# Categorical axioms for functional real-number computation

**Parts 1 & 2**

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# This is joint work with Alex Simpson published in LICS'2001

With some recent additions.

Mainly done in 1998-2000 while I was at Edinburgh and then St Andrews.

# Plan for the tutorial

1. Real-number computation in Gödel's system T. (Delivered by myself.)
2. Interval objects in categories with finite products. (Delivered by myself.)
3. Interval objects in categories of interest. (Delivered by Alex.)
4. Directions and questions. (Delivered by Alex.)

Although I will resist temptation as much as possible, there will be some spoilers.

Alex's slides are available at
http://homepages.inf.ed.ac.uk/als/Talks/simpson_map.pdf

Some Haskell code for my slides is available at
http://www.cs.bham.ac.uk/~mhe/.talks/map2011/

# Main idea

What is a line segment?

We exploit the notion of convexity to both:

1. Compute with line segments.
2. Define what line segments are.

# Compare with

What are the natural numbers?

Peano axioms, Lawvere's notion of natural numbers object.

Recursion/induction is exploited to both

1. Compute with natural numbers.
2. Define what the natural numbers are.

# Back to our main idea

We exploit the notion of convexity to both:

1. Compute with line segments.

2. Define what line segments are.

Convexity says: between any two points there is a line segment.

To define convexity.
We exploit the idea that between any two points there is a midpoint.

# Main idea

We exploit the notion of convexity to both:

1. Compute with line segments.

2. Define what line segments are.

Convexity says: between any two points there is a line segment.

To define convexity.
We exploit the idea that between any two points there is a midpoint.

And that taking midpoints can be infinitely iterared.

# Wishes

1. Want to compute with real numbers without knowing how they are represented.
2. Want a natural universal property for real numbers (induction, recursion).
3. Want a definition that applies to any category (with finite products).
4. The definition should work in Set (it does).
5. It should work in any topos (get the Cauchy completion of the Cauchy reals).
6. In Top, we should get the Euclidean topology (we do).
7. In Loc or formal spaces, we should get the localic line (we don't know).

## Related work

Higgs (using initial algebras) (1978). He gets the Scott topology on $[0, \infty]$.

Escardo-Streicher (using initial-final (co)algebras in categories of domains) (1997).

Pavlovic-Pratt (using final coalgebras) (1999).

Freyd (using final coalgebras) (1999, 2008).

Our starting point for this work is Higgs, with ideas from Escardo-Streicher.

But there are common ideas with Freyd, particularly the use of midpoint algebras.

# We consider a minimal higher-type programming language

Gödel's system T. Real numbers can be encoded in many ways.

1. We fix a secret, concrete encoding (e.g. binary notation with signed digits), which we may change (e.g. for the sake of efficiency).

2. We add an abstract type for $[0,1]$ or $[-1,1]$ or $[u,v]$.

Our theory is explicitly based on convexity.

It doesn't favour, or depend on, any particular choice of an interval.

# System T extended with an interval type

System I. Our types are

$$X, Y ::= \mathbb{N} \mid \mathbb{I} \mid 1 \mid X \times Y \mid X + Y \mid X \to Y$$

The type $\mathbb{I}$ is to be interpreted as $[0, 1]$ or $[-1, 1]$, or $[u, v]$ with $u < v$.

The terms for all types except $\mathbb{I}$ are standard:

1. Zero, successor and primitive recursion for $\mathbb{N}$.
2. Projection and pairing for finite products.
3. Injections and case analysis for binary sums.
3. Lambda-calculus for function spaces.

## Constants for the type $\mathbb{I}$ when $\mathbb{I} = [u, v]$

We have four constants, where $\mathbb{I}^{\mathbb{N}}$ abbreviates $\mathbb{N} \to \mathbb{I}$:

1. $u, v : \mathbb{I}$.

   Two extreme points.

2. $\mathrm{affine} : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \to \mathbb{I}$.

   Medial recursion.

   In the models we'll have $\mathrm{affine}_A : A \to A \to \mathbb{I} \to A$ where $A$ is any convex object.

3. $\mathrm{M} : \mathbb{I}^{\mathbb{N}} \to \mathbb{I}$.

   Medial convex combination of a sequence of points.

With only this, we can naturally go a long way, as I'll show you soon.

# Interpretation when $\mathbb{I} = [u, v]$

1. $u, v : \mathbb{I}$.
2. $\mathrm{affine} : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \to \mathbb{I}$.

   The term $\mathrm{affine}\, y z : \mathbb{I} \to \mathbb{I}$ is interpreted as the unique $f(x) = ax + b$ with $f(u) = y$ and $f(v) = z$.

3. $\mathrm{M} : \mathbb{I}^{\mathbb{N}} \to \mathbb{I}$.

   The term $\mathrm{M}\,\vec{x} = \mathrm{M}(x_0, x_1, x_2, \dots) = \underset{n}{\mathrm{M}}\, x_n$ is interpreted as the convex combination
$$\sum_{n \geq 0} x_n 2^{-n-1} = \frac{x_0}{2} + \frac{x_1}{4} + \frac{x_2}{8} + \cdots.$$

# Definability

We can define the midpoint operation $m\colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$,

$$m(x, y) = (x + y)/2,$$

by

$$m(x, y) = \mathrm{M}(x, y, y, y, y, \dots).$$

# Equational logic

The midpoint operation $m$ is a medial mean or midpoint algebra:

$$\begin{aligned} m(x,x) &= x, \\ m(x,y) &= m(y,x), \\ m(m(x,y),m(s,t)) &= m(m(x,s),m(y,t)). \end{aligned}$$

These are called idempotency, commutativity and transposition.

Semilattices satisfy this. But they don't have cancellation:

$$m(x,y) = m(x,z) \implies y = z.$$

# Equational logic

$$\underset{n}{\mathrm{M}}\, x_n = m\left(x_0, \underset{n}{\mathrm{M}}\, x_{n+1}\right).$$

Informally, $\mathrm{M}$ is the infinitely iterated midpoint operation:

$$\underset{n}{\mathrm{M}}\, x_n = m(x_0, m(x_1, m(x_2, m(x_3, \ldots)))).$$

# Equational logic

$$\mathop{\mathrm{M}}_{i} x = x,$$

$$\mathop{\mathrm{M}}_{i} \mathop{\mathrm{M}}_{j} x_{ij} = \mathop{\mathrm{M}}_{i} \mathop{\mathrm{M}}_{j} x_{ji},$$

$$m\left(\mathop{\mathrm{M}}_{i} x_i, \mathop{\mathrm{M}}_{i} y_i\right) = \mathop{\mathrm{M}}_{i} m(x_i, y_i).$$

We don't know whether these equations are complete in the absence of cancellation.

## A first glimpse of the universal property

The function $\mathrm{affine}\, xy \colon \mathbb{I} \to \mathbb{I}$ is the unique map $h \colon \mathbb{I} \to \mathbb{I}$ with

$$\begin{aligned} h(u) &= x, \\ h(v) &= y, \\ h(m(z,t)) &= m(h(z), h(t)). \end{aligned}$$

It is also (automatically) an $\mathrm{M}$ homomorphism:

$$h\left(\mathop{\mathrm{M}}_{n} z_n\right) = \mathop{\mathrm{M}}_{n} (h(z_n)).$$

Preliminary justification of the view that $\mathrm{affine}$ is a medial recursion combinator.

## Interpretation when $\mathbb{I} = [0, 1]$

Special case, which highlights the role of convexity in our axiomatization:

1. $0, 1 : \mathbb{I}$.
2. $\mathrm{affine} : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \to \mathbb{I}$ is binary convex combination:

$$\mathrm{affine}\, xyp = px + (1 - p)y.$$

3. $\mathrm{M} : \mathbb{I}^{\mathbb{N}} \to \mathbb{I}$ is still medial convex combination.

# Definability

$$
\begin{aligned}
1 - x &= \operatorname{affine} 10x, \\
xy &= \operatorname{affine} 0xy.
\end{aligned}
$$

Can also define all rational numbers in $\mathbb{I}$:

1. They have periodic binary expansions.
2. Define a sequence of *numbers* zero and one, then apply $\mathrm{M}$ to it.

## Medial power series

Suppose

$$f(x) = \sum_n a_n x^n.$$

Then

$$\begin{aligned} \frac{1}{2} f\left(\frac{x}{2}\right) &= \frac{1}{2} \sum_n a_n x^n 2^{-n} \\ &= \sum_n a_n x^n 2^{-n-1} \\ &= \underset{n}{\mathrm{M}}\, a_n x^n. \end{aligned}$$

Write $f_{\mathrm{M}}(x) = \frac{1}{2} f\left(\frac{x}{2}\right)$. This is the medial modification of $f$.

# The medial power series functional

We have a functional

$$\text{powerseries}\colon \mathbb{I}^{\mathbb{N}} \to (\mathbb{I} \to \mathbb{I})$$

defined by

$$\text{powerseries}\,\vec{a} = \lambda x. \underset{n}{\mathrm{M}}\, a_n x^n.$$

# Medial power series in continuous models

In a model of continuous functionals (e.g. compactly generated spaces, sequential spaces, QCB spaces), the function $\mathrm{M}$ and hence

$$\mathrm{powerseries}\colon \mathbb{I}^{\mathbb{N}} \to (\mathbb{I} \to \mathbb{I})$$

are continuous.

By the Tychonoff theorem, $\mathbb{I}^{\mathbb{N}}$ is compact. Hence so is the image of $\mathrm{powerseries}$.

(A natural compact set of continuous functions without invoking Arzela–Ascoli.)

## We often find $\mathbb{I} = [-1, 1]$ more convenient to work with

From now on I will adopt this notational convention.

# Definability in system T extended with $\mathbb{I} = [-1, 1]$

$$
\begin{aligned}
x \oplus y &= \frac{x+y}{2} = m(x,y) = \mathrm{M}(x,y,y,y,\ldots), \\
-x &= \mathrm{affine}\,1(-1)x, \\
0 &= -1 \oplus 1, \\
xy &= \mathrm{affine}(-x)xy, \\
f_{\mathrm{M}}(x) &= \underset{n}{\mathrm{M}}\, a_n x^n,
\end{aligned}
$$

Rational numbers.

The commutativity and associativity laws for multiplication are a consequence of the universal property of $\mathrm{affine}$, and so are $x(-1) = -x$, $x0 = 0$ and $x1 = x$.

## Some functions definable by medial power series

Automatically convergent and continuous ($\mathbb{M}$ is total and continuous).

$$
\begin{aligned}
\frac{1}{2-x} &= \mathop{\mathrm{M}}_n x^n \\
\exp_{\mathrm{M}}(x) &= \mathop{\mathrm{M}}_n x^n/n! \\
\sin_{\mathrm{M}}(x) &= \mathop{\mathrm{M}}_n \operatorname{parity}(n)(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor} x^n/n! \\
\frac{1}{2}\ln\left(1+\frac{x}{2}\right) &= \mathop{\mathrm{M}}_n (-1)^n x^{n+1}/(n+1). \\
\frac{1}{2}\sqrt{1+\frac{x}{2}} &= \mathop{\mathrm{M}}_n \frac{(-1)^n(2n)!}{(1-2n)(n!)^2 4^n} x^n
\end{aligned}
$$

And many others.

# Implementation of the interval abstract data type

One can use, among other representations of real numbers:

1. Cauchy sequences of rationals.
2. Rational continued fractions.
3. Binary notation with negative digits.
4. Nested sequences of rational intervals.

And for efficiency one can replace rational numbers by dyadic numbers.

# Implementation using binary notation with signed digits

```
type I = [Int]  -- Represents [-1,1] in binary using digits -1,0,1.
minusOne, one :: I
minusOne  = repeat (-1)
one       = repeat 1
type J = [Int]  -- Represents [-n,n] in binary using digits |d| <= n, for any n.
divideBy :: Int -> J -> I
divideBy n (a:b:x) = let d = 2*a+b
                      in if d < -n then -1 : divideBy n (d+2*n:x)
                   else if d >  n then  1 : divideBy n (d-2*n:x)
                                  else  0 : divideBy n (d:x)
mid :: I -> I -> I
mid x y = divideBy 2 (zipWith (+) x y)
bigMid :: [I] -> I
bigMid = (divideBy 4).bigMid' where bigMid'((a:b:x):(c:y):zs) = 2*a+b+c : bigMid'((mid x y):zs)
affine :: I -> I -> I -> I
affine a b x = bigMid  [h d | d <- x]
  where h (-1) = a
        h   0  = mid a b
        h   1  = b
```

## Implementation using the term algebra of $\mathbb{M}$

Even shorter (and way more inefficient in practice):

```
data I = MinusOne | One | M [I]

affine :: I -> I -> I -> I
affine x y = h
 where h MinusOne = x
       h One      = y
       h (M zs)   = M [h z | z <- zs]
```

The shortest ever implementation of an abstract data type for real numbers.

# Implementation using the term algebra of $\mathbb{M}$

Rename to compare with the previous implementation:

```
data I' = MinusOne | One | M [I']

affine' :: I' -> I' -> I' -> I'
affine' x y = h
 where h MinusOne = x
       h One      = y
       h (M zs)   = M [h z | z <- zs]
```

## These two implementations are inter-translatable

One translation: signed-digit binary expansions are particular $\mathbb{M}$ terms.

```
inclusion :: I -> I'
inclusion x = M [h d | d <- x]
 where h (-1) = MinusOne
       h   0  = M (MinusOne : repeat One)
       h   1  = One
```

The other: Evaluation of a term in a particular algebra.

```
eval :: I' -> I
eval MinusOne = minusOne
eval One      = one
eval (M xs)   = bigMid [eval x | x <- xs]
```

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```

Evaluation can also be seen as a “kind of” normalization procedure.

## Brief pause to quickly look at some Haskell code and run it

These algorithms are written in the system T fragment of Haskell.

And they run fast (but not fast enough to beat our competitors).

# Non-definability

Theorem. The *truncated doubling* function $\mathrm{double}\colon \mathbb{I} \to \mathbb{I}$,

$$\mathrm{double}(x) = \max(-1, \min(2x, 1)) = \begin{cases} -1 & \text{if } x \in \left[-1, -\frac{1}{2}\right], \\ 2x & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

is computable but not system I definable.

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is computable but not system I definable.

Lemma. If $f : \mathbb{I}^n \to \mathbb{I}$ is definable and $\vec{x}, \vec{y} \in \mathbb{I}^n$ are such that $x_i = y_i$ whenever $x_i \in \{-1, 1\}$, then $f(\vec{x}) \in \{-1, 1\}$ implies $f(\vec{x}) = f(\vec{y})$.

Hence: If $f : \mathbb{I} \to \mathbb{I}$ is definable and $f(x) \in \{-1, 1\}$ for some $x \in (-1, 1)$ then $f$ is a constant function.

## Definability from the truncated doubling function

Define the truncation map $\mathrm{I}\colon \mathbb{R} \to \mathbb{I}$ by $\mathrm{I}(x) = \max(-1, \min(x, 1))$ so that it is the identity on $\mathbb{I}$.

Lemma. The functions on the left-hand side are definable from $\mathrm{double}$.

$$
\begin{array}{rcll}
\mathrm{I}(x+y) & = & \mathrm{double}(x \oplus y) & \text{truncated sum} \\
x \ominus y & = & x \oplus (-y) & \\
\mathrm{I}(x-y) & = & \mathrm{double}(x \ominus y) & \text{truncated subtraction} \\
\max(0, x) & = & \mathrm{I}\left(\frac{1}{2}\,\mathrm{double}\left(\mathrm{I}\left(x - \frac{1}{2}\right)\right) + \frac{1}{2}\right) & \\
\max(x, y) & = & \mathrm{double}\left(\mathrm{I}\left(\frac{x}{2} + \max\left(0, y \ominus x\right)\right)\right) & \\
\min(x, y) & = & -\max(-x, -y) & \\
|x| & = & \max(-x, x) &
\end{array}
$$

# Natural to add double as primitive

1. The four primitive operations we have selected form the basis of our categorical definition of interval object.

2. But the doubling function will play an important role in definability.

One can also add the maximum-value functional

$$\mathrm{Max}\colon (\mathbb{I} \to \mathbb{I}) \to \mathbb{I}$$

But this goes beyond system T, amounting to a manifestation of the *fan functional*.

# Systems I and II

System I: system T extended with $(\mathbb{I}, -1, 1, \mathrm{affine}, \mathrm{M})$.

System II: system I extended with $\mathrm{double}$.

Read "system double-I".

# Definability in system II

In both systems, all definable functions $\mathbb{I}^n \to \mathbb{I}$ are of course computable.

Theorem. (First-order relative computational completeness.)

In system II,

1. Every computable $\mathbb{I}^n \to \mathbb{I}$ is definable relatively to some computable $\mathbb{N} \to \mathbb{N}$.
2. Every continuous $\mathbb{I}^n \to \mathbb{I}$ is definable relatively to some oracle $\mathbb{N} \to \mathbb{N}$.

Of course, system T doesn't define all computable functions $\mathbb{N} \to \mathbb{N}$.

# Definability in system II

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The proof applies a reduction of limit computations to medial convex combinations, and the Stone-Weierstrass approximation theorem.

Before the proof, we consider some corollaries.

# Definability in Gödel's system T

With signed-digit binary notation for the interval $\mathbb{I} = [-1, 1]$:

Corollary. (First-order *relative* computational completeness.)
Any computable $\mathbb{I}^n \to \mathbb{I}$ is T definable relatively to some computable $\mathbb{N} \to \mathbb{N}$.

Proof. Because $-1, 1, \mathrm{affine}, \mathrm{M}, \mathrm{double}$ are system T definable. QED

Corollary. (First-order *absolute* computational completeness.)
Any computable $\mathbb{I}^n \to \mathbb{I}$ is definable in system T with $\mu$-recursion.

Proof. Because all computable $\mathbb{N} \to \mathbb{N}$ are $\mu$-recursive. QED

## Definability of limits in system II

Lemma. If $(x_n)_{n>0}$ is a fast Cauchy sequence in $\mathbb{I}$, that is,

$$|x_m - x_n| \leq 2^{-n} \text{ for every } m \geq n,$$

then, defining $x_0 = 0$, we have

$$\lim_n x_n = \text{double}\left(\mathop{\mathrm{M}}_n \text{double}^{n+1}(x_{n+1} \ominus x_n)\right).$$

Proof. The sum is the limit of the partial sums, which are $\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \ldots$. QED

# Definability of the limit operator in system II

Corollary. There is a (total, continuous) system II definable functional

$$\lim\colon \mathbb{I}^{\mathbb{N}} \to \mathbb{I}$$

such that $\lim \vec{x}$ is the limit of $\vec{x}$ for any fast Cauchy sequence $\vec{x} \in \mathbb{I}^{\mathbb{N}}$.

This is a definable instance of Tietze's extension theorem, because the fast Cauchy sequences form a closed subspace of $\mathbb{I}^{\mathbb{N}}$.

# Definability of truncated polynomials

Recall the truncation map $\mathrm{I}\colon \mathbb{R} \to \mathbb{I}$ defined by $\mathrm{I}(x) = \max(-1, \min(x, 1))$.

Lemma. For every rational polynomial $p\colon \mathbb{I}^n \to \mathbb{R}$, the truncated polynomial $\mathrm{I} \circ p\colon \mathbb{I}^n \to \mathbb{I}$ is system II definable.

## Definability of truncated polynomials

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Lemma. For every rational polynomial $p\colon \mathbb{I}^n \to \mathbb{R}$, the truncated polynomial $\mathrm{I} \circ p\colon \mathbb{I}^n \to \mathbb{I}$ is system II definable.

Proof. Express the polynomial as a sum of monomials.
Consider $q(\vec{x}) =$ sum of the absolute values of the monomials.

Let $k$ be such that $2^k$ always exceeds $q(\vec{x})$ over the compact space $\mathbb{I}^n$.

Then one can use the truncated arithmetic operations to define the scaled polynomial $p(\vec{x})/2^k$, without any truncations arising.

The desired $\mathrm{I} \circ p(\vec{x})$ is then defined as $\mathrm{double}^k(p(\vec{x})/2^k)$. QED.

## Proof of the first-order definability theorem

Let $f\colon \mathbb{I}^n \to \mathbb{I}$ be continuous. By Stone–Weierstrass, there is a Cauchy sequence of rational polynomials $g_i\colon \mathbb{I}^n \to \mathbb{R}$ such that

$$||g_i - f|| \leq 2^{-i}$$

Then $f$ is definable from an oracle that enumerates the lists of rational coefficients of the $g_i$'s, because

$$f(\vec{x}) = \lim_i \mathrm{I}(g_i(\vec{x})).$$

The computable case uses a computable version of Stone–Weierstrass. QED

## Universal property of $\mathbb{I}$ in categories with finite products

For the sake of clarity, I will formulate the universal property:

1. Using finite products, natural numbers object $\mathbb{N}$ and exponentials $X^{\mathbb{N}}$, first.

2. Using finite products only, at a second stage.

## The free midpoint algebra over two generators in Set

It is the set of dyadic numbers $m/2^n$ in the unit interval.

We need a completeness axiom. It will be expressed in the form of infinite iteration.

# Iterative midpoint algebras

A midpoint algebra $(A, m)$ is iterative if there is a map $\mathrm{M} \colon A^{\mathbb{N}} \to A$ and

$$\mathop{\mathrm{M}}_n a_n = m(a_0, \mathop{\mathrm{M}}_n (a_{n+1}))$$

and such that

$$(\forall i(a_i = m(x_i, a_{i+1}))) \implies a_0 = M_i x_i.$$

The second condition makes $\mathrm{M}$ to be the unique map satisfying the first equation, but is stronger than saying that the first equation has a unique solution.

By uniqueness, any midpoint homomorphism of iterative midpoint algebras is automatically an $\mathrm{M}$ homomorphism.

# Iterative midpoint algebras

The equation

$$\mathop{\mathrm{M}}_{n} a_n = m(a_0, \mathop{\mathrm{M}}_{n} (a_{n+1}))$$

amounts to the diagram

$$\begin{array}{ccc}
A \times A^{\mathbb{N}} & \xrightarrow{\mathrm{id} \times \mathrm{M}} & A \times A \\
\uparrow{\scriptstyle \langle \mathrm{head}, \mathrm{tail} \rangle} & & \downarrow{\scriptstyle m} \\
A^{\mathbb{N}} & \xrightarrow[\mathrm{M}]{} & A.
\end{array}$$

# Iteration assuming binary products only

A midpoint algebra $(A, m)$ is iterative if for every map $\langle h, t\rangle \colon X \to A \times X$ there is a unique map $\mathrm{M} \colon X \to A$ such that

$$
\begin{array}{ccc}
A \times X & \xrightarrow{\mathrm{id} \times M} & A \times A \\
{\scriptstyle \langle h, t\rangle} \uparrow & & \downarrow {\scriptstyle m} \\
X & \xrightarrow[\mathrm{M}]{} & A.
\end{array}
$$

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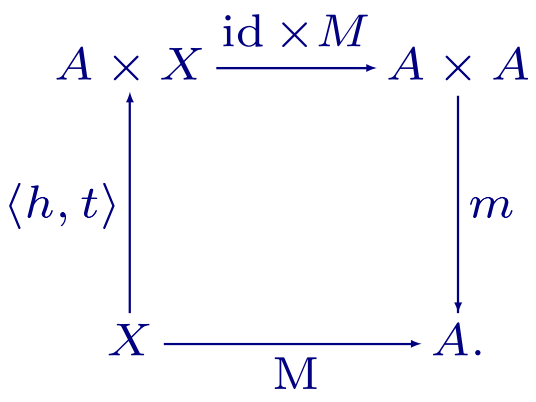


In equational form:

$$\mathrm{M}(x) = m(h(x), \mathrm{M}(t(x))).$$

## Iteration assuming binary products only

A midpoint algebra $(A, m)$ is iterative if for every map $\langle h, t \rangle \colon X \to A \times X$ there is a unique map $\mathrm{M} \colon X \to A$ such that

$$\begin{array}{ccc} A \times X & \xrightarrow{\mathrm{id} \times \mathrm{M}} & A \times A \\ \uparrow{\scriptstyle \langle h, t \rangle} & & \downarrow{\scriptstyle m} \\ X & \xrightarrow[\mathrm{M}]{} & A. \end{array}$$

If $A^{\mathbb{N}}$ exists then this is equivalent to the previous definition considering $X = A^{\mathbb{N}}$, and using the fact that $A^{\mathbb{N}}$ is a final co-algebra for the functor $- \times A$.

# Convex body

An (abstract) convex body is a cancellative, iterative midpoint algebra.

There are many concrete examples.

These examples and the role of cancellation will be elaborated in Alex's lectures.

## Interval object in a category with finite products

An interval object is a convex body freely generated by two global points:

1. Consider the category of convex bodies with two given points.
2. Midpoint-algebra homomorphisms that preserve the points.
3. An interval object is an initial object.

## Interval object in a category with finite products

An interval is a convex body $\mathbb{I} = [u, v]$ with points $u, v \colon 1 \to \mathbb{I}$ such that for any convex body $A$ with points $a, b \colon 1 \to A$ there is a unique $h \colon \mathbb{I} \to A$ such that

$$\begin{aligned} h(u) &= a, \\ h(v) &= b, \\ h(m(x, y)) &= m(h(x), h(y)). \end{aligned}$$

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$$\begin{aligned} h(u) &= a, \\ h(v) &= b, \\ h(m(x, y)) &= m(h(x), h(y)). \end{aligned}$$

In the absence of function spaces, we consider a parametric version of the notion.

In their presence, there is $\mathrm{affine} \colon A \to A \to \mathbb{I} \to A$ such that $h = \mathrm{affine}\, ab$.

# Convex bodies really are convex if there is an interval object

Let $\mathbb{I} = [0,1]$ be an interval object.

Then for any two points $a_0$ and $a_1$ of a convex body $A$ there is a parametrized line segment $h = \mathrm{affine}\, a_0 a_1 \colon \mathbb{I} \to A$ with $h(0) = a_0$ and $h(1) = a_1$.

Moreover, the axioms for binary convex combinations $\mathrm{affine}\, xyp$ with $x, y \in A$ and $p \in [0,1]$, suggestively written as

$$\mathrm{affine}\, xyp = px + (1-p)y,$$

are derivable from the universal property of $[0,1]$.

# Model of system I

Cartesian closed category with natural numbers object and interval object.

Thus, we see that in any such category one can define plenty of real arithmetic.

Cartesian closedness and a natural numbers object are not really necessary to get plenty of real arithmetic, provided we consider a parametric interval object.

# Examples of interval objects

Alex's lectures:

Sets: $[u, v]$ with standard midpoint.

Elementary topos with NNO: Cauchy completion of the Cauchy-reals interval.

Topological spaces: $[u, v]$ with the Euclidean topology.

Constructive sets, MLTT, etc: You'll see.

Locales and formal spaces: Don't know.

## Convex bodies freely generated by an object

There are plenty of examples.

1. The generating space is the $n$-point discrete space.
2. The generating space is the Sierpinski space.
3. The generating space is a domain.

And more (worked out or to be worked out).

Again in Alex's lectures.

# Thanks!

Alex's turn now.