When the principle of omniscience just holds

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Mathematics in dependent type theory

1. I'll work in intensional Martin-Löf type theory (MLTT).

2. I will make a number of remarks related to HoTT, in particular regarding -1-truncation and equivalence.

3. Sometimes I will use *function extensionality*.

   (Alternatively, I can assume that our hypothetical functions are extensional in a suitable sense, like Bishop did. However, this leads to the so-called *setoid hell*.)

4. I will work informally but rigorously.

   But I have also written formal versions of the proofs in the computer in Agda notation.

LPO

For any given $p : \mathbb{N} \to 2$, we can either find $n : \mathbb{N}$ with $p(n) = 0$, or else determine that $p(n) = 1$ for all $n : \mathbb{N}$.

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For any given $p : \mathbb{N} \to 2$, we can either find a root of p, or else determine that there is none.

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + \neg(\Sigma(n : \mathbb{N}).p(n) = 0)$$

Subsingleton version of LPO

Any $p : \mathbb{N} \to 2$ either has a root or it doesn't.

$$\Pi(p : \mathbb{N} \to 2).\|\Sigma(n : \mathbb{N}).p(n) = 0\| + \neg(\Sigma(n : \mathbb{N}).p(n) = 0)$$

No need to singleton-truncate the rightmost Σ, as the negation of a type is automatically a subsingleton.

Also, this truncation is definable in MLTT (by considering the existence of a minimal root).

The LPO types

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(or homotopically equivalent).

The second is a retract of the first.

(This doesn't use the HoTT formulation of the axiom of choice.)

(It is an instance of choice that just holds.)

LPO is undecided

$$\Pi(p : \mathbb{N} \to 2).(\Sigma(n : \mathbb{N}).p(n) = 0) + (\neg\Sigma(n : \mathbb{N}).p(n) = 0)$$

1. A meta-theorem is that MLTT doesn't inhabit LPO or $\neg$LPO.
2. Each of them is consistent with MLTT.

   Classical models validate LPO.

   Effective and continuous models validate $\neg$LPO.
3. LPO is undecided, and we'll keep it that way.
4. But we'll say it is a constructive taboo.

We now make $\mathbb{N}$ larger by adding a point at infinity

Let $\mathbb{N}_\infty$ be the type of decreasing binary sequences.

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}).\Pi(n : \mathbb{N}).\alpha(n) = 0 \to \alpha(n+1) = 0.$$

Side-remark:

1. $\mathbb{N}$ is the *initial algebra* of the functor $1 + (-)$.

2. $\mathbb{N}_\infty$ is the *final coalgebra* of this functor.

(This requires function extensionality.)

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1. The type $\mathbb{N}$ embeds into $\mathbb{N}_\infty$ by mapping the number $n : \mathbb{N}$ to the sequence $\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$.
2. A point not in the image of this is $\infty \stackrel{\text{def}}{=} 1^\omega$.
3. The assertion that every point of $\mathbb{N}_\infty$ is of one of these two forms is equivalent to LPO.
4. What is true is that no point of $\mathbb{N}_\infty$ is different from all points of these two forms.
5. The embedding $\mathbb{N} + 1 \to \mathbb{N}_\infty$ is an isomorphism iff LPO holds.
6. But the complement of its image is empty. We say it is dense.

Theorem

$$\Pi(p : \mathbb{N}_\infty \to 2).(\Sigma(x : \mathbb{N}_\infty).p(x) = 0) + \neg\Sigma(x : \mathbb{N}_\infty).p(n) = 0$$

1. This is LPO with $\mathbb{N}$ replaced by $\mathbb{N}_\infty$.

2. We don't use continuity axioms, which anyway are not available in MLTT.

3. However, this is motivated by topological (not homotopical) considerations.

   In Johnstone's *topological topos*, $\mathbb{N}_\infty$ gets interpreted as the one-point compactification of discrete $\mathbb{N}$.

   Here we are seeing a *logical manifestation* of *topological compactness*.

4. This theorem actually makes sense in any variety of constructive mathematics (JSL 2013).

WLPO is also undecided by MLTT

$$\Pi(p : \mathbb{N} \to 2).(\Pi(n : \mathbb{N}).p(n) = 1) + \neg\Pi(x : \mathbb{N}).p(n) = 1$$

(This implies that every Turing machine carries on for ever or it doesn't.)

But we have:

Theorem $\Pi(p : \mathbb{N}_\infty \to 2).(\Pi(n : \mathbb{N}).p(\underline{n}) = 1) + \neg\Pi(n : \mathbb{N}).p(\underline{n}) = 1$

1. The point is that now we quantify over $\mathbb{N}$, although the function p is defined on $\mathbb{N}_\infty$.

2. This again holds in any variety of constructive mathematics and doesn't rely on continuity axioms (JSL'2013).

Some consequences

1. Every function $f : \mathbb{N}_\infty \to \mathbb{N}$ is constant or not.

2. Any two functions $f, g : \mathbb{N}_\infty \to \mathbb{N}$ are equal or not.

3. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ has a minimum value, and it is possible to find the point at which the minimum value is attained.

4. For any function $f : \mathbb{N}_\infty \to \mathbb{N}$ there is a point $x : \mathbb{N}_\infty$ such that if f has a maximum value, the maximum value is x.

5. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is not continuous, or not-not continuous.

6. There is a non-continuous function $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.

Are there more types like $\mathbb{N}_\infty$?

1. Plenty.
2. Our business here is how to construct them.

What have we been doing?

Giving examples of types X and properties P of X such that the assertion

for all $x : X$, *either* $P(x)$ *or not* $P(x)$

just holds.

1. In classical mathematics, we assume excluded middle.
2. Here we investigate mathematically how much of it just holds.

Two notions

Definition (Omniscient type)
A type X is omniscient if for every $p : X \to 2$, the assertion that we can find $x : X$ with $p(x) = 0$ is decidable.

In symbols:

$$\Pi(p : X \to 2).(\Sigma(x : X).p(x) = 0) + (\neg\Sigma(x : X).p(x) = 0).$$

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Definition (Searchable type)
A type X is searchable if for every $p : X \to 2$ we can find $x_0 : X$, called a *universal witness* for p, such that if $p(x_0) = 1$, then $p(x) = 1$ for all $x : X$.

In symbols,

$$\Pi(p : X \to 2).\Sigma(x_0 : X).p(x_0) = 1 \to \Pi(x : X).p(x) = 1.$$

Their relationship

$$\text{omniscient}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2).(\Sigma(x : X).p(x) = 0) + (\neg\Sigma(x : X).p(x) = 0).$$

$$\text{searchable}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2).\Sigma(x_0 : X).p(x_0) = 1 \to \Pi(x : X).p(x) = 1.$$

NB. These types are not subsingletons in general.

Proposition A type X is searchable iff it has a point and is omniscient:

$$\text{searchable}(X) \iff X \times \text{omniscient}(X).$$

A few theorems rely on pointedness, using the notion of searchability.

Closure under Σ

If X is omniscient/searchable and Y is an X-indexed family of omniscient/searchable types, then so is its disjoint sum $\Sigma(x : X).Y(x)$.

Closure under Π

Not to be expected in general.

E.g. $\mathbb{N}_\infty$ and 2 are omniscient, but in continuous and effective models of type theory, the function space $\mathbb{N}_\infty \to 2$ is not.

In the topological topos, $\mathbb{N}_\infty \to 2$ is a countable discrete space.

Closure under finite products

Theorem A product of searchable types indexed by a finite type is searchable.

Brouwerian closure under countable products

Theorem Brouwerian intuitionistic axioms $\implies$
A countable product of searchable types is searchable.

This is a kind of Tychonoff theorem, if we think of searchability as a "synthetic" notion of compactness.

In particular, the Cantor type $2^{\mathbb{N}}$, which is interpreted as the Cantor space in the topological topos, is searchable.

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2. But validated in another effective model
   (realizability over Kleene's K_2),
   and in the topological topos.

   (I implemented this in Agda, by disabling the termination checker in a particular function. One can run interesting examples.)

We will need this form of closure under Π

Theorem (micro Tychonoff)
A product of searchable types indexed by a subsingleton type is itself searchable.

That is, if X is a subsingleton, and Y is an X-indexed family of searchable types, then the type $\Pi(x : X).Y(x)$ is searchable.

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This cannot be proved if searchability is replaced by omniscience (that is, if we don't assume that every $Y(x)$ is pointed).

This is easy with excluded middle, but we are not assuming it.

Theorem A subsingleton-indexed product of searchable types is searchable.

1. Let X subsingleton, $Y(x)$ searchable for every $x : X$.
2. $Z \stackrel{\text{def}}{=} \Pi(x : X).Y(x)$.

   We have $\Pi(x : X).(Z \simeq Y(x))$ and $(X \to 0) \to (Z \simeq 1)$.

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3. Let $p : Z \to 2$.
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   $p(z_0) = 1 \to \Pi(z : Z).p(z) = 1$. Q.E.D.

Disjoint sum with a point at infinity

Theorem
The disjoint sum of a countable family of searchable sets with a point at infinity is searchable.

We need to say how we add a point at infinity.

The type $1 + \Sigma(n : \mathbb{N}).X(n)$ won't do, of course.

We will do this in a couple of steps.

Injectivity of the universe of types

Theorem

For any embedding $e : A \to B$, every $X : A \to U$ extends to some $Y : B \to U$ along e, up to equivalence,

$$\Pi(a : A).(Y(e(a)) \simeq X(a)).$$

A map $e : A \to B$ is called an embedding iff its fibers $e^{-1}(b)$,

$$\Sigma(a : A).f(a) = b,$$

are all subsingletons.

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Two constructions:

1. We have the "maximal" extension $Y = X/e$.

$$\begin{aligned}(X/e)(b) &= \Pi\left(s : e^{-1}(b)\right).X(\mathrm{pr}_1\, s) \\ &\simeq \Pi(a : A).\ e(a) = b \ \to\ X(a).\end{aligned}$$

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2. And also the "minimal" extension $Y = X \setminus e$.

$$\begin{aligned}(X \setminus e)(b) &= \Sigma\left(s : e^{-1}(b)\right).X(\mathrm{pr}_1\, s) \\ &\simeq \Sigma(a : A).\ e(a) = b \;\times\; X(a).\end{aligned}$$

The first one works our purposes.

Injectivity of the universe of types

Let $e : A \to B$ be an embedding and $X : A \to U$.

Consider the extended type family $X/e : B \to U$ defined above:

$$(X \setminus e)(b) = \Pi \left(s : e^{-1}(b)\right) .X(\mathrm{pr}_1\, s)$$

We have

1. For all $b : B$ *not* in the image of the embedding,

$$(X/e)(b) \simeq 1.$$

2. If for all $a : A$, the type $X(a)$ is searchable too, then for all $b : B$ the type $(X/e)(b)$ is searchable, by micro-Tychonoff.

3. Hence if additionally B is searchable, the type $\Sigma(b : B).(X/e)(b)$ is searchable too.

4. We are interested in $A = \mathbb{N}$ and $B = \mathbb{N}_\infty$, which gives the disjoint sum of $X(a)$ with a point at infinity.

A map $L : (\mathbb{N} \to U) \to U$

Let $e : \mathbb{N} \to \mathbb{N}_\infty$ be the natural embedding.

Given $X : \mathbb{N} \to U$, first take $X/e : \mathbb{N}_\infty \to U$

This step adds a point at infinity to the sequence.

We then sum over $\mathbb{N}_\infty$, to get $L(X)$:

$$L(X) = \Sigma(u : \mathbb{N}_\infty).(X/e)(u).$$

Then L maps any sequence of searchable types to a searchable type.

Iterating this map $L : (\mathbb{N} \to U) \to U$

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1. Extensional means that any two elements with the same predecessors are equal.
2. The accessibility of points of X is inductively defined.

   We say that $x : X$ is accessible whenever every $y < x$ is accessible.

   The accessibility of a point is a subingleton.
3. $<$ is accessible if every $x : X$ is accessible.

   The accessibility of $<$ implies that it is subsingleton valued, and that X is set.

A functor $F : U \to U$

$F(X) = L(\lambda n.X)$, which is equivalent to $\Sigma(u : \mathbb{N}_\infty).\Pi(n : \mathbb{N}).X^{e(n)=u}$.

An equivalent conindductive definition of F is given by constructors

$$\begin{aligned} \mathrm{zero} &: X \to F(X), \\ \mathrm{succ} &: F(X) \to F(X). \end{aligned}$$

1. The Cantor type $2^{\mathbb{N}}$ is the carrier of a final coalgebra of F.
2. There is an initial algebra, whose carrier is the subset of Cantor consisting of the sequences with finitely many zeros, for a suitable notion of finiteness.

(Which is classically equivalent to the classical one.)

End