Continuity in constructive dependent type theory

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Brouwer's continuity principle

The value $f(\alpha)$ of a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ depends only on a finite prefix of the sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$.

NB. This is continuity in the topological sense if we endow $\mathbb{N}$ with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the product (=exponential) topology.

Question

How should one formulate Brouwer's continuity principle for functions

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in (intensional or extensional) Martin-Löf Type Theory?

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1. This question turns out to be subtler than it may seem at first sight.

   Even in the absence of function extensionality.

2. We of course don't expect a continuity principle to be provable.

3. But much less we expect it be disprovable.

4. However, perhaps surprisingly, its Curry–Howard interpretation actually is disprovable.

5. What does that mean, and what is the correct formulation of the continuity principle in MLTT?

Brouwer's continuity principle in predicate logic

$$\forall(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\ \forall(\alpha : \mathbb{N}^{\mathbb{N}}).\ \exists(n : \mathbb{N}).\ \forall(\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta.$$

Not provable in e.g. higher-type Heyting arithmetic (HA^ω).

But validated e.g. by realizability over Kleene's K_2 and by Johnstone's topological topos, among other well-known models.

Brouwer's continuity principle in dependent type theory

Take the Curry–Howard interpretation of the above:

$$\prod_{f:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}} \quad \prod_{\alpha:\mathbb{N}^{\mathbb{N}}} \quad \sum_{n:\mathbb{N}} \quad \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \quad \alpha =_n \beta \to f\alpha = f\beta.$$

1. This implies $0 = 1$.

   This implication is a theorem of intensional Martin-Löf type theory.

   With $\mathbb{N}, \Sigma, \Pi, \mathrm{Id}$ or alternatively $\mathbb{0}, \mathbb{1}, \mathbb{N}, \Sigma, \Pi, U$.

   By adaptation of an old argument due to Kreisel, originally relying on extensionality.

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   By adaptation of an old argument due to Kreisel, originally relying on extensionality.

2. Maybe shocking at first sight, but makes perfect topological sense.

   The above says explicitly that every f is continuous.

   But it also says implicitly that we can continuously find a modulus of continuity n of f at α as a function of f and α.

   It is the second, implicit continuity requirement that cannot hold.

Brouwer's continuity principle in dependent type theory

How do we formulate it in a consistent, and meaningful, way?

Brouwer's continuity principle in dependent type theory

$$\prod_{f:\mathbb{N}^\mathbb{N}\to\mathbb{N}} \quad \prod_{\alpha:\mathbb{N}^\mathbb{N}} \left\| \sum_{n:\mathbb{N}} \quad \prod_{\beta:\mathbb{N}^\mathbb{N}} \quad \alpha =_n \beta \to f\alpha = f\beta \right\|.$$

$\|X\|$ = quotient of the type X by the chaotic equivalence relation.

This has computational content,

and in particular it is has more information than $\neg\neg X$.

In the setoid model, this means replacing the given equivalence relation by the chaotic relation.

Present in HoTT, and in the cubical type theory developed and implemented by Cohen, Coquand, Huber, Mörtberg.

Brouwer's continuity principle in dependent type theory

$$\prod_{f:\mathbb{N}^\mathbb{N}\to\mathbb{N}} \ \prod_{\alpha:\mathbb{N}^\mathbb{N}} \left\| \sum_{n:\mathbb{N}} \ \prod_{\beta:\mathbb{N}^\mathbb{N}} \ \alpha =_n \beta \to f\alpha = f\beta \right\|.$$

1. $\|X\|$ = quotient of the type X by the chaotic equivalence relation.

2. $\|X\|$ is the truth value of the inhabitedness of X, without necessarily revealing an inhabitant.

3. Validated by the topological topos and some realizability toposes.

   In a topos, $\|X\|$ is the image of the unique map $X \to 1$.

4. We have $(\exists(x : X).A(x)) = \|\Sigma(x : X).A(x)\|$ in any topos.

The elimination rule of propositional truncation

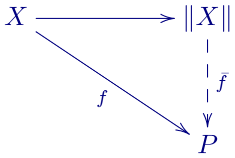
We (re)define a proposition to be a type with at most one element.

Also called a truth value.

$$\mathrm{isProp}\, X \stackrel{\mathsf{def}}{=} \Pi(x, y : X).\ x = y.$$

Here the equality sign denotes the identity type.

For any proposition P,



This is the (non-dependent) elimination rule.

The type $\|X\|$ is called the propositional truncation of X.

Example

The image of a function $f : X \to Y$ should be defined to be

$$\Sigma(y : Y).\, \|\Sigma(x : X).fx = y\| \,.$$

The type of all $y : Y$ such that there is $x : X$ with $fx = y$.

Exercise.

If you omit the truncation, the image of any f is isomorphic to X (even if e.g. Y is the unit type $\mathbb{1}$).

Uniform continuity

$$\forall (f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \exists (n\colon \mathbb{N}).\ \forall (\alpha, \beta\colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta.$$

1. Again not provable but consistent in HA^{ω}.
2. This time, its Curry–Howard interpretation

   $$\Pi(f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n\colon \mathbb{N}).\ \Pi(\alpha, \beta\colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$$

   is also consistent.
3. Moreover, it is logically equivalent to

   $$\Pi(f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \|\Sigma(n\colon \mathbb{N}).\ \Pi(\alpha, \beta\colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta\|,$$

   assuming function extensionality.
4. Chuangjie has also constructively developed a variation of the topological topos modelling this, and implemented it in Agda.

Summary of claims

1. Continuity is not provable in HA^ω, but is validated in some models.

2. The Curry–Howard interpretation of continuity is always false.

3. Consistent type-theoretic formulation via propositional truncation.

4. For uniform continuity, it doesn't make any difference whether we truncate Σ or not.

Failure of the Curry–Howard interpretation of continuity

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. The above axiom talks explicitly about functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ only.

   But it implicitly makes an assertion about *all* functions $X \to Y$.

2. If we have a "probe" $\mathbb{N}^{\mathbb{N}} \to X$ and an "observation" $Y \to \mathbb{N}$, then the composite $\mathbb{N}^{\mathbb{N}} \to X \to Y \to \mathbb{N}$ of the three functions has to be continuous according to the above axiom.

   Any function $X \to Y$ of any two types becomes empirically continuous by probing X and observing Y.

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   Any function $X \to Y$ of any two types becomes empirically continuous by probing X and observing Y.

   A remark is that in the model of Kleene–Kreisel continuous functionals, empirical continuity agrees with topological continuity.

   This remark is important for the intuition that guides the proof, but it doesn't feature in the proof, at least not explicitly.

Failure of the Curry–Howard interpretation of continuity

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1. The above axiom talks explicitly about functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ only.

   But it implicitly makes an assertion about *all* functions $X \to Y$.

2. Any function $X \to Y$ of any two types becomes continuous by probing X with a function $\mathbb{N}^{\mathbb{N}} \to X$ and observing Y with a function $Y \to \mathbb{N}$.
3. By projection, the continuity axiom gives a functional

   $$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$

   that assigns a modulus $n = M(f, \alpha)$ to f at the point α.

   Trouble: While all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ may be continuous, there can't be any continuous modulus-of-continuity functional M.

Proof of $0 = 1$

We set up an experiment to test the continuity of M.

1. Write $M(f) = M(f, 0^\omega)$ for the sake of brevity.
   0^ω is the infinite sequence of zeros, i.e. $\lambda i.0$.
   $0^n k^\omega$ consists of n zeros followed by infinitely many k's.
2. Let $m = M(\lambda\alpha.0)$.
   Define $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ to be $f(\beta) = M(\lambda\alpha.\beta(\alpha(m)))$, by probing M.
3. By expanding the definitions (which involves the ξ-rule), we get

   $$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha(m))) = M(\lambda\alpha.0) = m,$$

   and hence

   $$\Pi(\beta \colon \mathbb{N}^\mathbb{N}).0^\omega =_{Mf} \beta \to m = f\beta.$$

   For any $\beta \colon \mathbb{N}^\mathbb{N}$, by the continuity of $\lambda\alpha.\beta(\alpha m)$, we get
   $\Pi(\alpha \colon \mathbb{N}^\mathbb{N}).0^\omega =_{f\beta} \alpha \to \beta 0 = \beta(\alpha m)$.
4. Choosing $\beta = 0^{Mf+1}1^\omega$, we get $0^\omega =_{Mf+1} \beta$, and so $0^\omega =_{Mf} \beta$, and hence $f(\beta) = m$ and $\Pi(\alpha \colon \mathbb{N}^\mathbb{N}).0^\omega =_m \alpha \to \beta 0 = \beta(\alpha m)$.
5. Choosing $\alpha = 0^m(Mf+1)^\omega$, we have $0^\omega =_m \alpha$, and therefore
   $0 = \beta 0 = \beta(\alpha m) = \beta(Mf+1) = 1$. QED

Discussion

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. The problem with this formulation of the continuity axiom is the dependency of n on f and α, which is itself (empirically) continuous.

   This formulation of the axiom is saying more than we intended to say.

2. We have to break the implicit continuous dependency of the output n on the inputs f and α.

   A crude way to achieve this is to double-negate the conclusion:

   $$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\neg\neg\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

   But this is too weak. We can get more information.

The correct formulation of the continuity axiom should be

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\|\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta\|.$$

1. The axiom of choice is

   $(\Pi(x : X).\|\Sigma(y : Y).A(x, y)\|) \to \|\Sigma(f : X \to Y).\Pi(x : X).A(x, f(x))\|.$

2. Choice implies WLPO.

   (And even excluded middle if quotients are added to MLTT.)

   Continuity implies $\neg\mathrm{WLPO}$.

   Hence choice and continuity are together impossible.

   Extensionality considerations play no role in this argument.

We now discuss uniform continuity

The uniform continuity principle

$$\Pi(f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \|\Sigma(n\colon \mathbb{N}).\ \Pi(\alpha, \beta\colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta\|$$

is equivalent to its untruncated version

$$\Pi(f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n\colon \mathbb{N}).\ \Pi(\alpha, \beta\colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$$

using function extensionality.

Exiting propositional truncations

Often we have a *choice function* $\|X\| \to X$, even when we don't know whether X is empty or inhabited.

E.g. For any $f : \mathbb{N} \to \mathbb{N}$, we have $\|\sum_{n:\mathbb{N}} fn = 0\| \to \sum_{n:\mathbb{N}} fn = 0$.

If there is a root of f, then we can find one.

Because if there is a root, then there is a minimal root.

The type of minimal roots is a proposition, so we can eliminate.

So this is rather different from Markov's principle.

Exiting propositional truncations

However, global choice

$$\prod_{X:U} \|X\| \to X$$

implies that all types have decidable equality.

When there is a choice function $\|X\| \to X$, we have to work hard to get it.

Exit Lemma

Assume that $A(n)$ is a proposition for every $n : \mathbb{N}$.

If for any given n we have that $A(n)$ implies that $A(m)$ is decidable for all $m < n$, then we can eliminate

$$\|\Sigma(n : \mathbb{N}).\ A(n)\| \to \Sigma(n : \mathbb{N}).\ A(n).$$

For uniform continuity on $\mathbf{2}^{\mathbb{N}}$, we apply this lemma with

$$A(n) \overset{\text{def}}{=} \left(\Pi(\alpha.\beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \implies f\alpha = f\beta\right).$$

Summary and discussion

Summary:

1. Continuity is not provable in HA^ω, but is validated in some models.
2. The Curry–Howard interpretation of continuity is always false.

   (Proved in Agda.)
3. Correct type-theoretic formulation via propositional truncation.
4. For uniform continuity, it doesn't make any difference whether we truncate Σ or not. (Proved in Agda.)

Discussion:

1. What is, should be, or can be constructive existence?
2. One approach to achieve extensionality in MLTT is to add the equality-reflection rule.
3. Another one is to add $\|-\|$, which implies function extensionality, and also add propositional extensionality.
4. The ultimate extensionality axiom for MLTT is univalence, which is universe extensionality, and implies both function and propositional extensionality.

Some references related to continuity in type theory

1. Infinite sets that satisfy the principle of omniscience in any variety of constructive mathematics. JSL, 2013.

2. Constructive decidability of classical continuity. MSCS, 2014.

3. The inconsistency of a Brouwerian continuity principle with the Curry-Howard interpretation. TLCA, 2015, with Chuangjie Xu.

4. A constructive manifestation of the Kleene-Kreisel continuous functionals. Accepted for APAL, with Chuangjie Xu.

5. The universe is indiscrete. Accepted for APAL, with Thomas Streicher.