Totally separated types

Martin Hötzel Escardó
Theory Group, School of Computer Science
University of Birmingham

Types, Thorsten & Theories

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University of Nottingham

Question by Altenkirch, Anberrée & Li (2012)

Let X be any <u>definable</u> type in MLTT,

let $x_0, x_1 : X$ any two <u>provably distinct</u> elements.

Question Can we find $p : X \to 2$ with $p\,x_0 = 0$ & $p\,x_1 = 1$?

Answer (M.H.E 2012) There is a type X with elements $x_0 \neq x_1$ such that for any $p : X \to 2$, if $p\,x_0 = 0$ & $p\,x_1 = 1$, then WLPO holds.

(WLPO = every infinite binary sequence is constantly zero or not.)

The inspiration for the answer comes from classical topology

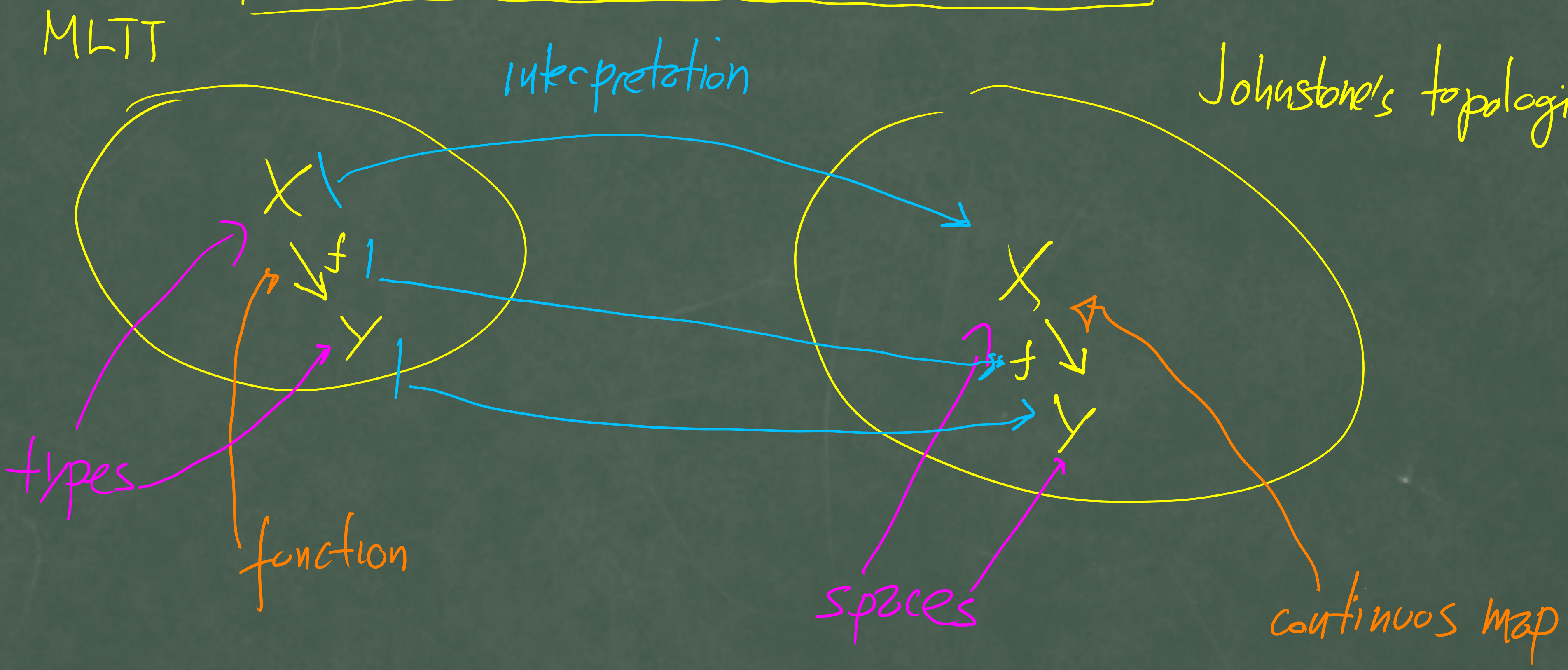
But our construction <u>doesn't</u> actually <u>use</u> topology.

1. clopen (closed and open) subsets of X correspond to continuous maps $X \to 2$.
2. <u>Potentially</u>, all maps are continuous in constructive mathematics.
3. WLPO is a discontinuity axiom. (WLPO $\Leftrightarrow$ there is a discontinuous function.)
4. A space is totally separated if the clopens separate the points.
5. There are plenty of <u>spaces</u> that fail to be totally separated.

$\Rightarrow$ Can we construct such <u>spaces</u> as <u>types</u> (with no extra structure)?

Have in mind something like this

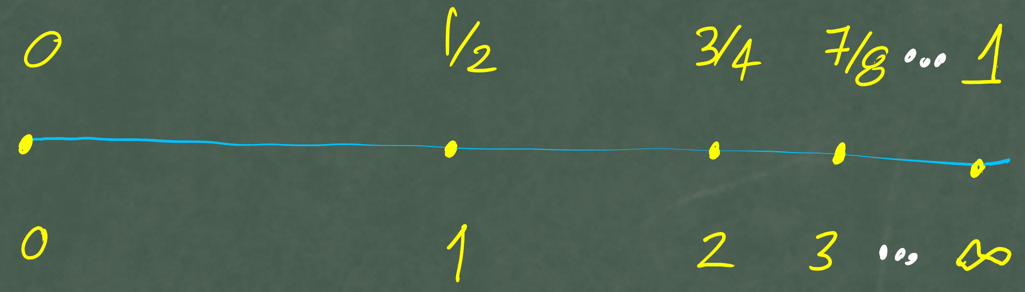
In my mind, types are spaces.



But actually it is enough to just use the 'intuition' of the above picture to construct and prove what we want.

Two examples from topology

1. One-point compactification of the discrete natural numbers.

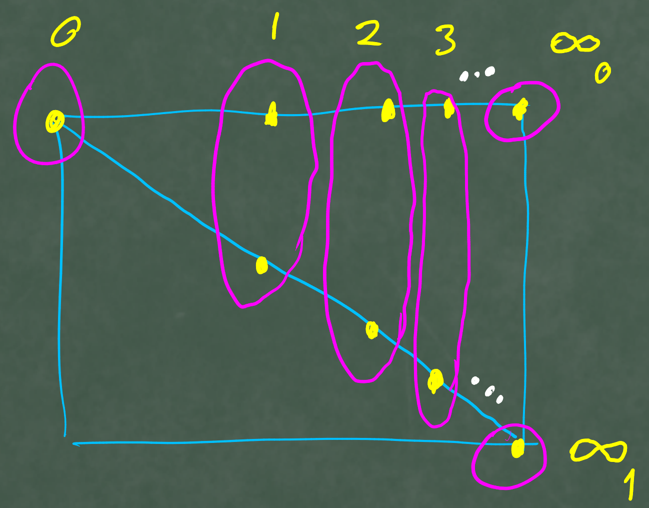


$\mathbb{N}_\infty$

This is compact & totally separated (Stone space)

Aka "generic convergent sequence".

2.



Quotient by the equivalence relation.

$\mathbb{N}_{\infty_0, \infty_1}$

This is compact, but not Hausdorff & hence not totally separated.

Constructing the above two examples as types

1. $\mathbb{N}_\infty := \sum_{\alpha : \mathbb{N} \to 2} \prod_{i : \mathbb{N}} \alpha_i \geq \alpha_{i+1}$.

Decreasing binary sequences

$\simeq$ conaturals

So $1^n 0^\omega$ represents $n : \mathbb{N}$, (count how many 1's we have)

and 1^ω represents ∞.

2. $\mathbb{N}_{\infty,\infty_1} := \sum_{x : \mathbb{N}_\infty} x = \infty \to 2$

identity type

when x is ∞, this type has <u>two</u> elements.

we create two copies of ∞: $\begin{cases} \lambda - . 0 & \& \\ \lambda - . 1 \end{cases}$

when x is finite, this type has <u>one</u> element.

$\boxed{\text{Theorem}}$ Assuming function extensionality:

1. The elements ∞_0 and ∞_1 of $\mathbb{N}_{\infty_0,\infty_1}$ are distinct.

Because any such function p is discontinuous

2. For any $p : \mathbb{N}_{\infty_0,\infty_1} \to 2$, if $p\,\infty_0 = 0$ and $p\,\infty_1 = 1$ then WLPO holds.

What can we say if we <u>don't</u> assume funext? $\boxed{\text{Meta-theorem}}$

Answer to original question

1. The elements ∞_0 and ∞_1 are <u>provably distinct</u>.
2. There is no <u>definable</u> $p : \mathbb{N}_{\infty_0,\infty_1} \to 2$ with $p\,\infty_0 = 0$ & $p\,\infty_1 = 1$.

Because this cannot be defined <u>with</u> funext, and hence cannot be defined with fewer assumptions.

Totally separated types

A type X is T.S. if it satisfies any of the following equivalent conditions:

1. $$\Pi\, x,y : X,\ \underbrace{\left(\Pi\, p : X \to 2,\ p\,x = p\,y\right)}_{x =_2 y} \to x = y$$

Boolean Leibniz principle

2. The apartness relation

$$x \#_2 y :\equiv \exists p : X \to 2,\ p\,x \neq p\,y$$

is tight, i.e. $\neg(x \#_2 y) \to x = y$.

3. The canonical map

$$X \longrightarrow ((X \to 2) \to 2)$$
$$x \longmapsto \lambda p.\, p\,x$$

is an embedding.

4. The quasi component $\Sigma\, y : X,\ x =_2 y$ of any $x : X$ is a singleton.

Properties and closure properties of T.S. types

1. T.S. types are sets in the sense of HoTT/UF.
2. T.S. types are $\neg\neg$-separated: $\neg\neg(x=y) \to x=y$.
3. T.S. types are closed under retracts.
4. $\mathbb{0}, \mathbb{1}, 2$, Fin n, $\mathbb{N}$ are T.S.
5. If $(X_i)_{i:I}$ is a family of T.S. types, then $\Pi\, i : I, X_i$ is T.S.

   (We don't require I to be T.S.)
6. So e.g. $\mathbb{N} \to 2$, $\mathbb{N} \to \mathbb{N}$, $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, etc. are T.S.
7. $\mathbb{N}_\infty$ is T.S. (theorem) but $\mathbb{N}_{\infty, \infty_1}$ is not provably T.S. (metatheorem).
8. So T.S. types are "not" closed under Σ.

cfd. Assuming funext & proptrunc:

Every type X has a T.S. reflection, given by the image TX of the canonical map $X \to ((X \to 2) \to 2)$:

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "f"'] & TX \arrow[d, dotted, "\exists!\, \bar{f}"] \\
& A
\end{tikzcd}
$$

A given arbitrary type $\to X$

TX: T.S. type (theorem)

$\exists!\, \bar{f}$ (theorem)

$A \leftarrow$ A given T.S. type

I hope you enjoyed the answer to your question & the addenda.

Happy birthday Thorsten!

Martin

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