Searchable sets, Dubuc-Penon compactness, Omniscience Principles, and the Drinker Paradox

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Abstract

1. A number of contenders for logical notion of compactness coincide.
2. Need the axiom of choice $\mathrm{AC}(X, 2)$ for one equivalence.
3. This is related to the topopological notion of total-separatedness.

Introduction

1. Higher-type computability.
2. Searchable sets must be compact, Tychonoff theorem.
3. Proof theory, Peirce translation, double negation shift.
4. Work by Dubuc and Penon in topos theory.

Logical notions of compactness

1. Drinker paradox.
2. Principle of omniscience.
3. Searchable sets.
4. Dubuc-Penon compactness.

Drinker paradox.

In every pub there is a person *a* such that if *a* drinks then everybody drinks.

A set X satisfies the *drinker paradox* iff

$$\forall p \colon X \to \Omega \;\; \exists a \in X \, (p(a) \implies \forall x \in X (p(x)))\,.$$

In classical logic, a set satisfies this condition if and only if it is non-empty.

Boolean drinker paradoxes

X satisfies the *boolean drinker paradox* iff

$$\forall p\colon X \to 2 \;\; \exists a \in X (pa = 0 \implies \forall x \in X (px = 0)).$$

Another version: In any pub there is a person a such that if somebody drinks then a drinks:

$$\forall p\colon X \to 2 \;\; \exists a \in X (\exists x \in X (px = 1)) \implies pa = 1.$$

Searchable sets.

X is *searchable* iff

$$\forall p\colon X \to 2 \;\; \exists a \in X \, (\neg\neg\exists x \in X (px = 1)) \implies pa = 1.$$

Remark

1. Considered a stronger definition in computability:

$$\exists \varepsilon \colon (X \to 2) \to X \;\; \forall p \colon X \to 2 (\neg\neg \exists x \in X (px = 1)) \implies p(\varepsilon p) = 1.$$

2. The axiom of choice gives the stronger version from the weaker one.
3. AC is is validated in realizability interpretations, and provable in ML type theory.
4. In this note it is more natural to take the weaker definition as the official one.

Principle of omniscience.

X satisfies the *principle of omniscience* iff

$$\forall p\colon X \to 2 \;\; (\exists x \in X(px = 1)) \vee (\forall x \in X(px = 0)).$$

Dubuc-Penon compactness

X is *Dubuc-Penon compact* iff

$$\forall A\colon \Omega \;\; \forall B\colon X \to \Omega \;\; (\forall x \in X (A \vee B(x))) \implies A \vee \forall x \in X (B(x)).$$

Boolean Dubuc-Penon compactness

We say that X is *boolean Dubuc-Penon compact* iff

$$\forall A \colon \Omega \;\; \forall B \colon X \to 2 \;\; (\forall x \in X (A \vee B(x))) \implies A \vee \forall x \in X (B(x)).$$

Summary of notions

BDP$_\forall$:
$\forall p\colon X \to 2 \;\; \exists a \in X(pa = 0 \implies \forall x \in X(px = 0)).$

BDP$_\exists$:
$\forall p\colon X \to 2 \;\; \exists a \in X(\exists x \in X(px = 1)) \implies pa = 1.$

searchable:
$\forall p\colon X \to 2 \;\; \exists a \in X\,(\neg\neg\exists x \in X(px = 1)) \implies pa = 1.$

PO:
$\forall p\colon X \to 2 \;\; (\exists x \in X(px = 1)) \vee (\forall x \in X(px = 0)).$

Dubuc-Penon compact:
$\forall A\colon \Omega \;\; \forall B\colon X \to \Omega \;\; (\forall x \in X(A \vee B(x))) \implies A \vee \forall x \in X(B(x)).$

BDP-compact:
$\forall A\colon \Omega \;\; \forall B\colon X \to 2 \;\; (\forall x \in X(A \vee B(x))) \implies A \vee \forall x \in X(B(x)).$

Theorem

The following are equivalent for any inhabited set X:

1. X is searchable.
2. X is boolean Dubuc-Penon compact.
3. X satisfies the boolean drinker paradox.
4. X satisfies the principle of omniscience.

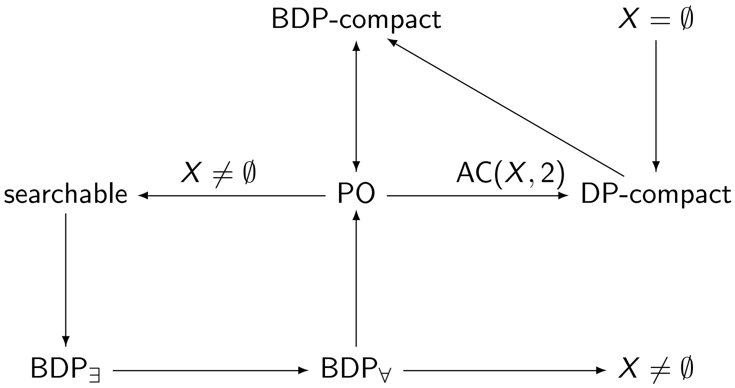
Moreover,

1. Dubuc-Penon compactness of X implies these conditions, and
2. if the axiom of choice $\mathrm{AC}(X, 2)$ holds then the converse is true.

Remark

1. In particular, this theorem holds in realizability over system T.
2. In Martin Löf type theory.
3. And we have developed it in Agda.

Proof structure and more details



Lemma

1. If $\mathrm{BDP}_{\forall}(X)$ then X is inhabited.
2. DP-compact($\emptyset$).

Proof.

Considering any predicate, say $p(x) = 0$, we get $a \in X$ by definition.

The left disjunct of the DP-compactness conclusion $A \vee \forall x \in X(B(x))$ holds vacuosly when X is empty. □

Lemma

$\text{searchable}(X) \implies \text{BDP}_{\exists}(X)$.

Proof.

$\text{BDP}_{\exists}(X)$ has a stronger premise and hence is weaker. □

Lemma

$\text{BDP}_\exists(X) \implies \text{BDP}_\forall(X)$.

Proof.

For any given $p\colon X \to 2$, the assumption produces $a \in X$ that satisfies $(\exists x \in X(px = 1)) \implies p(a) = 1$, and hence $p(a) = 0 \implies \forall x \in X(px = 0)$, and so $\text{BDP}_\forall(X)$ holds. $\square$

Lemma

$\mathrm{BDP}_{\forall}(X) \implies \mathrm{PO}(X)$.

Proof.

For any $p\colon X \to 2$, the assumption produces $a \in X$ such that $p(a) = 0 \implies \forall x \in X(px = 0)$.

Because $p(a) = 0$ is decidable, we can reason by cases.

If it holds, then $\forall x \in X(px = 0)$.

Otherwise $p(a) = 1$ and hence $\exists x \in X(p(x) = 1)$.

Therefore $\mathrm{PO}(X)$ holds. □

Lemma

PO(X) $\implies$ searchable(X) for X inhabited.

Proof.

Let $p\colon X \to 2$.

By PO(X), either $\exists x \in X(px = 1)$ or else $\forall x \in X(px = 0)$.

In the first case we take any a with $pa = 1$, and $\neg\neg\exists x \in X(px = 1) \implies pa = 1$ holds simply because the conclusion is true and so searchable(X) holds.

In the second case we have that $\neg\neg\exists x \in X(px = 1)$ is impossible,

and hence the implication $\neg\neg\exists x \in X(px = 1) \implies pa = 1$ holds for any $a \in X$,

which can be found by inhabitedness of X, and again searchable(X) holds. □

Lemma

BDP-compact(X) $\implies$ PO(X).

Proof.

Let $p\colon X \to 2$ and define $A = \exists x \in X.(px = 1)$ and $B(x) = (px = 0)$.

Then $A \vee B(x)$ holds for any $x \in X$.

In fact, because $B(x)$ is decidable, we can reason by cases.

If $B(x)$ holds, then $A \vee B(x)$.

Otherwise, $px = 1$ and hence A holds, and so does $A \vee B(x)$.

Hence $A \vee \forall x \in X(B(x))$ holds by DP-compactness of X,

which amounts to PO. □

Lemma

PO(X) $\implies$ BDP-compact(X).

Proof.

By PO, either $\exists x \in X(\neg Bx)$ or else $\forall x \in X(Bx)$.

In the first case A holds, and hence in both cases $A \vee \forall x \in X(B(x))$ holds,

which is the conclusion of boolean DP-compactness. □

If B is not decidable, then one cannot apply PO to B.

The following lemma instead applies PO to a suitable predicate constructed with the axiom of choice.

Lemma

$\mathrm{PO}(X) \implies \text{DP-compact}(X)$ if $\mathrm{AC}(X, 2)$ holds.

Proof.

Let $x \in X$ and assume that $A \vee B(x)$.

Then, reasoning by cases, there is $y \in 2$ such that $(y = 1 \implies A) \wedge (y = 0 \implies B(x))$.

By the axiom of choice, there is $p \colon X \to 2$ such that $(px = 1 \implies A) \wedge (px = 0 \implies B(x))$.

Now assume the premise $\forall x \in X(A \vee B(x))$ of DP-compactness.

By PO, either $\exists x \in X(px = 1)$ or else $\forall x \in X(px = 0)$.

In the first case A holds.

In the second case $\forall x \in X(B(x))$ holds.

Hence in both cases $A \vee \forall x \in X(B(x))$ holds, which is the conclusion of DP-compactness. □

Remark

Hence in the absence of the axiom of choice, DP-compactness is the strongest notion, for inhabited sets, among those considered here.

Remark about AC(X, 2).

Because existential quantification over 2 is disjunction, the axiom of choice AC(X, 2) amounts to

$$(\forall x\colon X(A(x,0) \vee A(x,1))) \implies \exists p\colon X \to 2(\forall x\colon X(A(x,p(x)))).$$

Hence another way of writing AC(X, 2) is

$$A_0 \cup A_1 = X \implies \exists B_0 \subseteq A_0, B_1 \subseteq A_1(B_0 \cap B_1 = \emptyset \wedge B_0 \cup B_1 = X),$$

considering $B_0 = p^{-1}(0)$ and $B_1 = p^{-1}(1)$.

Theorem

A set X is Dubuc-Penon compact if and only if

$$\forall C{,}B\colon X{\to}\Omega\ \ \forall x{\in}X(C(x){\vee}B(x)) \Longrightarrow \exists x{\in}X(C(x)){\vee}\forall x{\in}X(B(x)).$$

Proof.

($\Leftarrow$) consider $C(x) = A$.

($\Rightarrow$): Consider the proposition $A = \exists x \in X(C(x))$.

Then $\forall x \in X(C(x) \vee B(x))$ implies $\forall x \in X(A \vee B(x))$.

Thus DP-compactness transforms into $A \vee \forall x \in X(B(x))$, as required. $\square$

The axiom of choice and total separatedness

Discussion of previous work.

Totally separated sets.

X is *totally separated* if

$$\forall x, y \in X (\forall p \colon X \to 2 (p(x) = p(y)) \implies x = y.$$

Connected sets.

X is *connected* if all maps $X \to 2$ are constant.

If X is both connected and totally separated, then it has at most one point.

Hence total separatedness can be seen as a strong notion of disconnectedness.

(Weaker than total disconnectedness.)

Totally separated apartness relations.

To discuss a positive version of total separatedness, we consider apartness relations.

We say that an apartness relation $\sharp$ on X is *totally separated* if

$$\forall x, y \in X(x \sharp y \implies \exists p \colon X \to 2(p(x) \neq p(y))).$$

Apartness relation on a set X.

A binary relation $\sharp$ such that

1. $\neg(x \sharp x)$ (irreflexivity),
2. $x \sharp y \implies y \sharp x$ (symmetry),
3. $x \sharp y \implies z \sharp x \vee z \sharp y$ (co-transitivity).

Called sharp if

$$\neg(x \sharp y) \implies x = y.$$

Examples

1. The empty relation is an apartness relation that fails to be sharp but is totally separated in a trivial way,
2. If X has decidable equality then the negation $\neq$ of equality is a sharp apartness relation.
3. The reals have a sharp apartness relation.
4. A sharp apartness relation on the Cantor space $2^{\mathbb{N}}$ is given by

$$\alpha \sharp \beta \iff \exists i \in \mathbb{N}(\alpha_i \neq \beta_i).$$

   Moreover, this is totally separated, by considering $p(\gamma) = \gamma_i$ where i is a total separatedness witness.

Of course:

Lemma

If X has some totally separated, sharp apartness relation, then X is totally separated.

Proof.

Assume that $\forall p \colon X \to 2(p(x) = p(y))$.

The contra-positive of total separatedness of $\sharp$ gives the conclusion $\neg(x \sharp y)$,

which sharpness transforms into $x = y$. □

The step that relates choice to total separatedness is this:

Lemma

If AC$(X, 2)$ *holds, then any apartness relation on* X *is totally separated.*

Proof.

Assume that $x \sharp y$ and define $A(z, 0) \iff z \sharp y$ and $A(z, 1) \iff z \sharp x$.

Then, by co-transitivity, for every $z \in X$ there is $t \in 2$ such that $A(z, t)$.

By AC$(X, 2)$, there is $p \colon X \to 2$ such that $A(z, p(z))$ for all z,

which then satisfies $p(x) = 0$ and $p(y) = 1$, as required. □

Lemma

Any set X has an apartness relation given by

$$x \sharp_2 y \iff \exists p\colon X \to 2(p(x) \neq p(y)),$$

which is totally separated by construction.

Proof.

Irreflexivity and symmetry are immediate.

To prove co-transitivity, consider $p\colon X \to 2$ such that $p(x) \neq p(y)$, and let $z \in Z$.

By decidability of equality on 2, either $p(z) = p(y)$ or $p(z) = p(x)$.

In the first case $z \sharp_2 x$, and in the second case $z \sharp_2 y$, and hence $z \sharp_2 x$ or $z \sharp_2 y$, as required. □

Corollary

The relation $\sharp_2$ is the finest apartness relation if $\mathrm{AC}(X, 2)$ holds.

The apartness relation $\sharp_2$ doesn't need to be sharp. For example, if X is connected, then $\sharp_2$ is empty.

Lemma

The apartness relation $\sharp_2$ on X is sharp if and only if X is totally separated.

Proof.

($\Leftarrow$): Because $\neg(x \sharp_2 y)$ amounts to $\forall p \colon X \to 2(p(x) = p(y))$,

which total separatedness of X transforms into $x = y$.

($\Rightarrow$): Lemma 2. $\square$

Sharp sets.

Say that X is sharp if it has some sharp apartness relation.

By the above lemma, any totally separated set X is sharp, with sharpness witnessed by $\sharp_2$.

Putting the above together:

Theorem

If $\mathrm{AC}(X, 2)$ *holds,* X *is sharp if and only if it is totally separated.*

Moreover, as we have seen, in this case, any sharp apartness relation is totally separated, and $\sharp_2$ is the finest apartness relation, and is sharp.

Relevance to Dubuc-Penon compactness.

1. By the above discussion, if X is connected and has a sharp apartness relation and two distinct points, then $\mathrm{AC}(X, 2)$ fails.
2. The reals are not Dubuc-Penon compact in the models considered by Dubuc and Penon,.
3. They are boolean DP-compact in the same models because they are searchable.
4. Because these models validate connectedness of $\mathbb{R}$.

Concluding questions and speculative discussion.