Infinite sets that satisfy
the principle of omniscience
in all varieties of constructive mathematics

Martín Escardó
University of Birmingham, UK

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Principle of omniscience

$$\forall p\colon X \to 2\,(\exists x \in X\,(p(x) = 0) \vee \forall x \in X\,(p(x) = 1)).$$

Can be proved for X finite (not for X subfinite in general).

For $X = \mathbb{N}$ this is LPO, so can't be proved.

For $X = 2^{\mathbb{N}}$ can be proved from Brouwerian assumptions.
(Continuity, fan theorem. We don't do this in this talk.)

Omniscience can be proved for plenty of infinite sets

In spartan contructive mathematics

We'll look at omniscient subsets of the Cantor space $2^{\mathbb{N}}$.

They will be ordinals with respect to the lexicographical order.

Spartan constructive mathematics

Don't assume (or reject), among other things:

1. Choice.

2. Powerset.

3. Markov's principle.

4. Continuity, bar induction, fan theorem, double-negation shift.

5. Church's thesis.

7. Extensionality (with respect to extensional equality).

We do assume function types.

But we do need extensionality to prove omniscience theorems

We use extensionality as a hypothesis of theorems rather than as axiom.

$$\forall \text{ extensional } p\colon X \to 2\,(\exists x \in X\,(p(x) = 0) \vee \forall x \in X\,(p(x) = 1)).$$

Drinker paradox

In every pub there is a person a such that if a drinks then everybody drinks.

$$\forall \text{ extensional } p\colon X \to 2(\exists a \in X(p(a) = 1 \implies \forall x \in X(p(x) = 1))).$$

For X inhabited, this is equivalent to the omniscience of X.

Selection of roots of 2-valued functions

We want to avoid choice. So we build it in.

A selection function for a set X is a functional $\varepsilon\colon (X \to 2) \to X$ such that for all extensional $p\colon X \to 2$,

$$p(\varepsilon(p)) = 1 \implies \forall x \in X (p(x) = 1).$$

Equivalently, the function p has a root if and only if $\varepsilon(p)$ is a root.

$$p(\varepsilon(p)) = 0 \Longleftarrow \exists x \in X (p(x) = 0).$$

Searchable sets

We say that a set is searchable if it has a selection function.

The generic convergent sequence

$$\mathbb{N}_\infty = \{x \in 2^{\mathbb{N}} \mid \forall i \in \mathbb{N}(x_i \geq x_{i+1})\}.$$

Also known as the one-point compactification of the natural numbers.
It is the final co-algebra of the functor $X \mapsto 1 + X$.

Clearly, the set $\mathbb{N}_\infty$ has elements $\underline{n} = 1^n 0^\omega$ and $\infty = 1^\omega$.

However. $\mathbb{N}_\infty \subseteq \underline{\mathbb{N}} \cup \{\infty\} \implies$ LPO.

What we can say is that $\forall x \in \mathbb{N}_\infty \left(\forall n \in \mathbb{N}(x \neq \underline{n})\right) \implies x = \infty$.

Proof. For any i, if we had $x_i = 0$, then we would have $x = \underline{n}$ for some $n < i$, and so we must have $x_i = 1$.

First Omniscience Theorem

Theorem. $\mathbb{N}_\infty$ is searchable and hence omniscient.

Proof. Given $p\colon \mathbb{N}_\infty \to 2$ extensional, let

$$\varepsilon(p) = \lambda i. \min_{n \leq i} p(\underline{n}).$$

Clearly $\varepsilon(p) \in \mathbb{N}_\infty$ (it is a decreasing sequence). Also

$$(0) \qquad \forall n \in \mathbb{N}(\varepsilon(p) = \underline{n} \implies p(\underline{n}) = 0),$$

$$(1) \qquad \varepsilon(p) = \infty \implies \forall n \in \mathbb{N}(p(\underline{n}) = 1).$$

We need to show that $p(\varepsilon(p)) = 1 \implies \forall x \in \mathbb{N}_\infty(p(x) = 1)$.

Claim 0. $\boxed{p(\varepsilon(p)) = 1 \implies \forall n \in \mathbb{N}(\varepsilon(p) \neq \underline{n}).}$

Proof. We know that $\forall n \in \mathbb{N}(\varepsilon(p) = \underline{n} \implies p(\underline{n}) = 0)$.

But, for any $n \in \mathbb{N}$, if we had $\varepsilon(p) = \underline{n}$, we would have $p(\underline{n}) = 1$ by extensionality.

Claim 1. $\boxed{p(\varepsilon(p)) = 1 \implies \varepsilon(p) = \infty.}$

Proof. This follows from Claim 0 and the previous lemma that

$$\forall x \in \mathbb{N}_\infty (\forall n \in \mathbb{N}(x \neq \underline{n})) \implies x = \infty.$$

Claim 2. $\boxed{p(\varepsilon(p)) = 1 \implies \forall n \in \mathbb{N}(p(\underline{n}) = 1).}$

Proof. This follows from the previous fact $\varepsilon(p) = \infty \implies \forall n \in \mathbb{N}(p(\underline{n}) = 1)$.

Claim 1. $\boxed{p(\varepsilon(p)) = 1 \implies \varepsilon(p) = \infty.}$

Claim 2. $\boxed{p(\varepsilon(p)) = 1 \implies \forall n \in \mathbb{N}(p(\underline{n}) = 1).}$

Claim 3. $\boxed{p(\varepsilon(p)) = 1 \implies p(\infty) = 1.}$

Proof. This follows from Claim 1 and the extensionality of p.

Claim 4. $\boxed{p(\varepsilon(p)) = 1 \implies \forall x \in \mathbb{N}_\infty(p(x) = 1).}$

Proof. This follows from Claims 2 and 3, and from the density Lemma, formulated and proved below.

Q.E.D.

Density Lemma

For all extensional $p \colon \mathbb{N}_\infty \to 2$, if

1. $p(\underline{n}) = 1$ for every $n \in \mathbb{N}$, and

2. $p(\infty) = 1$,

then

3. $p(x) = 1$ for every $x \in \mathbb{N}_\infty$.

Proof. If we had $p(x) \neq 1$, then the extensionality of p would give $x \neq \underline{n}$ for every $n \in \mathbb{N}$ and $x \neq \infty$, which is impossible.

Addendum to the First Omniscience Theorem

$\varepsilon(p)$ is the infimum of the set of roots of p.

So it is the least root if p has a some root.

We work with the lexicographical order of the Cantor space and hence $\mathbb{N}_\infty$.

Easy closure properties of omniscient sets

1. Finite products.

2. Images.

3. Unions with an omniscient index set.

Omniscient sets are not closed under finite intersections.

A more powerful closure property will be discussed later.

Reformulations of previous theorems

1 . Every decidable subset of $\mathbb{N}_\infty$ is either empty or inhabited.

2 . Every decidable subset of $\mathbb{N}_\infty$ has an infimum.

3 . Every inhabited decidable subset of $\mathbb{N}_\infty$ has a least element.

3'. Every non-empty decidable subset of $\mathbb{N}_\infty$ has a least element.

Transfinite Induction Theorem

For every decidable predicate A on $\mathbb{N}_\infty$,

$$\forall x \in \mathbb{N}_\infty (\forall y < x (Ay)) \implies Ax,$$

implies

$$\forall x \in \mathbb{N}_\infty (Ax).$$

Proof. Density Lemma and case analysis on $\underline{\mathbb{N}} \cup \{\infty\}$.

So $\mathbb{N}_\infty$ is an ordinal

But with respect to decidable (extensional) predicates only.

Ordinal for our purposes

1. Linearly ordered set.

2. Any inhabited, decidable, extensional subset has a least element.

3. Any decidable, extensional subset satisfies transfinite induction.

We construct plenty of omniscient ordinals in the lexicographic order of $2^{\mathbb{N}}$.

Countable sums of omniscient ordinals

Not possible.

E.g. $\mathbb{N}$ is a countable sum.

But $\sum_i X_i + 1$ works if we define it properly.

Squashed sums

The crude definition, with $X_n \subseteq 2^{\mathbb{N}}$, is

$$\overline{\sum}_n X_n = \bigcup_n 1^n 0 X_n \cup \{\infty\}.$$

The refined definition is written down in the accompanying paper.

Second Omniscience Theorem

Theorem. The searchable subsets of $2^{\mathbb{N}}$ are closed under squashed sums.

Theorem. So are the ordinal subsets of $2^{\mathbb{N}}$.

Can reach any ordinal below ϵ_0

And higher using richer type systems.

We apply Coquand, Hancock and Setzer (CSL 1997).

Question. How far can we get?

Meta-mathematics

HA^ω is the minimal example of formalized spartan constructive mathematics.

Definition. A set is called full if its complement is empty.

Meta-Theorem. If you can prove that a set has no countable full subset, then you cannot prove it to be omniscient.

The proof uses the model of continuous functionals and variations.

Back-of-the-envelop argument for the moment. But I am prety confident it works.

Fun to formalize the proof of omniscience of $\mathbb{N}_\infty$ in Agda

The proofs of the theorem and main lemmas/claims formalized in one evening.

Those of trivial lemmas in two days.

History of the trick to define ε

See the last section of the paper with the same title as these slides.

Brouwer (1927), Kreisel–Lacombe–Shoenfield (1959), Bishop (1967), Grilliot (1971), Ishihara (1991).

But nobody seems to have established a constructive omniscience theorem.

The crucial Density Lemma seems to be a new observation.

THE END