An inductive-recursive universe of searchable ordinals

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The problem addressed here

Given a type X and $p\colon X \to 2$,

$2 = \{0, 1\} \simeq 1 + 1$

1. either exhibit $x : X$ such that $p(x) = 0$ (a root of p)
2. or else determine that p has no root.

For which infinite types X can this be done?

- Computation. Exhaustive search problem.
  Can we exhaustively search an infinite type mechanically?
- Topology. Compactness problem.
  Can we exhaustively search an infinite type continuously?
- Logic. Choice problem.
  Can we prove infinite instances of choice constructively?

Our type theory

Martin-Löf Type Theory (**MLTT**)

$0,\ 1,\ \mathbb{N},\ +,\ \times,\ \Sigma,\ \Pi,\ W,\ \mathsf{Id},\ \mathcal{U}$

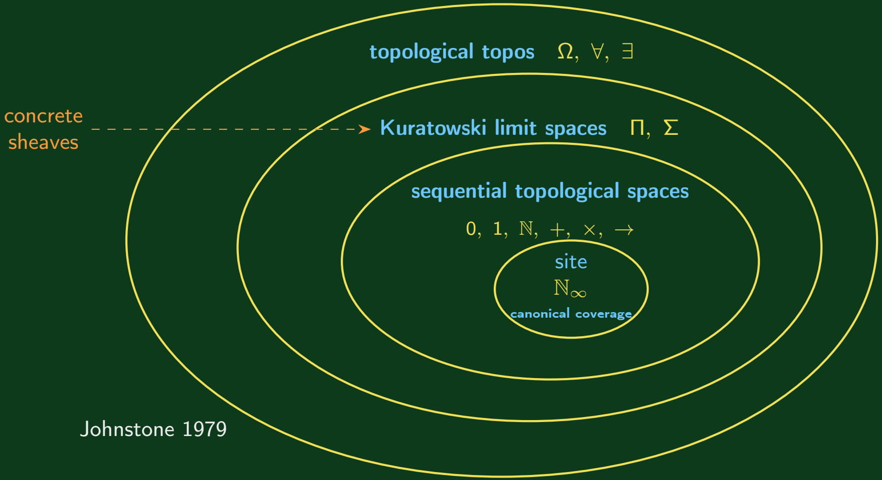
For some results towards the end, we include inductive–recursive types.

$+$ univalence (so in particular we have functional and propositional extensionality)
$+$ set quotients ($\iff$ propositional truncations $+$ set replacement)

Many models. We reason constructively, so our results hold in all models.

- Types are sets.
- Types are topological spaces.
- Types are homotopy types.
- Types are "sets with computational structure" (realizability).
- Types are the objects of a topos, or even an ∞-topos.

One particular model plays a guiding role



topological topos $\Omega,\ \forall,\ \exists$

concrete sheaves

Kuratowski limit spaces $\Pi,\ \Sigma$

sequential topological spaces

$0,\ 1,\ \mathbb{N},\ +,\ \times,\ \to$

site

$\mathbb{N}_\infty$

canonical coverage

Johnstone 1979

Examples of MLTT-definable objects of the topological topos

- $\mathbb{N}$ and $2 := 1 + 1$ get the discrete topology.
- $2^{\mathbb{N}} := (\mathbb{N} \to 2)$ is the Cantor space, and $\mathbb{N}^{\mathbb{N}} := (\mathbb{N} \to \mathbb{N})$ is the Baire space.
- $\mathbb{N}_\infty := \sum_{\alpha : \mathbb{N} \to 2} \prod_{i : \mathbb{N}} \alpha_i \geq \alpha_{i+1}$ is the one-point compactification of $\mathbb{N}$.

  $\underline{n} := 1^n 0^\omega, \qquad \infty := 1^\omega, \qquad (n \mapsto \underline{n}) : \mathbb{N} \hookrightarrow \mathbb{N}_\infty.$
- $\sum_{x : \mathbb{N}_\infty} ((x = \infty) \to 2)$ looks like this:

This is compact T_1 but not Hausdorff.

$$\{0, 1, \ldots, \infty_0\} \quad \cap \quad \{0, 1, \ldots, \infty_1\} \quad = \quad \mathbb{N}$$

compact

not compact

Mathematical expression of the problem in our system

For any given $p : X \to 2$, either find a root of p or determine there is none.

$$\prod_{p:X\to 2}\left(\left(\sum_{x:X} p\,x = 0\right) + \underbrace{\left(\prod_{x:X} p\,x = 1\right)}_{\iff \neg \sum_{x:X} p\,x = 0}\right)$$

- Stronger than excluded middle.
- We are making a choice.

We ask which types X satisfy this choice principle.

Definition. We call such types compact.

All types are compact $\iff$ global choice holds

Global choice. We can choose a point of every non-empty type.

$$\prod_{X:\mathcal{U}} \left(\underbrace{\neg\neg X}_{X \text{ is non-empty}} \to X \right)$$

- Stronger than choice, which is consistent with univalence.

  E.g. Voevodsky's model of simplicial sets.

- Contradicts univalence.
- But there are plenty of compact types in MLTT and HoTT/UF.
- The ones we are able to construct are all automatically equipped with well-orders.

Functions $p\colon X \to 2$

They classify complemented (aka detachable) subtypes of X.

$$X \simeq \left(\sum_{x:X} p\,x = 0\right) + \left(\sum_{x:X} p\,x = 1\right).$$

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, hook, "\text{complemented}"'] & 1 \arrow[d, "1"] \\
X \arrow[r, "p"'] & 2
\end{tikzcd}$$

- Topological topos. They classify clopen subspaces.

Totally separated types

Recall definition. A type X is called compact if

$$\prod_{p:X\to 2}\left(\left(\sum_{x:X} p\,x = 0\right) + \left(\prod_{x:X} p\,x = 1\right)\right).$$

This definition is not good unless there are plenty of maps $X \to 2$.

(Consider $X := \mathbb{R}$ as bad example.)

Definition. A type X is called totally separated if (Boolean Leibniz principle)

$$\prod_{x,y:X}\left(\prod_{p:X\to 2} p\,x = p\,y\right) \to x = y.$$

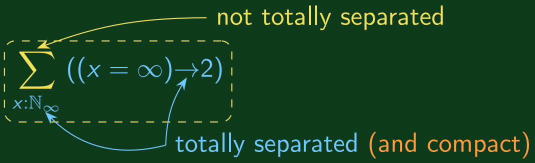
In the topological topos. The clopens separate the points.

(Topological notion with the same name.)

Some facts

1. Totally separated types are sets (elements can be equal in at most one way.)
2. They form an exponential ideal, and more generally are closed under Π, and are closed under $+$, $\times$, retracts, and include $0, 1, 2, \mathbb{N}, \mathbb{N}_\infty$ and all discrete types (those with decidable equality).
3. They are not closed under Σ in general.

Example. In the topological topos, the following type is not totally separated.



not totally separated

$$\sum_{x:\mathbb{N}_\infty} ((x = \infty) \to 2)$$

totally separated (and compact)

In type theory, from the assumption it is totally separated we conclude $\neg\neg$WLPO.

Some facts continued

4. Define the simple types to be the smallest collection of types including $0, 1, \mathbb{N}$ and closed under $\times, +, \to$.

   The simple types are all totally separated by (2) above.

5. **In the topological topos**, a closed subtype of a simple type is compact in the above type-theoretic sense iff it is compact in the topological sense. LMCS'2008

   In this case the inclusion is a section, and hence the subtype is itself totally separated.

6. Every type X has a totally separated reflection, given by the image of the evaluation map

$$\begin{aligned} X &\to ((X \to 2) \to 2) \\ x &\mapsto \lambda p.\ p\,x, \end{aligned}$$

   which is compact iff X is.

Counter-example to compactness

- The set $\mathbb{N}$ of natural numbers "fails" to be compact.
- The compactness of $\mathbb{N}$ amounts to Bishop's LPO. (Limited Principle of Omniscience)

More precisely, the compactness of $\mathbb{N}$ is independent of MLTT.

- False in realizability models (deciding/finding roots is not computable)
  in topological models (deciding/finding roots is not continuous)
- True in the model of classical sets (by the axiom of excluded middle)

Probably the simplest infinite example

$$\mathbb{N}_\infty := \sum_{\alpha:2^{\mathbb{N}}} \prod_{i:\mathbb{N}} \alpha_i \geq \alpha_{i+1}$$

This is the type of (non-strictly) decreasing binary sequences.

$$\underline{n} := 1^n 0^\omega$$
$$\infty := 1^\omega$$

We have an embedding

$$\mathbb{N} \hookrightarrow \mathbb{N}_\infty$$
$$n \mapsto \underline{n}$$

Theorem. (JSL'2013)

The type $\mathbb{N}_\infty$ is compact in our sense.

This was originally proved in a weaker system, namely Gödel's T.

Proof sketch

with the difficult part omitted

- Given $p\colon \mathbb{N}_\infty \to 2$ (not assumed to be continuous), define a binary sequence x by

$$x_n \;=\; \min(p\,\underline{0},\; p\,\underline{1},\; \ldots,\; p\,\underline{n}).$$

  Formula for the infimum of the roots.

- This is clearly decreasing, so $x\colon \mathbb{N}_\infty$.

- Now we check whether $p\,x = 0$ or $p\,x = 1$:

  0. If $p\,x = 0$, then we've found a root (in fact the smallest one).
  1. If $p\,x = 1$, then $p\,y = 1$ for all $y\colon \mathbb{N}_\infty$, and so there is no root.

     (In this case $x = \infty$.)

     (This is easy classically, but less so constructively.)

     In the pub $\mathbb{N}_\infty$ there is a person $x\colon \mathbb{N}_\infty$ such that if x drinks, then everybody drinks.

Some consequences

(decision procedures)

1. For every $p \colon \mathbb{N}_\infty \to 2$ either $\prod_{n:\mathbb{N}} p\,\underline{n} = 1$ or $\neg \prod_{n:\mathbb{N}} p\,\underline{n} = 1$. (JSL '2013)

   Quantification over $\mathbb{N}$ rather than $\mathbb{N}_\infty$! An instance of WLPO that just holds.

   Bishop's Weak Limited Principle of Omniscience is

$$\prod_{p:\mathbb{N}\to 2} \left(\left(\prod_{n:\mathbb{N}} p\,n = 1 \right) + \neg \left(\prod_{n:\mathbb{N}} p\,n = 1 \right) \right),$$

   which is independent of MLTT.

2. (a) For any given $f \colon \mathbb{N}_\infty \to \mathbb{N}$, we can decide whether it is not continuous. (MSCS '2015)

   (b) There is some discontinuous $f \colon \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.

Some applications of the compactness of $\mathbb{N}_\infty$

1. Dag Normann & William Tait. Springer '2017
   On the Computability of the Fan Functional

   They use the system T compactness of $\mathbb{N}_\infty$ to fill a gap in an unpublished but widely circulated 1958 manuscript by Tait.

2. Cécilia Pradic & Chad E. Brown. arXiv '2019
   Cantor–Bernstein implies excluded middle (Also implemented in Rocq.)

Compact sets in our type theory

1. 0, 1 and $\mathbb{N}_\infty$ are compact.
2. If X and Y are compact then so are $X + Y$.
3. If X and Y are compact then so is $X \times Y$. (Baby Tychonoff.)
4. If X is a compact type and A is a family of compact types indexed by X, then its disjoint union $\sum_{x:X} A\,x$ is a compact type.
5. If furthermore
   1. (a) the type X has at most one element (it is a proposition in the sense of HoTT/UF), and
   2. (b) we have a function that picks an element of $A\,x$ for any given $x : X$,

   then the cartesian product $\prod_{x:X} A\,x$ is compact. (Micro-Tychonoff.)

Can we prove full Tychonoff?

No.

- E.g. the compactness of the Cantor type $2^{\mathbb{N}}$ is independent.
  1. **True** in the topological topos (it is the Cantor space)
  2. **False** in Hyland's effective topos (Kleene tree to blame)
     (realizability topos over Kleene's K_1)
  3. **True** in the Kleene–Vesley topos
     (realizability over Kleene's K_2)

Perhaps amazingly, the toposes (1) and (3) have the same simple types.

More precisely, the full subcategories on the objects that arise as the interpretation of the simple types are equivalent.

Building more compact types

- The compact types we have constructed so far are all well-ordered:
  1. 0, 1, $\mathbb{N}_\infty$
  2. $X + Y$ everything in X smaller than everything in Y
  3. $X \times Y$ lexicographic order
  4. $\sum_{x:X} A\,x$ lexicographic order
- But we can't get very high ordinals with just the above.
- This is what we address next.

Well-ordered type

A type X equipped with a proposition-valued relation $<$ such that

1. The relation $<$ is transitive. (transitivity)
2. If two points have the same predecessors, then they are equal. (extensionality)
3. The order $<$ satisfies transfinite induction: (well-foundedness)

$$\left(\prod_{x:X}\left(\prod_{y:X} y < x \to P\,y\right) \to P\,x\right) \longrightarrow \prod_{x:X} P\,x.$$

Some consequences:

- X is automatically a set by (2) (equality types have at most one element).
- Trichotomy $x < y \vee x = y \vee x > y$ for all ordinals is equivalent to LEM.
- But there are plenty of trichotomous ordinals without assuming LEM.

The large type of all small ordinals

We also refer to well-ordered types as ordinals, and collect them in a (large) type Ord.

Univalence implies that the type Ord of small ordinals

1. is itself a large ordinal, and
2. has suprema of small-indexed families.
   (We'll discuss this further later.)

Ordinal sums

1. Let $\langle\alpha\rangle$ denote the underlying type of a well-ordered type $\alpha : \mathrm{Ord}$.
2. Given $\alpha\colon \mathrm{Ord}$ and $\beta\colon \langle\alpha\rangle \to \mathrm{Ord}$, we order the type

$$\sum_{x:\langle\alpha\rangle} \langle\beta\, x\rangle$$

lexigraphically:

$$(x, y) < (x', y') \quad :\equiv \quad (x < x') + \underbrace{\sum_{p\colon x = x'} \text{transport } p\, y < y'}_{\text{naively, } (x = x') \times (y < y')} .$$

Problem with lexicographic sums and possible solutions

- The lexicographic order is not extensional in general. (Can derive excluded middle.)
- It <u>is</u> extensional if
  1. The given orders have top elements. (Then so does the sum.)

     This is very lucky because our compact ordinals do have top.

  or 2. The given orders are trichotomous. (Then so is the sum.)

Suprema of families of ordinals

- Does every family $\alpha \colon I \to \mathrm{Ord}$ have a supremum constructively?
- As far as I know, this hadn't been answered before.

  Left open by Forsberg, Kraus & Xu (MFCS '21).
- Two answers, by Tom de Jong and myself independently.

  Both constructions and proofs implemented in Agda by Tom.
- Mine is as follows.

  The supremum is just the image of the function

$$\begin{aligned} \sum_{i:I} \langle \alpha_i \rangle &\longrightarrow \mathrm{Ord} \\ (i, x) &\longmapsto \alpha_i \downarrow x. \end{aligned}$$

  Although Ord is large, the image is small (assuming quotients).

Suprema of families of ordinals continued

Corollary. If I is compact and each α_i is compact, then so is $sup_{i:I}\ \alpha_i$.

Because $\sum_i \alpha_i$ is compact and images of compact types are compact.

Extending families of types

We need a further ingredient.

General situation:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dashed, "\overline{A}"] \\ & \mathcal{U} \end{tikzcd}$$

A: type family.
$\overline{A}$: extension of A along an embedding j.
$\mathcal{U}$: type universe.

Interested in:

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] \arrow[dr, "A"'] & \mathbb{N}_\infty \arrow[d, dashed, "\overline{A}"] \\ & \mathcal{U} \end{tikzcd}$$

canonical embedding.

Want: If $A\,x$ is compact for every $x : X$, then $\overline{A}\,y$ is compact for every $y : Y$.

Because then: If Y is also compact, then $\sum_{y:Y} \overline{A}\,y$ is compact too.

Family extension problem

MSCS '2021, "Injective types in univalent mathematics"

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dashed, "\overline{A}"] \\
 & \mathcal{U}
\end{tikzcd}$$

Consider

$$j^{-1}(y) \coloneqq \sum_{x:X} j\,x = y.$$

This type has at most one element (because j is an embedding).

Smallest solution (left Kan extension):

$$\underline{A}\,y \coloneqq \sum_{(x,-):\,j^{-1}(y)} A\,x$$

Largest solution (right Kan extension):

$$\boxed{\overline{A}\,y \coloneqq \prod_{(x,-):\,j^{-1}(y)} A\,x}$$

It is this that works for the wish of the previous slide, by Micro-Tychonoff.

Extending families of compact types to compact index sets

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dashed, "\overline{A}"] \\
& \mathcal{U}
\end{tikzcd}$$

Special case of interest:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, hook] \arrow[dr, "A"'] & \mathbb{N}_\infty \arrow[d, dashed, "\overline{A}"] \\
& \mathcal{U}
\end{tikzcd}$$

$$\overline{A}\,y := \prod_{(x,-):j^{-1}(y)} A\,x$$ (product of a family with at most one element).

Theorem. If the type $A\,x$ is compact for every $x : X$, then the type $\overline{A}\,y$ is compact for every $y : Y$.

Corollary. If additionally Y is compact, then so is $\sum_{y:Y} \overline{A}\,y$.

But this is not enough

The above says that type universes $\mathcal{U}$ are injective.

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dashed, "\overline{A}"] \\
 & \mathcal{U}
\end{tikzcd}$$

$$\overline{A}(y) = \prod_{(x,-) : j^{-1}(y)} A\,x$$

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "\alpha"'] & Y \arrow[d, dashed, "\overline{\alpha}"] \\
 & \mathrm{Ord}
\end{tikzcd}$$

Theorem. The type of (topped) ordinals also is injective.

Proof. That of the injectivity of a universe + additional construction of the order + checking it works.

We need to order $\overline{\alpha}(y)$.

We define, for $\mu, \nu : \langle \overline{\alpha}\, y \rangle$,

$$\mu < \nu :\equiv \sum_{\sigma : j^{-1}(y)} \mu\,\sigma < \nu\,\sigma.$$

Next

Using the previous machinery.

1. Compact ordinals induced by an inductive type of Brouwer ordinal expressions.
2. A generalization to an inductive-recursive universe à la Tarski of compact ordinals.

Brouwer ordinal codes

A type B inductively defined by constructors

$Z : B$	“zero”
$S : B \to B$	“successor”
$L : (\mathbb{N} \to B) \to B$	“limit”

A W type of countably branching trees.

Four interpretations of Brouwer codes as ordinals

0) standard interpretation

$$\begin{aligned} \llbracket Z \rrbracket_{\mathsf{sup}} &= 0 \\ \llbracket S\ b \rrbracket_{\mathsf{sup}} &= \llbracket b \rrbracket_{\mathsf{sup}} + 1 \\ \llbracket L\ b \rrbracket_{\mathsf{sup}} &= \sup_{i:\mathbb{N}} \llbracket b_i \rrbracket_{\mathsf{sup}} \end{aligned}$$

1) trichotomous interpretation

$$\begin{aligned} \llbracket Z \rrbracket_{\Sigma} &= 0 \\ \llbracket S\ b \rrbracket_{\Sigma} &= \llbracket b \rrbracket_{\Sigma} + 1 \\ \llbracket L\ b \rrbracket_{\Sigma} &= \sum_{i:\mathbb{N}} \llbracket b_i \rrbracket_{\Sigma} \end{aligned}$$

2) compact interpretation topped

$$\begin{aligned} \llbracket Z \rrbracket_{\overline{\mathsf{sup}}} &= 1 \\ \llbracket S\ b \rrbracket_{\overline{\mathsf{sup}}} &= \llbracket b \rrbracket_{\overline{\mathsf{sup}}} + 1 \\ \llbracket L\ b \rrbracket_{\overline{\mathsf{sup}}} &= \sup_{i:\mathbb{N}_\infty} \overline{\llbracket b_i \rrbracket_{\overline{\mathsf{sup}}}} \end{aligned}$$

3) compact totally separated topped

$$\begin{aligned} \llbracket Z \rrbracket_{\overline{\Sigma}} &= 1 \\ \llbracket S\ b \rrbracket_{\overline{\Sigma}} &= \llbracket b \rrbracket_{\overline{\Sigma}} + 1 \\ \llbracket L\ b \rrbracket_{\overline{\Sigma}} &= \sum_{i:\mathbb{N}_\infty} \overline{\llbracket b_i \rrbracket_{\overline{\Sigma}}} \end{aligned}$$

Overline means extension to $\mathbb{N}_\infty$ by injectivity.

Assuming excluded middle

$$\begin{array}{ccccc}
\text{standard} & [\![b]\!]_{\mathrm{sup}} & \leq & [\![b]\!]_{\Sigma} & \text{trichotomous} \\
 & \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} & \\
\text{compact} & [\![b]\!]_{\overline{\mathrm{sup}}} & \leq & [\![b]\!]_{\overline{\Sigma}} & \text{compact totally separated}
\end{array}$$

In the next slide we see what happens constructively here.

Why do we need excluded middle?

Because $(-) + 1$ is monotone $\iff$ excluded middle holds.

Theorems

Even better: every complemented subset is either empty or has a least element.

The ordinal $[\![b]\!]_{\Sigma}$

- is discrete, and moreover trichotomous
- is a retract of $\mathbb{N}$
- so countable
- not compact unless LPO holds

The ordinal $[\![b]\!]_{\overline{\Sigma}}$

- is compact
- is a retract of $\mathbb{N} \to 2$
- so totally separated
- not countable unless LPO holds
- not discrete unless LPO holds

- There is an order-preserving-reflecting embedding $[\![b]\!]_{\Sigma} \hookrightarrow [\![b]\!]_{\overline{\Sigma}}$ whose image has empty complement.
- LPO $\Rightarrow$ this embedding is a bijection $\Rightarrow$ WLPO.
- In models: the embedding doesn't have a computable/continuous inverse.

Illustration

Two manifestations of the ordinal $\omega + 1$.

$$\begin{tikzcd}
\mathbb{N} + 1 \arrow[r, hook] & \mathbb{N}_\infty \\
n \arrow[r, mapsto] & 1^n 0^\omega \\
* \arrow[r, mapsto] & 1^\omega
\end{tikzcd}$$

$\mathbb{N} + 1$ is

- discrete
- compact iff LPO
- countable

The map $\mathbb{N} + 1 \hookrightarrow \mathbb{N}_\infty$

- is a bijection iff LPO,
- but its image always has empty complement.

$\mathbb{N}_\infty$ is

- compact, totally separated
- discrete iff WLPO
- countable iff LPO

Every decreasing sequence is of one of the forms $1^n 0^\omega$ and 1^ω. Constructive taboo.

There is no decreasing sequence other than $1^n 0^\omega$ and 1^ω. Just true.

Generalizing and extending the above

In order to successively construct more and more compact ordinals in stages,

1. in the above, we take suprema/sums of $\mathbb{N}_\infty$-indexed families of ordinals, and
2. next we allow α-indexed families, where the compact ordinal α is contructed in previous stages.

   That is, we move from ω-branching trees to ordinal-branching trees.

Universes à la Tarski of compact ordinals

We define

$$\begin{array}{ll} E : \mathcal{U}_0, & \text{(infinitary) ordinal codes or expressions} \\ \Delta \colon E \to \mathrm{Ord}^{\top} & \text{discrete and trichotomous interpretation} \end{array}$$

by induction–recursion.

After that, we define, by recursion,

$$K \colon E \to \mathrm{Ord}^{\top} \qquad \text{compact (totally separated?) interpretation}$$

We define E, using Δ, inductively by the following constructors:

$$\begin{array}{l|l|l} \ulcorner 1 \urcorner \colon E & \_\ulcorner + \urcorner\_ \colon E \to E \to E & \ulcorner \Sigma \urcorner \colon (e : E) \to (\langle \Delta\, e \rangle \to E) \to E \\ \ulcorner \omega + 1 \urcorner \colon E & \_\ulcorner \times \urcorner\_ \colon E \to E \to E & \end{array}$$

Tarski universes continued

We simultaneously define

$$E : \mathcal{U}_0, \quad \Delta \colon E \to \mathrm{Ord}^{\top},$$

and then we simultaneously define

$$K \colon E \to \mathrm{Ord}^{\top}, \qquad \iota \colon (e : E) \to \langle \Delta\, e \rangle \to \langle K\, e \rangle,$$

$$\iota\text{-emb} \colon (e : E) \to \text{is-embedding}(\iota\, e).$$

Recall the E constructors:

$$\ulcorner 1 \urcorner, \ulcorner \omega + 1 \urcorner, \ulcorner + \urcorner, \ulcorner \times \urcorner, \quad \ulcorner \Sigma \urcorner \colon (e : E) \to (\langle \Delta e \rangle \to E) \to E.$$

We define $\Delta \ulcorner \mathit{brocoli} \urcorner = \text{brocoli}$ recursively.

K is defined as Δ except that rather than

$$\Delta(\ulcorner \Sigma \urcorner\, e\, A) \;=\; \sum_{x \colon \langle \Delta\ e \rangle} \Delta(A\, x) \qquad \text{we define} \qquad K(\ulcorner \Sigma \urcorner\, e\, A) \;=\; \sum_{x \colon \langle K\ e \rangle} K(\overline{A}\, x)$$

where $\overline{A}$ extends A along ι using the injectivity of $\mathrm{Ord}^{\top}$ as in the next page.

Tarski universes concluded

$$\begin{tikzcd}
\langle \Delta\ e \rangle \arrow[r, hook, "\iota\ e"] \arrow[d, "A"'] \arrow[dr] & \langle K\ e \rangle \arrow[d, dashed, "\overline{A}"] \\
E \arrow[r, "K"'] & \mathrm{Ord}^{\top}
\end{tikzcd}$$

Theorem. All compact well-ordered types we constructed above have the stronger property that every complemented subset is either empty or has a least element.

This is a property that well-ordered types don't have in general in a constructive setting.

The End