The topology of the universe

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MFPS, June 2012, Bath, UK

The topology of a Martin-Löf universe

The intrinsic topology of a Martin-Löf universe

The intrinsic topology of a Martin-Löf universe
with an application to
Rice's theorem for the universe

Martin-Löf type theory

1. A programming language with fancy types.

2. An alternative to ZFC, for constructive mathematics.

   E.g. it is claimed that it can formalize Bishop mathematics.

3. Variations/extensions implemented as NuPrl, Coq, Lego, Agda, . . .

4. Like set theory, ML type theory can be treated *naively* rather than *formally*.

   With the understanding that whatever we do should be formalizable.

   This is the approach taken in this talk. (But I have formalized the claims.)

The universe U is a large type of small types

Some typical uses of the universe include:

1. U is the collection of propositions. E.g. excluded middle is expressed as

$$\prod_{A\,:\,U} A + (A \to \mathbf{0}).$$

2. Large type of small types with a distinguished element and a binary operation:

$$\sum_{X:U} X \times (X \times X \to X).$$

3. Large type of small monoids:

$$\mathrm{Monoid} = \sum_{X:U} \sum_{e:X} \sum_{(\cdot):X\times X\to X} [e \text{ and } \cdot \text{ have suitable properties}].$$

Questions addressed in this talk

1. It is well known that types in programming languages behave like spaces.

   | What is the intrinsic topology of the universe in Martin-Löf type theory? |

2. Does the universe of types have non-trivial, extensional, decidable properties?

   That is, | does the universe have non-trivial intrinsically clopen sets? |

3. | If not, can we add clopen sets by adding elimination rules to the universe? |

   This is sometimes called large elimination.

We consider intensional Martin-Löf theory

1. A type of natural numbers $\mathbb{N}$ that supports primitive recursion.

2. Types $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ with zero, one and two elements.

3. Products $\prod$, sums Σ, identity types $=$.

4. In particular, products $\times$, sums $+$, and function types $\to$.

5. A universe U closed under the above constructions, regarded as a large type.

And perhaps more constructions and axioms (e.g. W-types, univalence, . . .).

BHK interpretation of logic

A proposition is identified with its type of "witnesses" or "realizers" or "proofs":

1. $A \wedge B = A \times B$,

2. $A \vee B = A + B$,

3. $A \Rightarrow B = A \to B$,

4. $\neg A = A \to \mathbf{0}$,

4. $\forall x \colon X (A(x)) = \prod_{x \colon X} A(x)$,

5. $\exists x \colon X (A(x)) = \sum_{x \colon X} A(x)$.

Rather than the truth of a proposition, one considers type inhabitation.

Propositional equality, or identity type

1. For any type X and elements $x, y : X$, we have the identity type $\boxed{x =_X y}$.

2. Intuitively, the type $x = y$ has at most one element.

   It is a singleton if and only if x and y are the same.

   As usual, the intuition is wrong and too naive, but still often successful.

3. More precisely, $=_X$ is the least reflexive relation.

   This is formulated by a certain induction principle J.

Is intensional MLTT really intensional?

1. We have $f = g \implies \forall x \colon X (f(x) = g(x))$.

2. Don't have $\forall x \colon X (f(x) = g(x)) \implies f = g$ or its negation.

   (Axiom of extensionality.)

Extensionality is undecided.

(Univalence decides it positively. Of course, univalence itself is undecided.)

Excluded middle

The proposition $\forall A \colon U (A \vee \neg A)$ amounts to $\prod_{A \colon U} (A + (A \to \mathbf{0}))$.

1. It cannot be proved. (Because cannot be realized in a recursive model.)

2. Its negation cannot be proved either. (Because classical sets are a model.)

We are happy to keep it undecided (hence compatible with both models).

WLPO, a relevant special case of excluded middle

The weak limited principle of omniscience.

Every binary sequence is constantly one or it isn't.

$$\forall \alpha \colon 2^{\mathbb{N}} \ (\forall n \colon \mathbb{N}(\alpha_n = 1) \ \vee \ \neg \forall n \colon \mathbb{N}(\alpha_n = 1)).$$

Also independent for the same reasons.

(In the recursive model it solves the Halting Problem.)

We are happy to also keep it undecided.

But it is a constructive taboo.

Intrinsic convergence

Making calculus students happy:

A sequence x_n converges to a limit l if and only if $x_\infty = l$.

Except, of course, that it doesn't make sense to write x_∞.

But students are seldom absolutely wrong (I boldly claim).

Extend the type $\mathbb{N}$ to a type $\mathbb{N}_\infty$ so that:

1. The sequence $0, 1, 2, \ldots, n, \ldots$ converges to the limit $\infty : \mathbb{N}_\infty$.

2. A sequence $x : \mathbb{N} \to X$ converges to a limit $x_\infty : X$ if and only if it extends to a function

$$\hat{x} : \mathbb{N}_\infty \to X$$

that maps ∞ to x_∞.

This works because all functions are "secretly" continuous.

(We don't postulate continuity axioms, and we can't prove or refute them.)

(Continuity is undecided.)

There are models that make this good

1. Johntone's topological topos.

   The universe can be modeled using a general technique by Streicher.

2. Kuratowski limit spaces. (Submodel.)

   Normann and Waagbø.

   But I will not use any model.

The type $\mathbb{N}_\infty$

Taken to be that of decreasing binary sequences.

1. We imagine $1^n 0^\omega \longrightarrow 1^\omega$, having the models in mind.

2. We notationally identify $n \sim 1^n 0^\omega$ and $\infty \sim 1^\omega$.

3. In Martin-Löf type theory, subtypes are defined using $\sum$:

$$\mathbb{N}_\infty = \sum_{\alpha \,:\, 2^{\mathbb{N}}} \prod_{n \,:\, \mathbb{N}} (\alpha_n = 0 \to \alpha_{n+1} = 0).$$

Remark

Assuming extensionality:

1. The type $\mathbb{N}$ is the initial algebra of the functor $1 + (-)$,

   as is well known.

2. The type $\mathbb{N}_\infty$ is the final co-algebra of the same functor,

   as you might have guessed.

   `www.cs.bham.ac.uk/~mhe/agda/CoNaturals.html`

Sensible to consider convergence up to isomorphism in U

We say that a sequence of types

$$X : \mathbb{N} \to U$$

converges to a limit $X_\infty : U$ if and only if it extends to a function

$$\hat{X} : \mathbb{N}_\infty \to U$$

with

$$\hat{X}_n \cong X_n, \qquad \hat{X}_\infty \cong X_\infty.$$

Theorem. The universe is indiscrete

> Every sequence of types converges to any type.

This is a theorem of Martin-Löf type theory,
assuming the axiom of extensionality as a hypothesis.

It holds for any extension of Martin-Löf type theory.

Formalized in Agda, but I want to give you an informal, rigorous proof here.

`www.cs.bham.ac.uk/~mhe/agda/TheTopologyOfTheUniverse.html`

Rice's Theorem for the Universe

We say that $P \colon U \to 2$ is extensional if $X \cong Y \implies P(X) = P(Y)$.

> For any extensional $P \colon U \to 2$ and $X, Y \colon U$, if $P(X) \neq P(Y)$ then WLPO.

It is a taboo to say that the universe has a non-trivial, extensional, decidable property.

This is a corollary of the Universe Indiscreteness Theorem.

Also holds for any extension of Martin-Löf type theory.

`www.cs.bham.ac.uk/~mhe/agda/RicesTheoremForTheUniverse.html`

Meta-theorem about Martin-Löf type theory

Without assuming extensionality this time.

> For all closed terms $P \colon U \to 2$ and $X, Y \colon U$ with a given proof of extensionality of P, there is no closed term of type $P(X) \neq P(Y)$.

WLPO is not provable, hence such a term is not definable from extensionality.

Without assuming the axiom of extensionality, fewer closed terms are definable.

This holds for any extension of Martin-Löf type theory with a recursive model.

We conclude by giving the short proofs in three slides

Every sequence of types converges to the type 1

Given $X : \mathbb{N} \to U$, define $\hat{X} : \mathbb{N}_\infty \to U$ by

$$\hat{X}_u = \prod_{k\,:\,\mathbb{N}} (u = k \to X_k).$$

Then

$$\hat{X}_n = \prod_{k\,:\,\mathbb{N}} (n = k \to X_k) \cong X_n,$$

and

$$\hat{X}_\infty = \prod_{k\,:\,\mathbb{N}} (\infty = k \to X_k) \cong \prod_{k\,:\,\mathbb{N}} (\mathbf{0} \to X_k) \cong \prod_{k\,:\,\mathbb{N}} \mathbf{1} \cong \mathbf{1}.$$

This is actually rather non-trivial, requiring subtle identity-type inductions.

Every sequence of types converges to any type

Let $X \colon \mathbb{N} \to U$ and $Y \colon U$ be given.

(i) $\boxed{X_n \longrightarrow \mathbf{1}}$ Previous slide.

(ii) $\boxed{(\mathbf{0})_n \longrightarrow \mathbf{1}}$ Special case of (i).

(iii) $\boxed{(\mathbf{0})_n \longrightarrow Y}$ Multiply (ii) by Y, and use $\mathbf{0} \times Y \cong \mathbf{0}$ and $\mathbf{1} \times Y \cong Y$.

(iv) $\boxed{(\mathbf{1})_n \longrightarrow \mathbf{0}}$ Compose (ii) with $(- \to \mathbf{0})$ and use $(\mathbf{0} \to \mathbf{0}) \cong \mathbf{1}$ and $(\mathbf{1} \to \mathbf{0}) \cong \mathbf{0}$.

(v) $\boxed{X_n \longrightarrow \mathbf{0}}$ Multiply (i) and (iv), and use $X \times \mathbf{1} \cong X$ and $\mathbf{1} \times \mathbf{0} \cong \mathbf{0}$.

(vi) $\boxed{X_n \longrightarrow Y}$ Add (iii) and (v), and use $\mathbf{0} + X_n \cong X_n$ and $Y + \mathbf{0} \cong Y$.

$$\boxed{P \colon U \to 2 \ \textbf{extensional}, \ \ X, Y \colon U, \quad P(X) \neq P(Y) \implies \textbf{WLPO}}$$

Assume w.l.o.g. that $P(X) = 0$ and $P(Y) = 1$.

By the Universe Indiscreteness Theorem, there is $Q \colon \mathbb{N}_\infty \to U$ with

$$\forall n \colon \mathbb{N}(Q(n) \cong X), \qquad Q(\infty) \cong Y.$$

Let $p \colon \mathbb{N}_\infty \to 2$ be $P \circ Q$. By the extensionality of P, we have that

$$\forall n \colon \mathbb{N}(p(n) = 0), \qquad p(\infty) = 1.$$

Hence for any given $x \colon \mathbb{N}_\infty$ we can decide whether $x = \infty$ by checking the decidable condition
$$p(x) = 1.$$

This amounts to WLPO.

Summary and more

1. The universe is intrinsically indiscrete.

   Up to isomorphism, assuming extensionality.
   We only need to know that it has very basic closure properties.

2. Hence it satisfies the conclusion of Rice's Theorem.

   So there is no hope of extending the theory with elimination rules $U \to Y$
   where Y is a small type, unless Y is indiscrete too.

   But notice that there are plenty of definable functions $U \to U$.

3. Thomas Streicher has looked at what happens in the models.

   In the topological topos, U only has constant (extensional or not) maps into 2.
   And into the Sierpinski space as well. (Indiscrete topological reflection.)
   We are joining these results in a single paper.