding with one then the other separately.

I. Monoidal categories

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

Theorem 18. *A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.*

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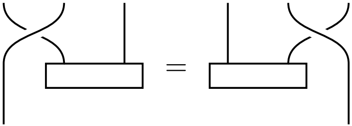
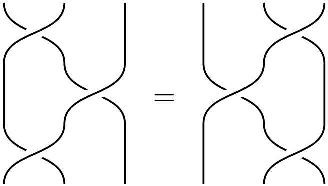
The coherence theorem is very powerful. For example, the following equations hold:





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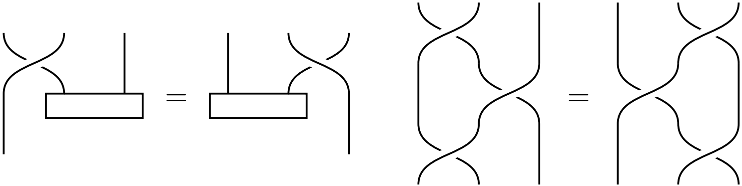

The second equation is called the *Yang–Baxter equation*, which plays an important role in the mathematical theory of knots.

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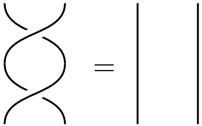

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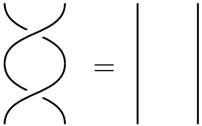

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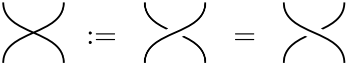
Lemma 21. In a symmetric monoidal category $\sigma_{A,B} = \sigma_{B,A}^{-1}$, with the following graphical representation:



Part II

Higher categories

II. Higher categories

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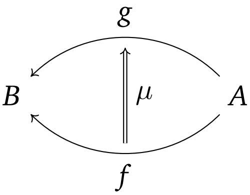
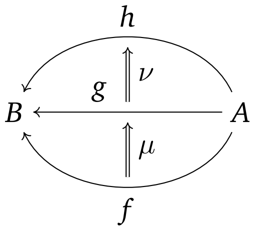
- a collection $\mathrm{Ob}(\mathbf{C})$ of *objects*;
- for any two objects A, B, a category $\mathbf{C}(A, B)$, with objects called *1-morphisms* drawn as $A \xrightarrow{f} B$, and morphisms μ called *2-morphisms* drawn as $f \overset{\mu}{\Longrightarrow} g$, or in full form as follows:



II. Higher categories

- for 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $g \overset{\nu}{\Longrightarrow} h$, an operation called *vertical composition* given by their composite as morphisms in $\mathbf{C}(A, B)$:



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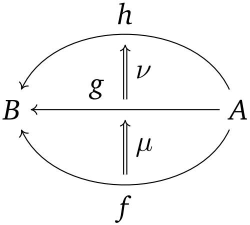
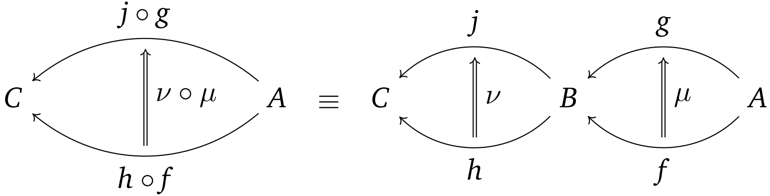


- for any triple of objects A, B, C a *horizontal composition* functor:

$$\circ : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \to \mathbf{C}(A,C)$$



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A 2-category is *strict* just when every λ_f, ρ_f, $\alpha_{h,g,f}$ is an identity.

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The transformations α, λ and ρ are the same for both structures.

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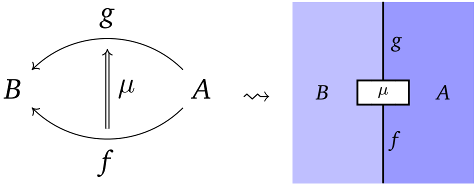
In this more general graphical calculus, objects are represented by regions, 1-morphisms by vertically-oriented lines, and 2-morphisms by vertices:



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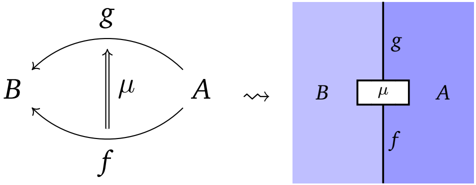

The graphical calculus is the *dual* of the pasting diagram notation.

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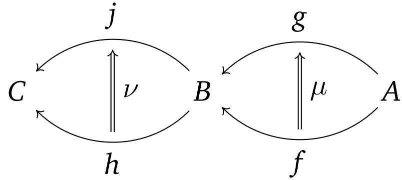
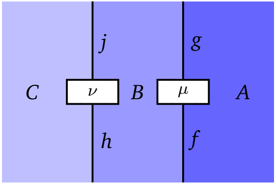
Horizontal and vertical composition is represented like this:



$\rightsquigarrow$



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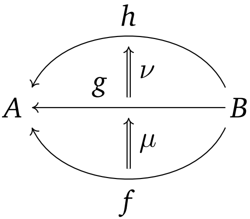

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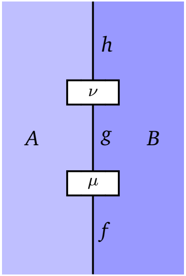

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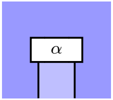
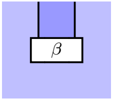
If we have only a single object *A*, which we may as well denote by a region coloured white, then the graphical calculus is identical to that of a monoidal category.

II. Higher categories

We can use the graphical calculus to define equivalence.

Definition. In a 2-category, an *equivalence* is a pair of 1-morphisms $A \xrightarrow{F} B$ and $B \xrightarrow{G} A$, and 2-morphisms $G \circ F \overset{\alpha}{\Longrightarrow} \mathrm{id}_A$ and $\mathrm{id}_B \overset{\beta}{\Longrightarrow} F \circ G$:





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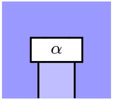
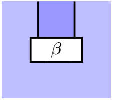
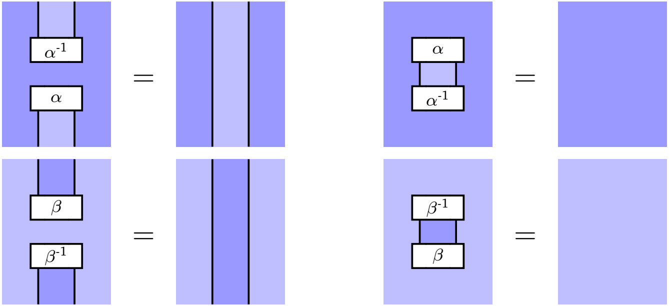




They must satisfy the following equations:



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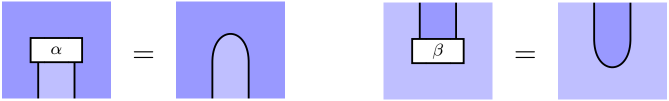
Definition. In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a *right dual* $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \xRightarrow{\alpha} \mathrm{id}_A$ and $\mathrm{id}_B \xRightarrow{\beta} F \circ G$



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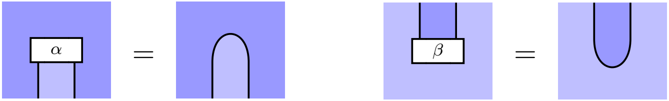

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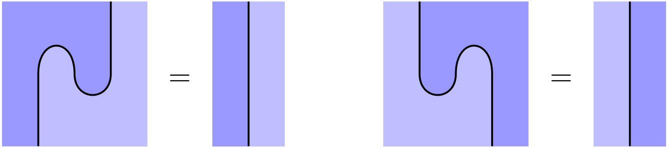

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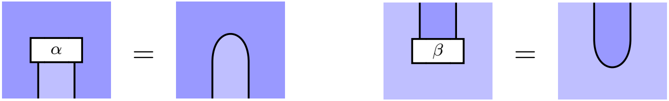

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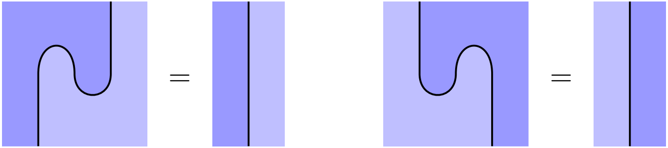

Theorem. In **Cat**, a duality $F \dashv G$ is exactly an adjunction $F \dashv G$ between F and G as functors.

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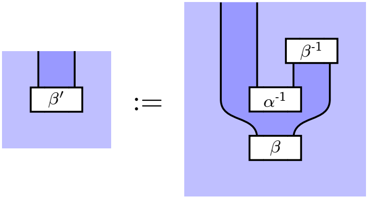
Proof. Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms α and β. Then we will build a new equivalence witnessed by α and β', with β' defined like this:



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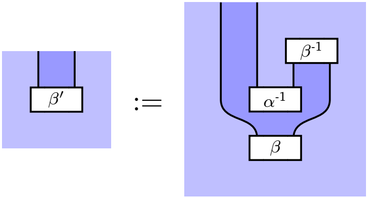
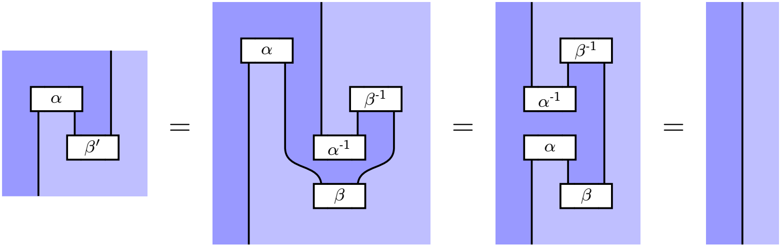


Since α' is composed from invertible 2-morphisms it must itself be invertible, and so it is clear that α' and β still give an equivalence.

II. Higher categories

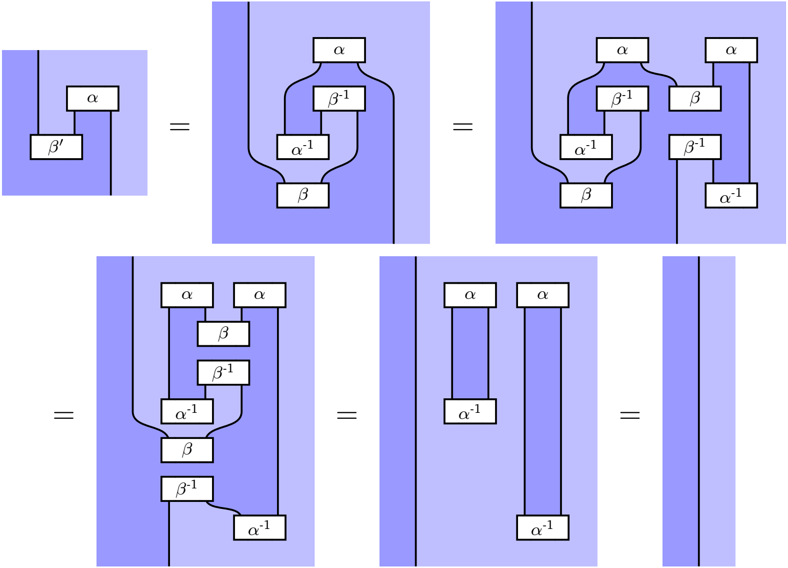
We now demonstrate that the adjunction equations are satisfied.

The first adjunction equation takes following form:



II. Higher categories

The second is demonstrated as follows:



II. Higher categories

Since monoidal categories are just 2-categories with one object, we immediately have the following corollary.

Corollary. In a monoidal category, if $A \otimes B \simeq B \otimes A \simeq I$, then $A \dashv B$ and $B \dashv A$.

II. Higher categories

Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!

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- objects correspond to planes;
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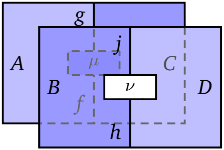
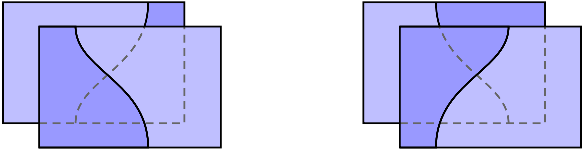
Tensor product. Given 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $h \overset{\nu}{\Longrightarrow} j$, the their *tensor product* 2-morphism $\mu \boxtimes \nu$ is given like this:



II. Higher categories

Interchange. Components can move freely in their separate layers.

The order of 1-morphisms in separate sheets can be *interchanged*:



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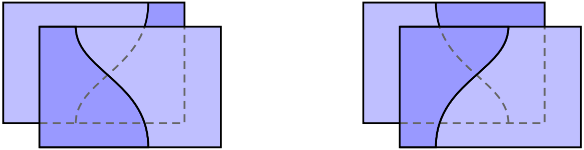

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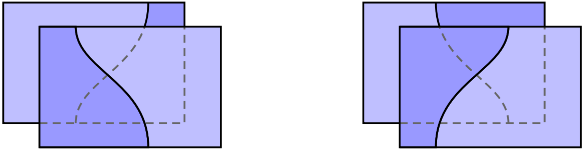

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These two interchangers are inverse to each other.

Unit object. A monoidal 2-category has a *unit object* I, represented by a 'blank' region.

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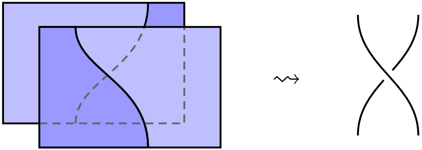
Something interesting happens when we combine interchangers and the unit object. Consider the interchanger diagram, but with all 4 planar regions labelled by the unit object:



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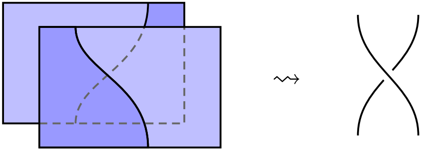

We obtain the graphical representation of a *braiding*.

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Theorem. *A monoidal category is a 2-category with one object.*

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The emerging pattern here is called the *periodic table*, and was predicted by Baez and Dolan in 1995.

Part III

Coherence

III. Coherence

Some monoidal categories have a particularly simple structure.

Definition 22. A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, λ_A and ρ_A are all identities.

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This seems like a very useful thing. *But beware!* This is not enough:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad\qquad I \otimes A = A = A \otimes I$$

In particular, it does not ensure that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.
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Theorem. Not every monoidal category is monoidally equivalent to a strict monoidal skeletal category.

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making the following diagrams commute:

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\big(F(A) \otimes F(B)\big) \otimes F(C) & \xrightarrow{\alpha_{F(A),F(B),F(C)}} & F(A) \otimes \big(F(B) \otimes F(C)\big) \\
{\scriptstyle (F_2)_{A,B} \otimes \mathrm{id}_{F(C)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F(A)} \otimes (F_2)_{B,C}} \\
F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\
{\scriptstyle (F_2)_{A \otimes B,C}} \downarrow & & \downarrow {\scriptstyle (F_2)_{A,B \otimes C}} \\
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F(A) \otimes I & \xrightarrow{\rho_{F(A)}} & F(A) \\
{\scriptstyle \mathrm{id}_{F(A)} \otimes F_0}\downarrow & & {\scriptstyle F(\rho_A^{-1})}\downarrow \\
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\end{array}
\qquad
\begin{array}{ccc}
I \otimes F(A) & \xrightarrow{\lambda_{F(A)}} & F(A) \\
\downarrow{\scriptstyle F_0 \otimes \mathrm{id}_{F(A)}} & & \downarrow{\scriptstyle F(\lambda_A^{-1})} \\
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$$(R_2)_{m,n} : |i\rangle \otimes |j\rangle \mapsto |ni + j\rangle$$
$$R_0 : 1 \mapsto 1$$

This is full, faithful and essentially surjective, and satisfies the monoidal functor conditions. □

III. Coherence

We now prove the strictification theorem.

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Proof sketch. Let **C** be a monoidal category, and define **D** like this:

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- a morphism $(F, \gamma) \to (F', \gamma')$ is $\theta\colon F \Longrightarrow F'$ such that:

$$\begin{array}{ccc} F(A) \otimes B & \xrightarrow{\gamma_{A,B}} & F(A \otimes B) \\ {\scriptstyle \theta_A \otimes \mathrm{id}_B} \downarrow & & \downarrow {\scriptstyle \theta_{A \otimes B}} \\ F'(A) \otimes B & \xrightarrow[\gamma'_{A,B}]{} & F'(A \otimes B) \end{array}$$

III. Coherence

Proof sketch (continued).

- the tensor product is $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where δ is

$$F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).$$

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We can then calculate these products:

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Now build a monoidal functor $L\colon \mathbf{C}\to\mathbf{D}$ in the following way:

$$L(A) := (A\otimes -, \alpha_{A,-,-})$$

You can show that L is full and faithful.

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You can show that L is full and faithful.

Finally, restrict $\mathbf{D}$ to the strict monoidal subcategory containing objects isomorphic to those in the image of L. Then L is a monoidal equivalence of $\mathbf{C}$ with a strict monoidal category. □

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The final topic in this chapter is *coherence*: any well-formed equation built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, id, $\otimes$ and $\circ$ holds.

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To make this precise, let a *bracketing* be a fixed way to bracket a list of objects of a given length, including empty brackets. For example, we could define the following bracketings v, w:

$$v(A,B,C,D) = ((A \otimes B) \otimes ()) \otimes (C \otimes D)$$
$$w(A,B,C,D) = \big(() \otimes (A \otimes (B \otimes C))\big) \otimes (() \otimes (() \otimes D)))$$

Then we can consider transformations of bracketings $\theta, \theta' : \nu \Rightarrow \mu$.

III. Coherence

We now give a proof of the coherence theorem.

Theorem 30. Let v, w be bracketings; then any two transformations $\theta, \theta' \colon v \Rightarrow w$ built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, id, $\otimes$, and $\circ$ are equal.

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Proof. We can define a canonical morphism

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$$\begin{array}{ccc} v(L(A), \dots, L(Z)) & \xrightarrow{\theta_{(L(A),\dots,L(Z))}} & w(L(A), \dots, L(Z)) \\ {\scriptstyle L_v{}^{-1}} \uparrow & & \downarrow {\scriptstyle L_w} \\ L(v(A, \dots, Z)) & \xrightarrow[L(\theta_{(A,\dots,Z)})]{} & L(w(A, \dots, Z)) \end{array}$$

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using the fact that L is a monoidal functor, and similarly for w. Then the following diagram commutes, for both θ and θ':

$$\begin{array}{ccc} v(L(A), \ldots, L(Z)) & \xrightarrow{\theta_{(L(A),\ldots,L(Z))}} & w(L(A), \ldots, L(Z)) \\ {\scriptstyle L_v^{-1}} \uparrow & & \downarrow {\scriptstyle L_w} \\ L(v(A, \ldots, Z)) & \xrightarrow[L(\theta_{(A,\ldots,Z)})]{} & L(w(A, \ldots, Z)) \end{array}$$

But $\theta_{(L(A),\ldots,L(Z))} = \theta'_{(L(A),\ldots,L(Z))} = \text{id}$! So $L(\theta_{(A,\ldots,Z)}) = L(\theta'_{(A,\ldots,Z)})$, and hence $\theta_{(A,\ldots,Z)} = \theta'_{(A,\ldots,Z)}$, since L is faithful. $\square$

Part IV

Duals in monoidal categories

IV. Duals in monoidal categories

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Definition 31. An object L is *left-dual* to an object R, and R is *right-dual* to L, written $L \dashv R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

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$$\begin{array}{ccccc} L & \xrightarrow{\rho_L^{-1}} & L \otimes I & \xrightarrow{\mathrm{id}_L \otimes \eta} & L \otimes (R \otimes L) \\ \downarrow{\scriptstyle \mathrm{id}_L} & & & & \downarrow{\scriptstyle \alpha^{-1}_{L,R,L}} \\ L & \xleftarrow[\lambda_L]{} & I \otimes L & \xleftarrow[\varepsilon \otimes \mathrm{id}_L]{} & (L \otimes R) \otimes L \end{array}$$

$$\begin{array}{ccccc} R & \xrightarrow{\lambda_R^{-1}} & I \otimes R & \xrightarrow{\eta \otimes \mathrm{id}_R} & (R \otimes L) \otimes R \\ \downarrow{\scriptstyle \mathrm{id}_R} & & & & \downarrow{\scriptstyle \alpha_{R,L,R}} \\ R & \xleftarrow[\rho_R]{} & R \otimes I & \xleftarrow[\mathrm{id}_R \otimes \varepsilon]{} & R \otimes (L \otimes R) \end{array}$$

IV. Duals in monoidal categories

We draw an object L as a wire with an upward-pointing arrow, and a right dual R as a wire with a downward-pointing arrow.

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We draw an object L as a wire with an upward-pointing arrow, and a right dual R as a wire with a downward-pointing arrow.

The unit $I \xrightarrow{\eta} R \otimes L$ and counit $L \otimes R \xrightarrow{\varepsilon} I$ are drawn as bent wires:

This notation is chosen because of the attractive form it gives to the duality equations:

They are also called the *snake equations*.

IV. Duals in monoidal categories

The monoidal category **FHilb** has all duals. Every finite-dimensional Hilbert space H is both right dual and left dual to its dual Hilbert space H^*, in a canonical way.

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The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is defined like this:

$$\varepsilon\colon |\phi\rangle \otimes \langle\psi| \mapsto \langle\psi|\phi\rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $|i\rangle$:

$$\eta\colon 1 \mapsto \sum_i \langle i| \otimes |i\rangle$$

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These definitions sit together rather oddly: η seems basis-dependent, while ε is clearly not.

In fact the same value of η is obtained whatever orthonormal basis is used, as we will see below.

Infinite-dimensional spaces do not have duals.

IV. Duals in monoidal categories

In **Rel**, every object is its own dual, even sets of infinite cardinality. The unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ can be defined like this:

$$\bullet \sim_\eta (s,s) \text{ for all } s \in S$$
$$(s,s) \sim_\varepsilon \bullet \text{ for all } s \in S$$

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$$\bullet \sim_\eta (s,s) \text{ for all } s \in S$$
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In $\mathbf{Mat}_{\mathbb{C}}$, every object n is its own dual, with a canonical choice of η and ε given as follows:

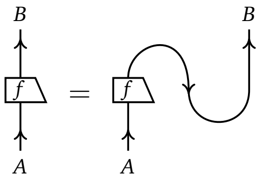
$$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle \qquad\qquad \varepsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij} 1$$

IV. Duals in monoidal categories

The category **Set** only has duals for sets of size 1. Let's see why.

Definition 32. In a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$, given a morphism $A \xrightarrow{f} B$, we define its *name* $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes B$ and *coname* $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ as the following morphisms:

Morphisms can be recovered from their names or conames:



In **Set** 1 is terminal, and so all conames $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} 1$ must be equal. If **Set** had duals this would imply all functions $A \to B$ were equal.

IV. Duals in monoidal categories

We first show duals are well-defined up to canonical isomorphism.

Lemma 33. In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.

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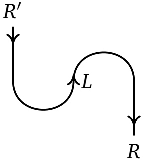
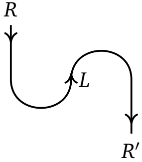
Proof. If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ as follows:





The snake equations imply that these are inverse.

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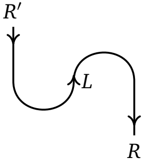
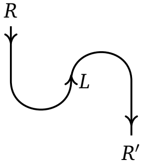
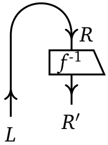
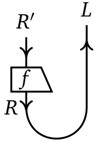




The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \xrightarrow{f} R'$ is invertible, we can construct a duality $L \dashv R'$:





□

IV. Duals in monoidal categories

Given a duality, the unit determines the counit, and vice-versa.

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Lemma 34. In a monoidal category, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta, \varepsilon')$ both exhibit a duality, then $\varepsilon = \varepsilon'$. Similarly, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta', \varepsilon)$ both exhibit a duality, then $\eta = \eta'$.

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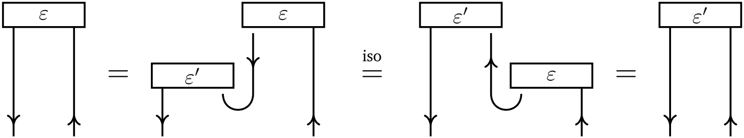
Proof. For the first case, we use the following graphical argument.



The second case is similar.

IV. Duals in monoidal categories

The following lemma shows that dual objects interact well with the monoidal structure.

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Lemma 35. In a monoidal category, $I \dashv I$.

Proof. Taking $\eta = \lambda_I^{-1} \colon I \to I \otimes I$ and $\varepsilon = \lambda_I \colon I \otimes I \to I$ shows that $I \dashv I$. The snake equations follow from the coherence theorem. □

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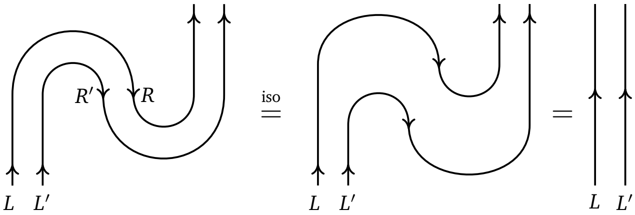
Lemma 36. In a monoidal category, $L \dashv R, L' \dashv R' \;\Rightarrow\; L \otimes L' \dashv R' \otimes R$.

Proof. Suppose that $L \dashv R$ and $L' \dashv R'$. We make the new unit and counit maps from the old ones, and compute as follows:



The other snake equation follows similarly. □

IV. Duals in monoidal categories

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

Lemma 37. In a braided monoidal category, $L \dashv R \;\Rightarrow\; R \dashv L$.

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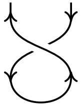
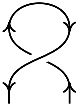


$I \xrightarrow{\eta'} L \otimes R$



$R \otimes L \xrightarrow{\varepsilon'} I$

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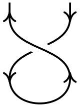

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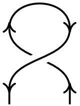
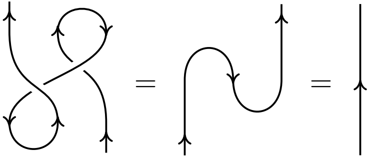


$R \otimes L \xrightarrow{\varepsilon'} I$

We can then test the snake equations:



The other snake equation can be proved in a similar way. □

IV. Duals in monoidal categories

Next we will prove some nice theorems showing the relationship between duals and monoidal functors.

To understand them, we will need to develop a graphical calculus for monoidal functors.

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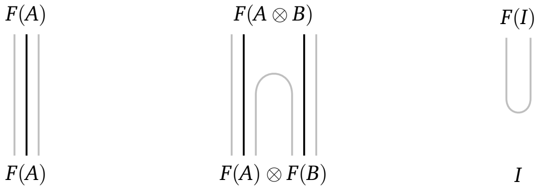
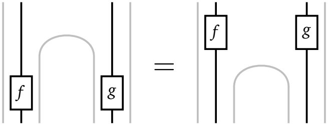
We depict a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ and the isomorphisms $(F_2)_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B)$ and $F_0\colon I \to F(I)$ like this:



IV. Duals in monoidal categories

Naturality means that morphisms can pass through the gaps:



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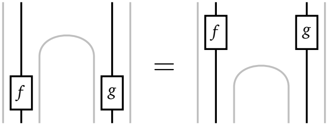

The coherence equations look like this:

They have a nice topological flavour.

IV. Duals in monoidal categories

Let's prove our first theorem using these techniques.

Theorem 38. Monoidal functors preserve duals.

IV. Duals in monoidal categories

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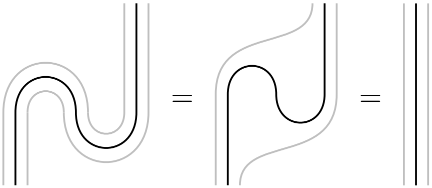
Theorem 38. Monoidal functors preserve duals.

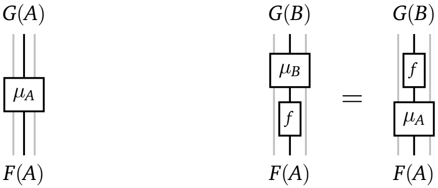
Proof. If we apply our monoidal functor to the unit and counit, we can show that the duality equations are still satisfied:



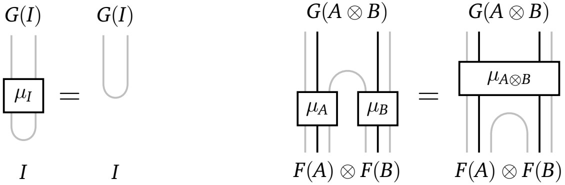
The other duality equation can be proved in a similar way. □

IV. Duals in monoidal categories

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$ and a natural transformation $\mu\colon F \Longrightarrow G$, we can denote it like this:



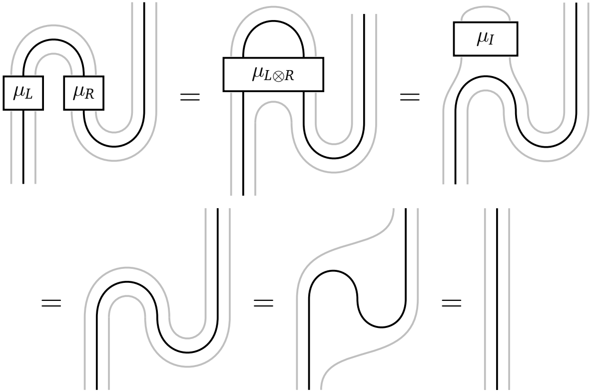
If **C**, **D**, F, G and μ are monoidal, then we have following extra properties:



IV. Duals in monoidal categories

Theorem 39. Let $\mu\colon F \Longrightarrow G$ be a monoidal natural transformation. If $A \in \mathrm{Ob}(\mathbf{C})$ has a left or a right dual, $F(A) \xrightarrow{\mu_A} G(A)$ is invertible.

Proof. Choose $A = L$ with $L \dashv R$ in **C**. Then we perform the following computation:



The rest of the proof uses similar techniques. □

IV. Duals in monoidal categories

Choosing duals for objects extends functorially to morphisms.

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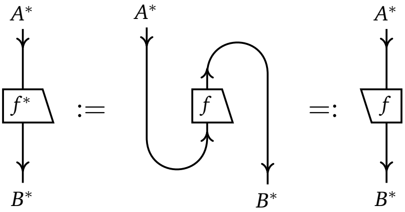
Definition 40. For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the *right dual* $B^* \xrightarrow{f^*} A^*$ is defined in the following way:



We represent this graphically by rotating the box representing f, as shown in the third image above.

IV. Duals in monoidal categories

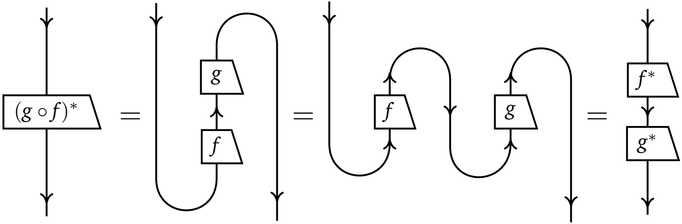
The dual can 'slide' along the unit and counit.

Lemma 41. In a monoidal category with chosen dualities $A \dashv A^*$ and $B \dashv B^*$, the following equations hold for all morphisms $A \xrightarrow{f} B$:

Proof. Let's write it out on the board. □

Lemma 42. If a monoidal category has assigned right duals, the right-duals construction $(-)^*$ defines a functor.

Proof. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$. Then we perform the following calculation:



Similarly, $(\mathrm{id}_A)^* = \mathrm{id}_{A^*}$ follows from the snake equations. □

Example 43. Let's see how the right duals functor acts for our example categories, with chosen right duals as given earlier.

IV. Duals in monoidal categories

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- In **FVect** and **FHilb**, the right dual of a morphism $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of W^*.

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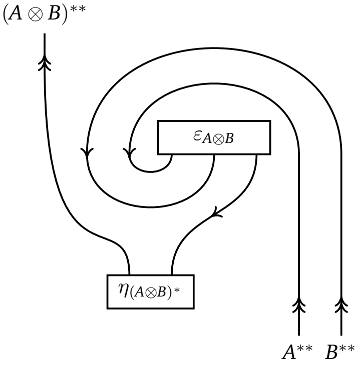
- In $\mathbf{Mat}_{\mathbb{C}}$, the dual of a matrix is its transpose.
- In **Rel**, the dual of a relation is its converse. So the right duals functor and the dagger functor have the same action: $R^* = R^\dagger$ for all relations R.

IV. Duals in monoidal categories

Lemma 44. For a monoidal category with chosen right duals for objects, the double duals functor $(-)^{**} : \mathbf{C} \to \mathbf{C}$ is monoidal.

Proof. The isomorphism $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$ looks like this:



Showing this satisfies the monoidal functor axioms is a monster! □

IV. Duals in monoidal categories

Definition 45. A monoidal category with right duals is *pivotal* when it is equipped with a monoidal natural transformation $A \xrightarrow{p_A} A^{**}$.

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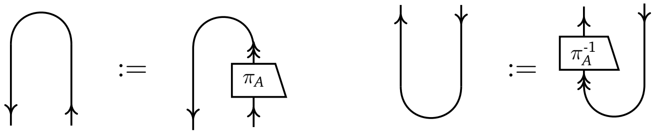
In a pivotal category, we extend the graphical calculus:



We can use this to rotate boxes arbitrarily.

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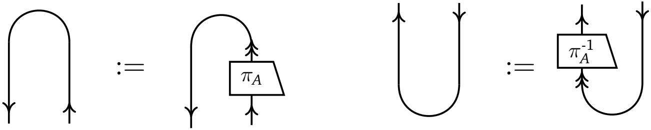
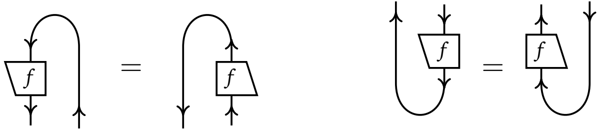


We can use this to rotate boxes arbitrarily.

Lemma. In a pivotal category, the following equations hold for all morphisms $A \xrightarrow{f} B$:



Proof. Let's write it out on the board. □

IV. Duals in monoidal categories

We can formalize this as follows.

Theorem 46. A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

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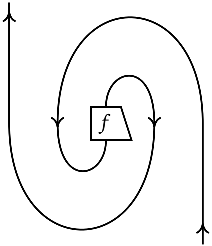
The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:

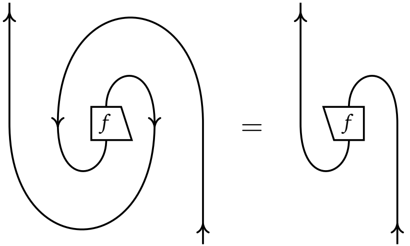


IV. Duals in monoidal categories

We can formalize this as follows.

Theorem 46. A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:



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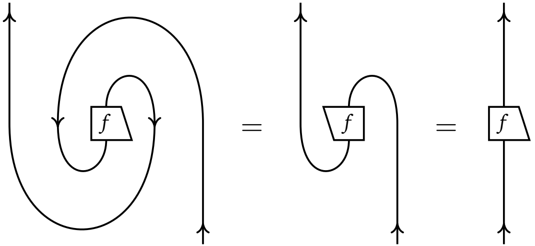

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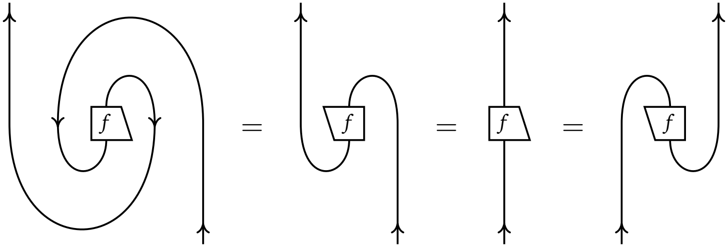
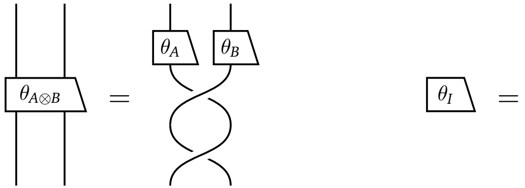


IV. Duals in monoidal categories

Definition 47. A braided monoidal category is *balanced* when it is equipped with a natural isomorphism $\theta_A : A \to A$ called a *twist*, satisfying the following equations:



The second equation here says $\theta_I = \mathrm{id}_I$.

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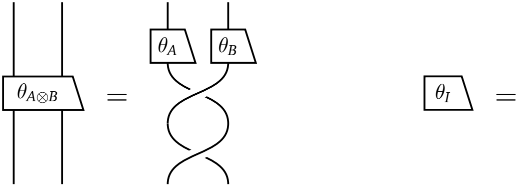

The second equation here says $\theta_I = \mathrm{id}_I$.

These equations look strange—we will see later what they mean!

IV. Duals in monoidal categories

Theorem 48. For a braided monoidal category with duals, a pivotal structure uniquely induces a twist structure, and vice versa.

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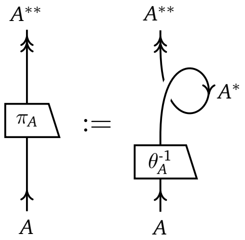
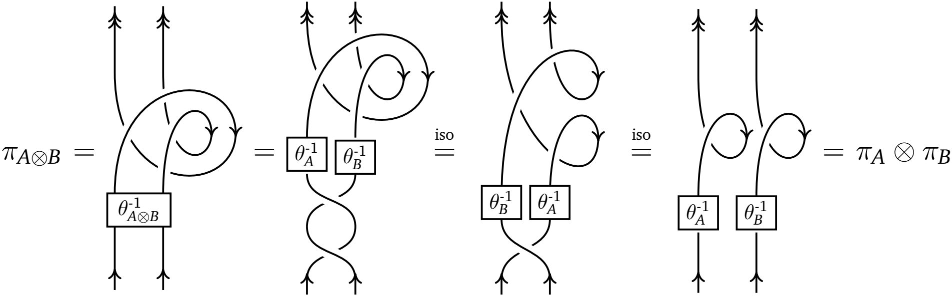
Proof. Suppose we have a twist structure $\theta_A : A \rightarrow A$. Then define a pivotal structure as follows:



We must verify that it is a monoidal natural transformation, and that it is natural.

IV. Duals in monoidal categories

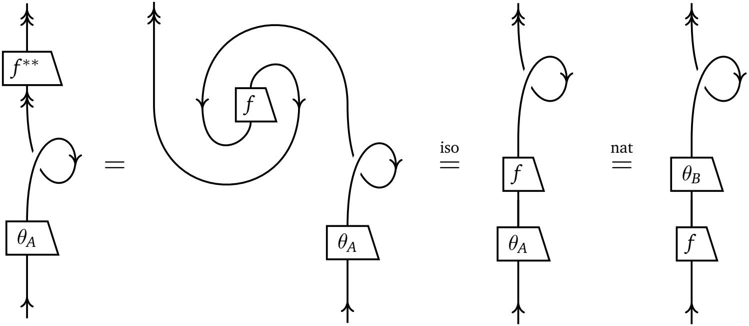
For the monoidal property, perform the following calculation:



For simplicity we have ignored the isomorphism $(A \otimes B)^{**} \simeq A^{**} \otimes B^{**}$.

IV. Duals in monoidal categories

To check naturality, we perform the following calculation:



Conversely, we can use a pivotal structure to define a twist. □

IV. Duals in monoidal categories

A symmetric monoidal category with duals has a canonical twist.

Definition 49. A *compact category* is a pivotal symmetric monoidal category with duals where the canonical twist is the identity $\theta_A = \mathrm{id}_A$.

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Our example categories **FHilb**, **FVect** and **Rel** are all compact categories.

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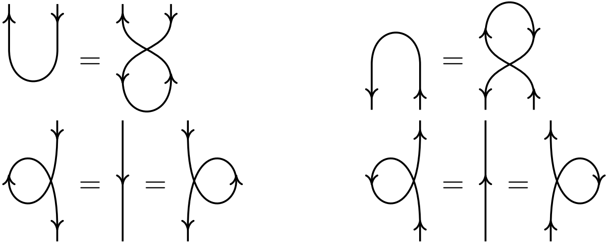
Our example categories **FHilb**, **FVect** and **Rel** are all compact categories.

Note that *in general*, other balancings may exist: that is, it is possible for a symmetric monoidal category with duals and a twist *not* to be a compact category.

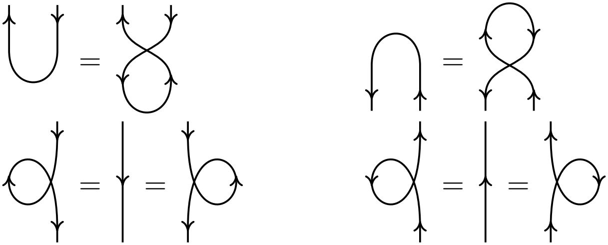
IV. Duals in monoidal categories

Lemma 50. In a compact category, the following equations hold:

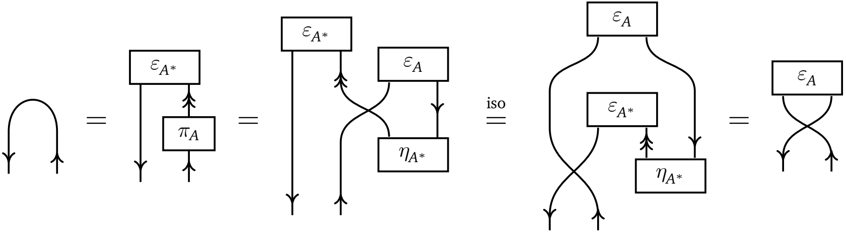


IV. Duals in monoidal categories

Lemma 50. In a compact category, the following equations hold:



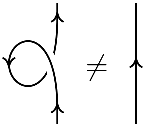
Proof. Let's prove the second equation in the top row:



The others can be proved in a similar way. □

IV. Duals in monoidal categories

In a braided pivotal category, we must be careful with loops:



IV. Duals in monoidal categories

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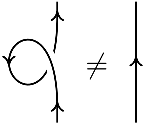
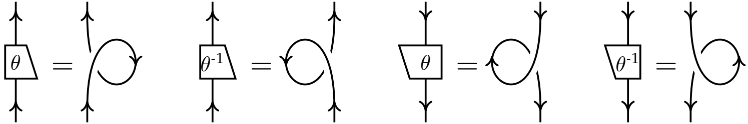


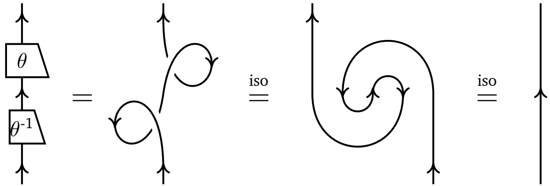
In fact, a loop on a single strand is directly related to the twist.

Lemma 51. In a braided pivotal category, the following hold:



IV. Duals in monoidal categories

Proof. Let's verify the expression for θ^{-1}:



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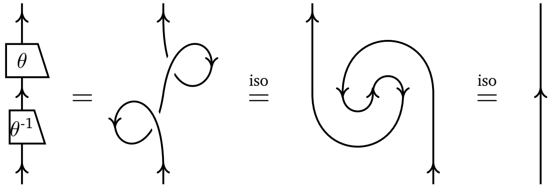

The equation $\theta \circ \theta^{-1} = \mathrm{id}$ can be checked in a similar way. Since inverses in a category are unique, this proves θ^{-1} is correct.

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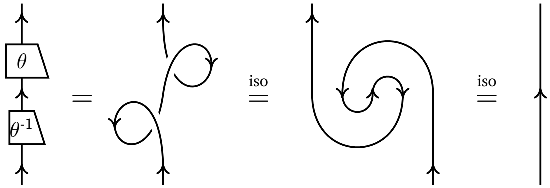
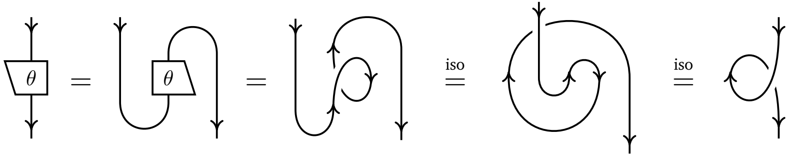


The equation $\theta \circ \theta^{-1} = \mathrm{id}$ can be checked in a similar way. Since inverses in a category are unique, this proves θ^{-1} is correct.

We demonstrate the graphical form of θ^{*} as follows:



The rest of the theorem can be proved similarly. □

IV. Duals in monoidal categories

Thinking about ribbons inspires the following definition.

Definition 52. A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_{A^*}$.

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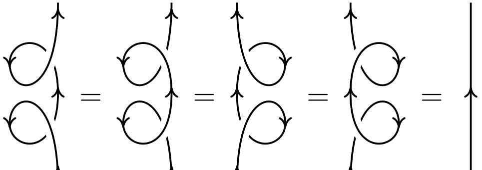
Lemma 54. In a ribbon category, the following equations hold:



IV. Duals in monoidal categories

These are the equations we would expect to be satisfied by *ribbons*.

Theorem 55. A well-formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in three dimensions.

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'Framed isotopy' is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires.

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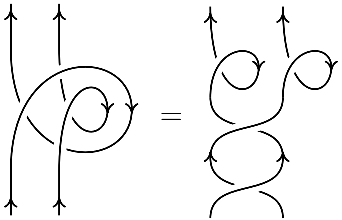
'Framed isotopy' is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires.

To get a feeling for framed isotopy, use ribbons to verify the following equations:

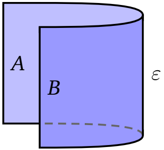


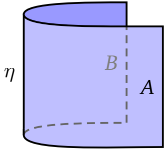
Part V

Duals in higher categories

V. Duals in higher categories

Definition. In a monoidal 2-category, an object A has a *right dual* B when it can be equipped with 1-morphisms called *folds*

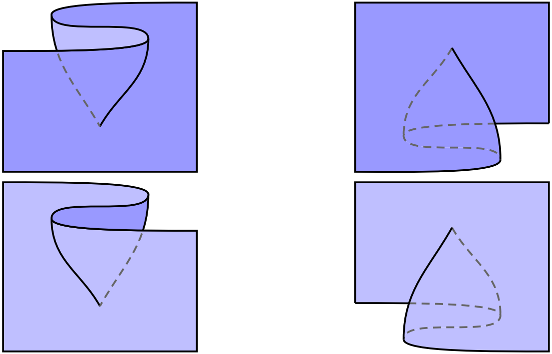




V. Duals in higher categories

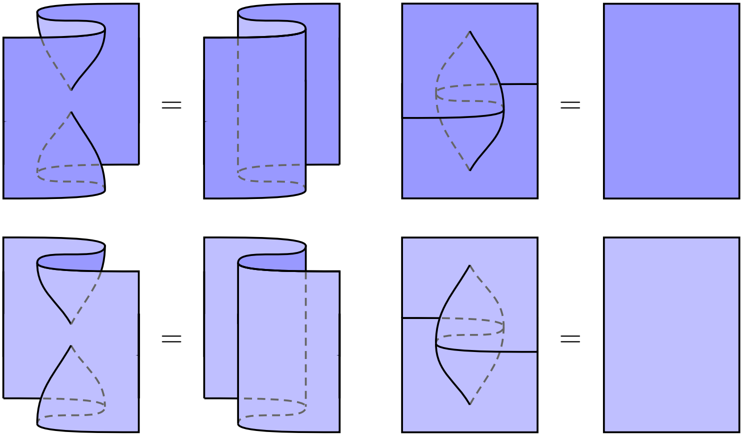
Definition. In a monoidal 2-category, an object A has a *right dual* B when it can be equipped with 1-morphisms called *folds*

and invertible 2-morphisms called *cusps*:



V. Duals in higher categories

The invertibility equations look like this:



It’s just like deforming a piece of fabric!

V. Duals in higher categories

To capture all the structure of oriented manifolds, we must require that our fold morphisms *themselves* have duals.

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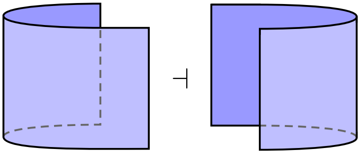
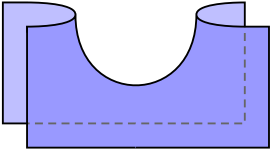
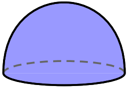
To see what happens, let's investigate this duality:



V. Duals in higher categories

It has a unit and counit, which we draw like this:





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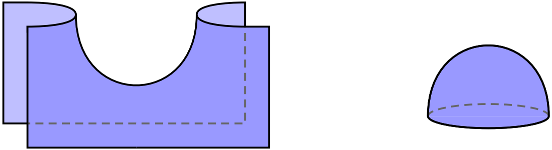
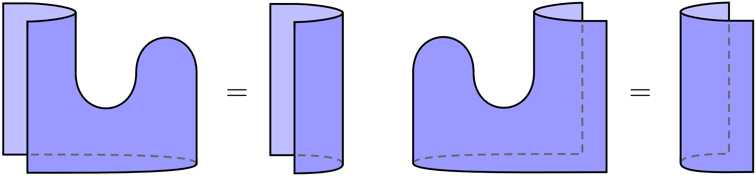


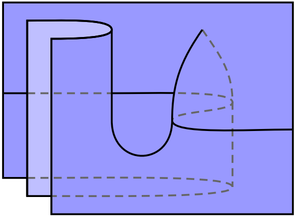
The snake equations for the duality then look like this:



Again, this makes sense in terms of deformations of surfaces!

V. Duals in higher categories

There is only one set of equations left to completely specify the behaviour of oriented surfaces. They look like this:



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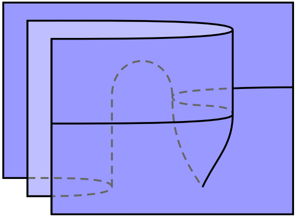

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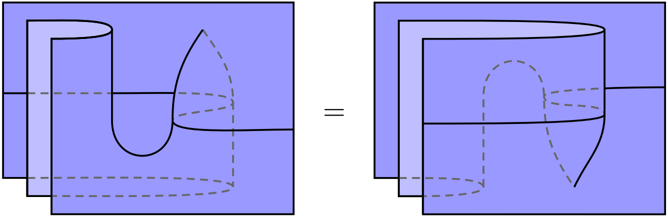

These are called the *cusp-flip equations*.

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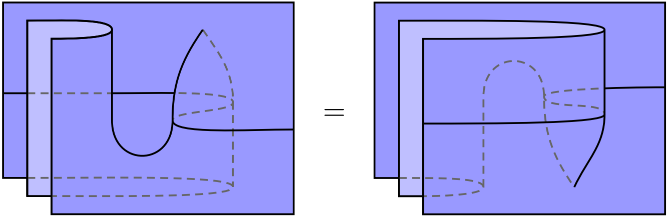

These are called the *cusp-flip equations*.

The *Cobordism Hypothesis* says that you can describe *n*-dimensional manifolds in a similar way.