Type Theory and Constructive Mathematics

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Equality in Logic and Mathematics

Frege (1879) and Russell (1903) define $x = y$ as

$$\forall P.\ P(x) \leftrightarrow P(y)$$

Leibnitz's identity of indiscernibles

This is an example of impredicative definition: we define a proposition $x = y$ by quantification over all predicates

Equality in Logic and Mathematics

Should the equality satisfy the extensionality principle?

This question was formulated by Russell as: if two properties (propositional functions) are extensionally equal, are they equal?

If we have

$$\forall x. P(x) \leftrightarrow Q(x)$$

do we also have $P = Q$?

Equality in Logic and Mathematics

In the Introduction of the present edition we have assumed that a function can only enter into a proposition through its values ... The uses we have made of this assumption can be validated by definition, even if the assumption is not universally true. That is to say, we can decide that mathematics is to confine itself to functions of functions which obey the above assumption. This amounts to saying that mathematics is essentially extensional rather than intensional

Appendix C of the second edition of Principia Mathematica

Equality in Logic and Mathematics

Principia Mathematica intensional (1910), then extensional (1927) under the influence of Wittgenstein

Russell's remarks that a function only "occurs through its values" have been expressed as mathematical theorems in the work of R. Gandy (1956) interpreting extensional type theory in intensional type theory

The first axiom of ZF set theory is the axiom of extensionality

MLTT: intensional 1970-1979, extensional 1979-1986, intensional 1986-?

Equality in Type Theory

Introduced in 1975

Strengthening of usual laws of equality

$$\frac{}{x = x} \qquad \frac{x = y \qquad P(x)}{P(y)}$$

or

$$\frac{}{x = x} \qquad \frac{x = y \qquad \forall z. C(z, z)}{C(x, y)}$$

Equality in Type Theory

In type theory

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{Id}_A\ t\ t} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{Ref}\ t : \mathsf{Id}_A\ t\ t}$$

$$\frac{\Gamma \vdash C : (\Pi x_0\ x_1 : A)(\mathsf{Id}_A\ x_0\ x_1 \to U) \qquad \Gamma \vdash d : (\Pi x : A) C\ x\ x\ (\mathsf{Ref}\ x)}{\Gamma \vdash \mathsf{J}\ d : (\Pi x_0\ x_1 : A)(\Pi p : \mathsf{Id}_A\ x_0\ x_1) C\ x_0\ x_1\ p}$$

$$\mathsf{J}\ d\ x\ x\ (\mathsf{Ref}\ x) = d\ x : C\ x\ x\ (\mathsf{Ref}\ x)$$

Equality in Type Theory

Notice

1. strong elimination rule
2. $\mathsf{Id}_A\ x_0\ x_1$ is a new *type*

Thus we can iterate this type forming operation, and if $p\ q : \mathsf{Id}_A\ x_0\ x_1$ we can form

$$\mathsf{Id}_{\mathsf{Id}_A\ x_0\ x_1}\ p\ q$$

Equality in Type Theory

Not so canonical? Ch. Paulin-Mohring formulated the variation

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{Id}_A\ t\ t} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{Ref}\ t : \mathsf{Id}_A\ t\ t}$$

$$\frac{\Gamma \vdash a : A \qquad C : (\Pi x : A)(\mathsf{Id}_A\ a\ x \to U) \qquad \Gamma \vdash d : C\ a\ (\mathsf{Ref}\ a)}{\Gamma \vdash \mathsf{C}\ d : (\Pi x : A)(\Pi p : \mathsf{Id}_A\ a\ x) C\ x\ p}$$

$$\mathsf{C}\ d\ a\ (\mathsf{Ref}\ a) = d : C\ a\ (\mathsf{Ref}\ a)$$

Equality in Type Theory?

The two formulations are equivalent (one direction is not easy)

Which formulation is the most basic?

Equality in Type Theory?

If we have $\vdash p : \mathrm{Id}_A\ t_0\ t_1$ then we have $t_0 = t_1$ and p has to be convertible to $\mathrm{Ref}\ t_0$

The equality type breaks the *extensionality principle*, which states that if we have $f\ g : N \to N$ and we have a proof of $Eq_{N\to N}\ f\ g$ then if $\varphi(f)$ is provable so is $\varphi(g)$

We can take $\varphi(h)$ to be $\mathrm{Id}_{N\to N}\ f\ h$, then we should have that $Eq_{N\to N}\ f\ g$ implies $\mathrm{Id}_{N\to N}\ f\ g$

This is not the case since we may have a proof of $Eq_N\ f\ g$ without having $f = g$

Equality in Type Theory

The strong form of the elimination rule for identity type implies that any type has a *groupoid* structure

We can define a composition operation $\mathrm{Id}_A\ x_0\ x_1 \rightarrow \mathrm{Id}_A\ x_1\ x_2 \rightarrow \mathrm{Id}_A\ x_0\ x_2$

Let us write $p \cdot q : \mathrm{Id}_A\ x_0\ x_2$ if $p : \mathrm{Id}_A\ x_0\ x_1$ and $q : \mathrm{Id}_A\ x_1\ x_2$

This operation is associative (up to propositional equality) and has a neutral element (the proof of reflexivity)

We can prove the equality of $p \cdot (q \cdot r)$ and $(p \cdot q) \cdot r$ and of $p \cdot (\mathrm{Ref}\ x_1)$ and p and of $(\mathrm{Ref}\ x_0) \cdot p$ and p

Equality in Type Theory

There is also an inverse operation $\mathsf{Id}_A\ x_0\ x_1 \to \mathsf{Id}_A\ x_1\ x_0$ and we can prove the equality of $p \cdot p^{-1}$ and Ref x_0 and of $p^{-1} \cdot p$ and Ref x_1

In each case the proof is a direct application of the strong elimination rule for identity type (and of the associated computation rule)

All this was found around 1992 (Thorsten Altenkirch, Martin Hofmann) and represented in the system LEGO

Equality in Type Theory

Any function $f : A \to B$ can be seen as a functor between the associated groupoid

All the functor laws are directly provable using the strong form of the elimination rule for equality

Equality in Type Theory

It was then realized (Hofmann, Streicher) that set theoretic groupoids form a model of type theory

This model shows that we cannot prove proof irrelevance in type theory

$$\mathrm{PI}_A = (\Pi x_0\ x_1 : A)(\Pi p\ q : \mathrm{Id}_A\ x_0\ x_1)\mathrm{Id}_{\mathrm{Id}_A\ x_0\ x_1}\ p\ q$$

This principle can also be formulated as (Streicher)

$$(\Pi x : A)(\Pi p : \mathrm{Id}_A\ x\ x)\mathrm{Id}_{\mathrm{Id}_A\ x\ x}\ p\ (\mathrm{Ref}\ x)$$

Equality in Type Theory

It was possible to show PI_A in some cases, for instance for $A = N_0, N_1, N$

All this was generalized by the following result

Theorem: (Hedberg, 1998) *If the equality on A is decidable i.e. if we can prove*

$$(\Pi x_0\ x_1 : A)(\mathrm{Id}_A\ x_0\ x_1 + \neg(\mathrm{Id}_A\ x_0\ x_1))$$

then PI_A is provable

Equality in Type Theory

Lemma: *For any type A if we have*

$$f : (\Pi x_0\ x_1 : A)(\mathsf{Id}_A\ x_0\ x_1 \to \mathsf{Id}_A\ x_0\ x_1)$$

then we have a proof of

$$(\Pi x_0\ x_1 : A)(\Pi p : \mathsf{Id}_A\ x_0\ x_1)\ \mathsf{Id}\ (f\ p)\ ((f\ (\mathsf{Ref}\ x_0)) \cdot p)$$

The statement is well-typed

The proof is immediate (we only to have to look at the case $p = \mathsf{Ref}\ x_0$)

Equality in Type Theory

The motivation was formalization of domain theory (Hedberg's PhD thesis)

Given $f_n : X_n \to X_{n+1}$ we want to define $f_{n,m} : X_n \to X_m$ for $n \leqslant m$

In type theory $f_{n,m,p} : X_n \to X_m$ where p is a proof of $n \leqslant m$

$n \leqslant m$ is defined as $(\Sigma r : N)\mathrm{Id}_N\ (n + r)\ m$

The argument p should be proof irrelevant: the result should not be dependent on the way we prove $n \leqslant m$

Equality in Type Theory

If we have $P : U \ (x : A)$ then we have an action of the groupoid A on the fibers

$$(\Pi x_0 \ x_1 : A) P(x_0) \to \mathsf{Id}_A \ x_0 \ x_1 \to P(x_1)$$

This map is given by the elimination rule of identity type

Equality in Type Theory

This strongly suggests a connection with algebraic topology and the notion of fibrations

Given a map $p : E \to B$ between topological space, view this map as defining a family of spaces $p^{-1}(b)$, the *fibers*, varying with $b : B$

This map is called a *Hurewicz fibration* iff any path $\sigma : I \to B$ and point $e : E$ such that $p(e) = \sigma(0)$ can be lifted to a path $\tilde{\sigma} : I \to E$ such that $\tilde{\sigma}(0) = e$ and $p \circ \tilde{\sigma} = \sigma : I \to B$

This is called a *regular* fibration if a constant path is lifted to a constant path.

Equality in Type Theory

A deeper connection with algebraic topology was investigated by S. Awodey and his students

The connection is between the rules of equality in type theory and the notion of *Quillen model structure*, an abstract setting to represent homotopy theory

Equality proofs as paths

The type $\mathsf{Id}_A\ x_0\ x_1$ should be/can be thought of as a type of *paths* between x_0 and x_1

The proof of reflexivity is thought of to be a constant path

From now on, we write $\mathsf{Path}_A\ x_0\ x_1$ instead of $\mathsf{Id}_A\ x_0\ x_1$

Space of path

Remarkably the notion of space of path has been introduced in algebraic topology and plays a very important role.

A classical example of a fiber space is the triple (B^I, B, p) *where* B *is an arbitrary space and* p *assigns to each path* $\omega \in B^I$ *its origin* $\omega(0)$

(Hurewicz, 1955)

Space of path

It is even strongly connected to Ch. Paulin-Mohring's elimination rule!

(J.P.Serre) *when I was working on homotopy groups (around 1950), I convinced myself that, for a space X, there should exist a fiber space E, with base X, which is* contractible*; such a space would allow me (using Leray's methods) to do lots of computations on homotopy groups... But how to find it? It took me several weeks (a very long time, at the age I was then) to realize that the* space of "paths" *on X had all the necessary properties-if only I dared call it a "fiber space". This was the starting point of the loop space method in algebraic topology.*

Space of path

In type theory, this correspond to the fibers $\mathrm{Path}_A\ a\ x$ seen as a family of types above $x : A$

The "total space" associated to this family is $E = (\Sigma x : A)\mathrm{Path}_A\ a\ x$

The fact that this is contractible corresponds to the fact that we have

$$(\Pi(x, p) : E)\ \mathrm{Path}_E\ (a, \mathrm{Ref}\ a)\ (x, p)$$

or

$$(\Pi x : A)(\Pi p : \mathrm{Path}_A\ a\ x)\ \mathrm{Path}_E\ (a, \mathrm{Ref}\ a)\ (x, p)$$

which is another formulation of the elimination rule of identity type

The computation rule corresponds to the regularity of this fibration

Homotopy group

To any point $a : A$ we can associate a "group" $\Omega_1(A, a) = \mathsf{Path}_A\ a\ a$ which is the group of loops from a to itself

One can iterate this construction $\Omega_{n+1}(A, a) = \Omega_n(\mathsf{Path}_A\ a\ a, \mathsf{Ref}\ a)$

A classical result (Čech, 1932) is that each group $\Omega_n(A, a)$ is commutative for $n \geqslant 2$

This can be translated as a pure type theoretical result

Proposition: *If X with a binary operation and an element $e : X$ which is both a left and right unit for this operation then the group $\Omega_1(X, e) = \mathsf{Path}_X\ e\ e$ is commutative*

Homotopy group

Streicher's characterisation of types A satisfying PI as

$$(\Pi x : A)(\Pi p : \mathsf{Path}_A\ x\ x)\mathsf{Id}_{\mathsf{Path}_A\ x\ x}\ p\ (\mathsf{Ref}\ x)$$

can be understood in the following way

Any space defines a groupoid, and there is at most one arrow between two objects iff there is only one loop at any point

Equality in Sigma type

Given $A : U$ and $P : U \ (x : A)$ we define $T = (\Sigma x : A)P$

We have $b \cdot p : P(a_1)$ if $b : P(a_0)$ and $p : \mathrm{Path}_A \ a_0 \ a_1$

Given $a_0 : A, \ b_0 : P(a_0)$ and $a_1 : A, \ b_1 : P(a_1)$ we have two maps

$$\mathrm{Path}_T \ (a_0, b_0) \ (a_1, b_1) \to (\Sigma p : \mathrm{Path}_A \ a_0 \ a_1)\mathrm{Path}_{P(a_1)} \ (b_0 \cdot p) \ b_1$$

and

$$(\Sigma p : \mathrm{Path}_A \ a_0 \ a_1)\mathrm{Path}_{P(a_1)} \ (b_0 \cdot p) \ b_1 \quad \to \mathrm{Path}_T \ (a_0, b_0) \ (a_1, b_1)$$