# On the versatile selection monad
## with applications to proof theory

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# A selection of work with Paulo Oliva and other people on *selection functions*

M.E. Synthetic topology of program types and classical spaces. ENTCS'2004.

M.E. Infinite sets that admit fast exhaustive search. LICS'2007.

M.E. Exhaustible sets in higher-type computation. LMCS'2008.

M.E.& P.O. Selection functions, bar induction and backward induction. MSCS'2010.

M.E.& P.O. The Peirce translation and the double negation shift. CiE'2010, LNCS extended version in APAL.

M.E.& P.O. Computational interpretations of analysis via products of selection functions. CiE'2010, LNCS.

M.E.& P.O. What Sequential Games, the Tychonoff Theorem and the Double-Negation Shift have in Common. MSFP 2010, ACM SIGPLAN.

M.E.& P.O. Sequential games and optimal strategies. Proceedings of the Royal Society A, 2011.

M.E. & P.O. & Thomas Powell System T and the product of selection functions. CSL'2011.

M.E. & P.O. Bar recursion and products of selection functions. Journal of Symbolic Logic, 80(1):1-28, 2015.

## Seemingly disparate constructions

1. Topology. Tychonoff Theorem:

   $X_i$ compact $\implies \prod_i X_i$ also compact.

2. Higher-type computation. Computational Tychonoff Theorem:

   $X_n$ exhaustively searchable $\implies \prod_n X_n$ also exhaustively searchable.

   The point is that we get infinite exhaustively searchable sets.

3. Game theory.

   Optimal plays of sequential games of unbounded length. Nash equilibria.

4. Proof theory. Double Negation Shift:

   $\forall n \in \mathbb{N}(\neg\neg A(n)) \implies \neg\neg\forall n \in \mathbb{N}(A(n))$.

## What do they have in common?

Implemented/realized by a certain countably infinite product of selection functions.

Given $R$, define a type of *selection functions* on $X$ by

$$J(X) = ((X \to R) \to X).$$

Then we have a countable-product functional

$$\bigotimes\colon \prod_n J(X_n) \to J\left(\prod_n X_n\right).$$

This is uniquely determined by the equation

$$\bigotimes_n \varepsilon_n = \varepsilon_o \otimes \bigotimes_n \varepsilon_{n+1},$$

in certain models, for a certain binary product of selection functions

$$\otimes : J(X) \times J(Y) \to J(X \times Y).$$

This higher-type functional that can be implemented in practice in e.g. the functional programming language Haskell.

```
type J r x = (x -> r) -> x
bigotimes :: [J r x] -> J r [x]
bigotimes (e:es) p = a : bigotimes es (p.(a:))
 where a = e(\x -> p(x : bigotimes es (p.(x:))))
```

# Selection functions $(X \to R) \to X$

$X$ set of things.

Goods in a store; possible moves of a game; proofs of a proposition; points of a space.

$R$ set of values.

Prices; outcomes win, lose, draw; how much money you win; true or false; proofs again.

$X \xrightarrow{p} R$ value judgement.

How you value it; how much it costs you; pay-off of a move; propositional function.

$(X \to R) \xrightarrow{\varepsilon} X$ selects something according to some criterion.

The best, the cheapest, any, something odd.

# Example 1

$X$ set of goods.

$R$ set of prices.

$X \xrightarrow{p} R$ table of prices.

$(X \to R) \xrightarrow{\varepsilon} X$ selects a cheapest good in a given table.

$(X \to R) \xrightarrow{\phi} R$ determines the lowest price in a given table.

Fundamental equation:

$$p(\varepsilon(p)) = \phi(p).$$

This says that the price of a cheapest good is the lowest in the table.

$$\phi = \mathrm{minvalue} \qquad \varepsilon = \mathrm{argmin},$$

$$p(\mathrm{argmin}(p)) = \mathrm{minvalue}(p).$$

# Example 2

$X$ set of individuals.

$R$ set of booleans $\mathrm{false} = 0 < 1 = \mathrm{true}$.

$X \xrightarrow{p} R$ property.

$(X \to R) \xrightarrow{\varepsilon} X$ selects an individual with the highest truth value.

$(X \to R) \xrightarrow{\phi} R$ determines the highest value of a given property.

Fundamental equation:

$$p(\varepsilon(p)) = \phi(p)$$

$$\begin{aligned} \phi &= \mathrm{maxvalue} = \exists \\ \varepsilon &= \mathrm{argmax} = \mathrm{arg}\text{-}\exists = \text{Hilbert's choice operator} \end{aligned}$$

$p(\varepsilon(p)) = \exists(p)$ Hilbert's definition of $\exists$ in his $\varepsilon$-calculus

## Maximum-Value Theorem

Let $X$ be a compact non-empty topological space.

| Any continuous function $p\colon X \to \mathbb{R}$ attains its maximum value. |
| --- |

This means that there is $a \in X$ such that

$$\sup p = p(a).$$

However, the proof is non-constructive when e.g. $X = [0, 1]$.

A maximizing argument $a$ cannot be algorithmically calculated from $p$.

Of course, there is a Minimum-Value Theorem too.

## Mean-Value Theorem

Any continuous function $p\colon [0,1] \to \mathbb{R}$ attains its mean value.

There is $a \in [0,1]$ such that

$$\int p = p(a).$$

Again this $a$ cannot be found from $p$ using an algorithm.

# Universal-Value Theorem

Let $X$ be a non-empty set and $2 = \{0, 1\}$ be the set of booleans.

| |
|---|
| Any $p\colon X \to 2$ attains its universal value. |

There is $a \in X$ such that

$$\forall p = p(a).$$

This is again a classical statement if the set $X$ is infinite.

This is usually formulated as the Drinker Paradox:

| |
|---|
| In any inhabited pub there is a person $a$ s.t. if $a$ drinks then everybody drinks. |

We've also met the Existential-Value Theorem.

# General situation

With $\phi$ among $\exists, \forall, \sup, \inf, \int, \ldots$, we have that

$$\boxed{\phi(p) = p(a)}$$

for some $a$ depending on $p$.

In favourable circumstances, $a$ can be calculated as

$$a = \varepsilon(p),$$

so that

$$\boxed{\phi(p) = p(\varepsilon(p))}$$

# Selection function

Definition.

A selection function for a (logical, arithmetical, . . . ) quantifier

$$\phi \colon (X \to R) \to R$$

is a functional

$$\varepsilon \colon (X \to R) \to X$$

such that

$$\phi(p) = p(\varepsilon(p)).$$

## Monad morphism

Every $\varepsilon\colon (X \to R) \to X$ is the selection function of some $\phi\colon (X \to R) \to R$.

Namely $\phi = \overline{\varepsilon}$ defined by

$$\overline{\varepsilon}(p) = p(\varepsilon(p)).$$

This construction defines a monad morphism $\theta\colon J \to K$:

$$\overbrace{(X \to R) \to X}^{JX} \xrightarrow{\theta} \overbrace{(X \to R) \to R}^{KX}$$

$$\varepsilon \longmapsto \overline{\varepsilon}$$

This is a morphism from the selection monad to the quantifier monad.

# Units of the monads

$$\begin{array}{ccc} X & \stackrel{\eta}{\longrightarrow} & KX \\ x & \longmapsto & \lambda p.p(x). \end{array}$$

| Quantifies over the singleton $\{x\} \subseteq X$ |
|---|

$\eta(x) = \exists_{\{x\}} = \forall_{\{x\}}$.

$$\begin{array}{ccc} X & \stackrel{\eta}{\longrightarrow} & JX \\ x & \longmapsto & \lambda p.x. \end{array}$$

| Produces a selection function for the singleton quantifier |
|---|

$\overline{\eta^J(x)} = \eta^K(x)$.

## Functors of the monads

$$\begin{array}{rcl} X & \xrightarrow{f} & Y \\ KX & \xrightarrow{Kf} & KY \\ \phi & \longmapsto & \lambda p.\phi(\lambda x.p(f(x))) \end{array}$$

If $\phi$ quantifies over a set $S \subseteq X$, then $Kf(\phi)$ quantifies over the set $f(S) \subseteq Y$.

$$\begin{array}{rcl} JX & \xrightarrow{Jf} & JY \\ \varepsilon & \longmapsto & \lambda p.f(\varepsilon(\lambda x.p(f(x)))). \end{array}$$

If $\varepsilon$ is a selection function for $\phi$, then $Jf(\varepsilon)$ is a selection function for $Kf(\phi)$.

$$\overline{Jf(\varepsilon)} = Kf(\overline{\varepsilon}).$$

# Multiplications

$$\begin{array}{rcl} KKX & \stackrel{\mu}{\longrightarrow} & KX \\ \Phi & \longmapsto & \lambda p.\Phi(\lambda\phi.\phi(p)). \end{array}$$

If $\Phi$ existentially quantifies over a set of existential quantifiers that quantify over some sets, then the resulting quantifier quantifies over the union of these sets.

$$\begin{array}{rcl} JJX & \stackrel{\mu}{\longrightarrow} & JX \\ E & \longmapsto & \lambda p.E(\lambda\varepsilon.\overline{\varepsilon}(p))(p). \end{array}$$

Use the selection function $E$ to find a selection function $\varepsilon$ such that $\overline{\varepsilon}(p)$, and apply this resulting selection function to $p$ to find an element of $X$.

$$\overline{\mu^J(E)} = \mu^K(\overline{E}).$$

## Monad algebras

$K A \to A$.

$((A \to R) \to R) \to A$.

Double-negation elimination.

Explains the Gödel–Gentzen translation of classical into intuitionistic logic.

$J A \to A$.

$((A \to R) \to A) \to A$.

Peirce's Law.

Get different proof translation of classical into intuitionistic logic.

## Aside: we get a more conceptual explanation of call/cc

The type of the CPS translation of call/cc can be written as $JKX \to KX$.
(An instance of Peirce's Law, as first observed by Tim Griffin.)

Its $\lambda$-term can be reconstructed as follows:

1. $KX$ is a $K$-algebra, with structure map $\mu\colon KKX \to KX$.

2. Because we have a morphism $J \xrightarrow{\theta} K$, every $K$-algebra is a $J$-algebra:

$$JA \xrightarrow{\theta_A} KA \xrightarrow{\alpha} A.$$

3. Call/cc is what results for $A = KX$ and $\alpha = \mu$:

$$JKX \xrightarrow{\theta_{KX}} KKX \xrightarrow{\mu} KX.$$

# Strengths

$$\begin{array}{rcl} X \times KY & \xrightarrow{t} & K(X \times Y) \\ (x, \phi) & \longmapsto & \lambda p.\phi(\lambda y.p(x,y)). \end{array}$$

If $\phi$ quantifies over $S \subseteq Y$, then $t(x,\phi)$ quantifies over $\{x\} \times S \subseteq X \times Y$.

$$\begin{array}{rcl} X \times JY & \xrightarrow{t} & J(X \times Y) \\ (x, \varepsilon) & \longmapsto & \lambda p.(x, \varepsilon(\lambda y.p(x,y))). \end{array}$$

This produces a selection function for the above quantifier.

# We have monoidal-monad structures

Because we have strong monads $T = J$ and $T = K$ on a ccc.

$$
\begin{array}{rcll}
TX \times TY & \xrightarrow{\otimes} & T(X \times Y) & \\
(u, v) & \longmapsto & (T(\lambda x.t_{X,Y}(x, v)))(u) & \longleftarrow \text{we want this one,} \\
(u, v) & \longmapsto & (T(\lambda y.t_{Y,X}(u, x)))(v) & \longleftarrow \text{not this one.}
\end{array}
$$

The monads are not commutative.

The order in which you do things matters. (Illustrated below.)

# Examples

$$\begin{array}{ccc} KX \times KY & \xrightarrow{\otimes} & K(X \times Y) \\ (\exists_A, \exists_B) & \longmapsto & \exists_{A\times B}. \end{array}$$

$$\begin{array}{ccc} KX \times KY & \xrightarrow{\otimes} & K(X \times Y) \\ (\forall_A, \exists_B) & \longmapsto & \lambda p.\forall x \in A.\exists y \in B.p(x,y). \end{array}$$

The other choice of $\otimes$ concatenates the quantifiers in reverse order.

## Because we have a strong monad morphism:

$$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}.$$

In other words:

Theorem.

If

$\varepsilon \in JX$ is a selection function for the quantifier $\phi \in KX$,

$\delta \in JY$ is a selection function for the quantifier $\gamma \in KY$,

then

$\varepsilon \otimes \delta$ is a selection function for the quantifier $\phi \otimes \gamma$.

## Binary product of quantifiers and selection functions

In every pub there are a man $b$ and a woman $c$ such that if $b$ buys a drink for $c$ then every man buys a drink for some woman.

## Binary product of quantifiers and selection functions

In every pub there are a man $b$ and a woman $c$ such that if $b$ buys a drink for $c$ then every man buys a drink for some woman.

If $X =$ set of men and $Y =$ set of women, and if we define $\phi = \forall \otimes \exists$, i.e.

$$\phi(p) = (\forall x \in X \; \exists y \in Y \; p(x, y)),$$

then our claim amounts to

$$\phi(p) = p(a)$$

for a suitable pair $a = (b, c) \in X \times Y$,

This is calculated as $a = (\varepsilon \otimes \delta)(p)$ where $\overline{\varepsilon} = \forall_X$ and $\overline{\delta} = \exists_Y$, using the theorem.

## The infinite strength of the selection monad

In certain categories of interest

There is a countable monoidal-monad structure

$$\bigotimes : \prod_n JX_n \to J \prod_n X_n$$

uniquely determined by the equation

$$\bigotimes_n \varepsilon_n = \varepsilon_o \otimes \bigotimes_n \varepsilon_{n+1}.$$

Turns out to be a form of *Bar recursion.*

# The continuation monad lacks infinite strength

However, if a sequence of quantifiers $\phi_n$ have selection functions $\varepsilon_n$,

then their product can be defined as

$$\bigotimes_n \phi_n = \overline{\bigotimes_n \varepsilon_n}$$

and uniquely satisfies

$$\bigotimes_n \phi_n = \phi_o \otimes \bigotimes_n \phi_{n+1}.$$

This is useful for various applications, including the double negation shift.

# What does $\bigotimes$ do?

Many things!

1. Designed to implement a computational version of the countable Tychonoff Theorem.
2. It turns out to compute optimal plays of sequential games.
3. It happens to realize the double-negation shift.
4. In the finite case, it gives Bekic's construction of fixed-point operators.
5. Among other things.

What does it do in general? Maybe (2) subsumes all cases.

# Program extraction from classical proofs with choice

Start with intuitionistic choice

$$\forall x \in X\,(\exists y \in Y_x\,(A(x,y))) \implies \exists f \in \prod_x Y_x\,(\forall x \in X\,(A(x, fx)))\,.$$

Apply the $T$-translation, say for $T = K$ or $T = J$:

$$\forall x \in X\,\big(T\exists y \in Y_x\,\big(A^T(x,y)\big)\big) \implies T\exists f \in \prod_x Y_x\,\big(\forall x \in X\,\big(A^T(x, fx)\big)\big)\,.$$

Is that realizable? Does it have a proof term?

## The $J$-shift

Think of $JA = ((A \to R) \to A)$ as a logical modality.

Theorem

The product functional $\bigotimes\colon \prod_n JX_n \to J\left(\prod_n X_n\right)$ realizes the $J$-shift

$$\forall n(J(A(n)) \to J\left(\forall n(A(n))\right).$$

To guess the theorem, apply Curry–Howard.

To prove it, use bar induction and continuity.

## Realizing the $J$-translation of countable choice

1. Start again with intuitionistic choice, but countable this time:

$$\forall n \in \mathbb{N}\,(\exists x \in X_n\,(A(n,x))) \implies \exists f \in \prod_n X_n\,(\forall n \in \mathbb{N}\,(A(n, fn)))\,.$$

2. Apply the functor $J$:

$$\underline{J(\forall n \in \mathbb{N}\,(\exists x \in X_n\,(A(n,x))))} \implies J\exists f \in \prod_n X_n\,(\forall n \in \mathbb{N}\,(A(n, fn)))\,.$$

3. Finally pre-compose with the instance of the $J$-shift

$$\forall n \in \mathbb{N}\,(J\exists x \in X_n\,(A(n,x))) \implies \underline{J\,(\forall n \in \mathbb{N}\,(\exists x \in X_n\,(A(n,x))))}.$$

**Theorem.** The $J$-translation of countable choice is realizable.

## Realizing the $K$-translation of countable choice

It suffices to realize the $K$-shift.

However, we saw that $K$ doesn't have a countable strength.

| Reduce the problem to the $J$-shift. |
|---|

We can go from $J$ to $K$ with the monad morphism.

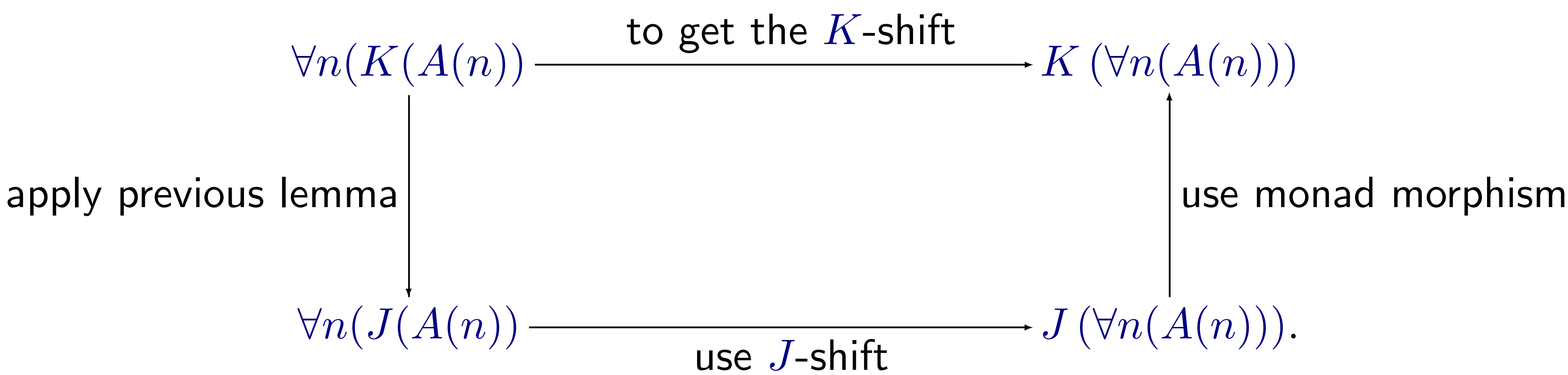
In general there is no way back.

But for formulas $A$ in the image of the $K$-translation, we have $R \to A$.

Lemma. $(R \to A) \to KA \to JA$.

Proof. Transitivity of implication.

# $K$-shift from $J$-shift




The $J$-shift is stronger than the $K$-shift, because it works for all formulas, not just the ones in the image of the translation.

## Running a classical proof with choice in Agda

We programmed/proved the infinite pigeonhole principle in Agda.

Agda is a dependently typed functional programming language based on intensional Martin-Löf Type Theory (MLTT).

Proofs literally are programs.

## This is what we do in our agda files

From a classical proof of an infinitary statement, we discuss how to get a proof of a finitary statement that we can run.

There are two main steps:

1. Friedman's A-translation and Friedman's trick, for pure classical logic.

2. Use a proof term for classical countable choice.

   We disable the termination checker in the module that defines the proof term of the $J$-shift.

# Infinite Pigeonhole Principle: mathematical formulation

If we colour the integers with finitely many colours, then infinitely many integers get same colour.

This holds classically but fails constructively.

How do we identify colour that occurs infinitely often?

We'll consider two colours without loss of generality, called $0$ and $1$.
Then $2 = \{0, 1\}$ is the set of colours.

## Infinite Pigeonhole Principle: symbolic formulation

$$\forall \alpha \in 2^{\mathbb{N}} \quad \exists b \in 2 \quad \exists g \colon \mathbb{N} \to \mathbb{N} \quad \forall i \in \mathbb{N} \quad (gi < g(i+1) \wedge \alpha(gi) = b).$$

1. The given colouring is the infinite sequence $\alpha$.
2. The colour shared by infinitely many integers is $b$.
3. The subsequence of integers with that colour is described by $g$.

## Infinite Pigeonhole Principle: classical proof

1. Either from some point on the given sequence $\alpha$ is constantly 0 or not.

2. If it is, we've found the desired colour and monochromatic subsequence.

3. Otherwise:

   (a) For every $n$ there is $i \geq n$ such that the colour of $i$ is $1$.
   (b) By the axiom of countable choice, there is $f : \mathbb{N} \to \mathbb{N}$ such that $fn \geq n$ for every $n$ and the colour of position $fn$ is $1$.
   (c) Then the desired monochromatic subsequence can be given by induction as

$$\begin{aligned} g(0) &= f(0) \\ g(n+1) &= gn + 1 + f(gn+1). \end{aligned}$$

Q.E.D.

## Infinite Pigeonhole Principle: symbolic version of the proof

1. Let $A = \exists n \; \forall i \geq n \; (\alpha_i = 0)$. By excluded middle, $A \vee \neg A$.

2. Assume $A$. Then the desired monochromatic subsequence can be taken as $g(i) = n + i$, with colour $b = 0$.

3. Assume $\neg A$. Then:

   (a) $\forall n \; \exists i \geq n \; (\alpha_i = 1)$.

   (b) By the axiom of countable choice, $\exists f : \mathbb{N} \to \mathbb{N} \; \forall n (fn \geq n \wedge \alpha_{fn} = 1)$.

   (c) Then the desired monochromatic subsequence can be taken to be

$$\begin{aligned} g(0) &= f(0) \\ g(n+1) &= gn + 1 + f(gn+1). \end{aligned}$$

Q.E.D.

## Infinite Pigeonhole Principle: avoiding "$\geq$" for simplicity

1. Let $A = \exists n \, \forall i \, (\alpha_{n+i} = 0)$. By excluded middle, $A \vee \neg A$.

2. Assume $A$. Then the desired monochromatic subsequence can be taken as $g(i) = n + i$, with colour 0.

3. Assume $\neg A$. Then, classically,

   (a) $\forall n \, \exists i \, (\alpha_{n+i} = 1)$.

   (b) By the axiom of countable choice, $\exists f : \mathbb{N} \to \mathbb{N} \, \forall n (\alpha_{n+fn} = 1)$.

   (c) Then the desired monochromatic subsequence can be taken to be

$$
\begin{aligned}
g(0) &= f(0) \\
g(n+1) &= gn + 1 + f(gn + 1).
\end{aligned}
$$

Q.E.D.

## Infinite Pigeonhole Principle: negative translation

General recipe:

Insert "$\neg\neg$" in front of "$\exists$", "$\vee$" and "$=$"

When we can get away with it, we place fewer "$\neg\neg$" than formally required.

In our example:

$$\forall \alpha \in 2^{\mathbb{N}} \quad \neg\neg\exists b \in 2 \quad \exists g\colon \mathbb{N} \to \mathbb{N} \quad \forall i \in \mathbb{N} \quad (gi < g(i+1) \wedge \neg\neg(\alpha(gi) = b)).$$

# Facts about the negative translation

1. For any formula $B$ in the image of the translation, $\neg\neg B \implies B$ has an intuitionistic proof.

2. For any formula $A$ whatsoever, $\neg\neg(A \vee \neg A)$ has an intuitionistic proof.

3. In particular, for any formula $A$ whatsoever, and any $B$ in the image of the translation, $(A \to B) \wedge (\neg A \to B) \implies B$ has an intuitionistic proof.

4. From a classical proof of a formula, we automatically get an intuitionistic proof of its translation.

   This fails if we have axioms that don't have as consequences their own translations, such as countable choice.

   That's why we had to spend some effort constructing a proof term for the translation of countable choice.

## Infinite Pigeonhole Principle: proof of negative translation

1. Let $A = \exists n \, \forall i \, \neg\neg(\alpha_{n+i} = 0)$. By intuitionistic logic, $\neg\neg(A \vee \neg A)$.

2. Assume $A$. Then the desired monochromatic subsequence can be taken as $g(i) = n + i$, with colour 0.

3. Assume $\neg A$. Then, by intuitionistic logic,

   (a) $\forall n \, \neg\neg\exists i \, \neg\neg(\alpha_{n+i} = 1)$.

   (b) By classical countable choice, $\neg\neg\exists f\colon \mathbb{N} \to \mathbb{N} \; \forall n \neg\neg(\alpha_{n+fn} = 1)$.

   (c) To conclude $\neg\neg$-existence, we can take $b = 1$ and

$$g(0) = f(0), \qquad g(n+1) = gn + 1 + f(gn + 1).$$

4. Therefore $\neg\neg\exists b \in 2 \quad \exists g\colon \mathbb{N} \to \mathbb{N} \quad \forall i \in \mathbb{N} \quad (gi < g(i+1) \wedge \neg\neg(\alpha(gi) = b))$ by intuitionistic logic. Q.E.D.

# Infinite Pigeonhole Principle: Friedman's A-translation

General recipe: Choose an arbitrary proposition $R$, playing the role of false, and define $KA = ((A \to R) \to R)$ (generalized double negation).

Insert "$K$" in front of "$\exists$", "$\vee$" and "$=$"

When we can get away with it, we place fewer "$K$" than formally required.

In our example:

$$\forall \alpha \in 2^{\mathbb{N}} \quad K\exists b \in 2 \quad \exists g \colon \mathbb{N} \to \mathbb{N} \quad \forall i \colon \mathbb{N} \quad (gi < g(i+1) \wedge K(\alpha(gi) = b)).$$

## Facts about the generalized double negation modality $K$

For any two formulas $A, B$, the following hold in intuitionistic logic:

1. $(A \to B) \to (KA \to KB)$ (strong functor)
2. $A \to KA$ (unit)
3. $KKA \to KA$ (multiplication)

It follows intuitionistically from the above that:

1. $(A \to KB) \to (KA \to KB)$ (Kleisli extension)
2. $(A \wedge KB) \to (KA \wedge KB)$ (strength)

# Facts about Friedman's translation

1. For any formula $B$ in the image of the translation, $R \implies B$ has an intuitionistic proof (ex falso quodlibet). The role of $\bot$ is played by $R$.

2. For any formula $B$ in the image of the translation, $KB \implies B$ has an intuitionistic proof.

   Translated formulas are algebras of the monad.

3. For any formula $A$ whatsoever, $K(A \vee (A \to R))$ has an intuitionistic proof.

4. In particular, for any formula $A$ whatsoever, and any $B$ in the image of the translation, $(A \to B) \wedge ((A \to R) \to B) \implies B$ has an intuitionistic proof.

5. From a classical proof of a formula, we automatically get an intuitionistic proof of its translation.

## Infinite Pigeonhole Principle: Friedman's translation

1. Let $A = \exists n\ \forall i\ K(\alpha_{n+i} = 0)$. By intuitionistic logic, $K(A \vee (A \to R))$.

2. Assume $A$. Then the desired monochromatic subsequence can be taken as $g(i) = n + i$, with colour 0.

3. Assume $(A \to R)$. Then, intuitionistically,
   (a) $\forall n\ K\exists i\ K(\alpha_{n+i} = 1)$.
   (b) By Klassical countable choice, $K\exists f \colon \mathbb{N} \to \mathbb{N}\ \forall n K(\alpha_{n+fn} = 1)$.
   (c) To conclude $K$-existence, we can take $b = 1$ and

$$g(0) = f(0), \qquad g(n+1) = gn + 1 + f(gn + 1).$$

4. Therefore $K\exists b \in 2 \quad \exists g \colon \mathbb{N} \to \mathbb{N} \quad \forall i \in \mathbb{N} \quad (gi < g(i+1) \wedge K(\alpha(gi) = b))$ by intuitionistic logic. Q.E.D.

## Corollary: Finitary version

Every infinite sequence of two colours has finite monochromatic subsequences of arbitrary length.

## Finite pigeonhole principle: symbolic version

$$\forall \alpha \in 2^{\mathbb{N}} \quad \forall m \in \mathbb{N} \quad \exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1} \qquad \forall i < m \quad (s_i < s_{i+1}) \\ \wedge \quad \forall i \leq m \quad (\alpha_{s_i} = b).$$

Using the Infinite Pigeonhole principle, we get

$$\forall \alpha \in 2^{\mathbb{N}} \quad \forall m \in \mathbb{N} \quad K\exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1} \qquad \forall i < m \quad (s_i < s_{i+1}) \\ \wedge \quad \forall i \leq m \quad K(\alpha_{s_i} = b).$$

Just take $s$ to be the restriction of $g$. Now intuitionistic logic gives

$$\forall \alpha \in 2^{\mathbb{N}} \quad \forall m \in \mathbb{N} \quad K\exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1} \qquad \forall i < m \quad (s_i < s_{i+1}) \\ \wedge \quad \forall i \leq m \quad (\alpha_{s_i} = b).$$

Can shift $K$ out of finite quantifiers and arbitrary existential quantifiers.

## Friedman's trick

Used to get rid of the last remaining $K$.

1. Suppose you have proved intuitionistically $K\exists x(Ax)$ for arbitrary $R$ and that $R$ doesn't occur in $A(x)$. (As we did.)

2. Then you get $\exists x(Ax)$ intuitionistically.

3. Indeed, $K\exists x(Ax)$ amounts to $((\exists x(Ax)) \to R) \to R$.

4. Now just take $R = \exists x(Ax)$ and we are done.

# Finite Pigeonhole from Infinite Pigeonhole

In our example,

$$\forall \alpha \in 2^{\mathbb{N}} \quad \forall m \in \mathbb{N} \quad K\exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1} \qquad \forall i < m \quad (s_i < s_{i+1}) \\ \wedge \quad \forall i \leq m \quad (\alpha_{s_i} = b),$$

take $R(\alpha, m) = \exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1}(\forall i < m \quad (s_i < s_{i+1})) \wedge \forall i \leq m \quad (\alpha_{s_i} = b)$
to get, intuitionistically,

$$\forall \alpha \in 2^{\mathbb{N}} \quad \forall m \in \mathbb{N} \quad \exists b \in 2 \quad \exists s \in \mathbb{N}^{m+1} \qquad \forall i < m \quad (s_i < s_{i+1}) \\ \wedge \quad \forall i \leq m \quad (\alpha_{s_i} = b).$$

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