A Smyth-like dictionary for $\boxed{\text{Stone Types}}$ in MLTT

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Stone spaces classically

Def.

A stone space is a
compact
totally separated (the clopens separate the points)
topological space.

Stone spaces are zero dimensional (base of clopens).

Examples

1. The Cantor space $2^{\mathbb{N}}$.
2. The one-point compactification $\mathbb{N}_\infty$ of the discrete natural numbers.
3. The topological product $\prod_{n \in \mathbb{N}} \{0, \cdots, n\}$ ← discrete
4. The set of decreasing sequences in $\mathbb{N}^{\mathbb{N}}$ bounded by K where the Baire space $\mathbb{N}^{\mathbb{N}}$ has the product topology.

1. We want to define Stone type, totally separated type, compact type, discrete type in MLTT.

2. Give (counter) examples in MLTT.

3. Prove theorems about them in MLTT.

4. Don't assume "topological" axioms.

5. Interpret the above in e.g. the topological topos and the Effective topos.

$\boxed{\text{Extensionality}}$ is unavoidable

There are two current ways to deal with it:

1. Gandy / Bishop / Martin Hofmann.
   "setoid model"

2. "HoTT/UF"
   This includes even "universe extensionality" (univalence), which we don't need here.

We adopt the HoTT/UF way of dealing with it.

Clopen

two-point discrete space

1. In classical topology

$$\text{clopens } X \simeq \text{continuous maps } X \to 2$$

2. In MLTT, we define

$$\text{clopens } X := X \to 2.$$

$2 := 1 + 1$
$0 = \text{inl } *$
$1 = \text{inr } *$

So a clopen of X is just a detachable or decidable subtype of X.

Discrete type

1. In classical topology X is discrete iff its diagonal is clopen

2. In MLTT

identity type

$$X \text{ is discrete} :\equiv \prod x, y : X . \, (x = y) + (x \neq y).$$

For example, 2, $\mathrm{Fin}\ n$ and $\mathbb{N}$ are discrete, of course.

Totally separated type

1. In classical topology X is totally separated if for every $x \neq y$ in X there is a clopen C with $x \in C$ and $y \notin C$.

   - Equivalently, if two points have the same clopen neighbourhoods, then they are equal.

2. In MLTT, (Boolean Leibniz principle)

   X is totally separated

   $$:= \prod x, y : X, \left(\prod p : X \to 2, \; px = py\right) \to x = y.$$

1. $x =_2 y := \prod p : X \to 2,\ px = py.$

2. quasi-component $x := \sum y : X,\ x =_2 y.$

3. $x \#_2 y := \exists p : X \to 2,\ px \neq py$

This is an apartness relation.

- $\neg(x \# x)$
- $x \# y \to y \# x$
- $x \# y \to x \# z \vee y \# z$

- $\neg(x \#_2 y) \to x = y$

4. Theorem. T.F.A.E.

(i) X is totally separated.

(ii) The quasi component of any point is a singleton.

(iii) The apartness relation $\_\#_2\_$ is tight

(iv) The evaluation map $X \to ((X \to 2) \to 2)$ is an embedding.

Totally separated reflection

Theorem. Let X be any type.

1. If X is totally separated, then it has $\neg\neg$-stable equality and hence X is a set in the sense of HoTT/UF.

2. X has a totally separated reflection:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\forall f"'] & TX \arrow[d, dashed, "\exists!\, \bar{f}"] \\ & A \text{ totally separated} \end{tikzcd}$$

composition with η is an equivalence.

$$(TX \to A) \simeq (X \to A)$$

Idea for (2): $TX :=$ image of the evaluation map.

$\eta :=$ corestriction of the evaluation map.

Examples of totally separated types

1. Any discrete type.
2. $2^{\mathbb{N}}$
3. More generally, products of totally separated types are totally separated.
4. So the simple types are totally separated.
5. Binary sums $X+Y$ of totally separated types X and Y.

⚡ Arbitrary sums $\Sigma\, i : I,\ X_i$ of totally separated types are <u>not</u> in general totally separated.

Compact type

This is trickier, and I want to make a long story short.

1. In classical topology, if X is sufficiently nice and totally separated, then X is compact iff the map

$$A : 2^X \longrightarrow 2$$

$$A\,p = 1 \text{ iff } \forall x : X.\ p\,x = 1$$

is continuous.

is-decidable $A := A + \neg A$

2. In MLTT, we define

$$X \text{ is } \Pi\text{-compact} := \prod p : X \to 2,\ \text{is-decidable}\left(\prod x : X,\ p\,x = 1\right)$$

Variations of the notion of compact type

1. $\prod p : X \to 2,\ \text{is-decidable}\left(\prod x : X,\ p x = 1\right)$ $\prod$-compact

2. $\prod p : X \to 2,\ \text{is-decidable}\left(\exists\, x : X,\ p x = 0\right)$ $\exists$-compact

3. $\prod p : X \to 2,\ \text{is-decidable}\left(\Sigma\, x : X,\ p x = 0\right)$ Σ-compact

- We have that (3) $\Rightarrow$ (2) $\Rightarrow$ (1).
- All our examples given later satisfy the stronger condition (3).

The case $X := \mathbb{N}$

1. $\prod p : \mathbb{N} \to 2$, is-decidable$(\prod x : \mathbb{N}, px = 1)$ $\iff$ Bishop's WLPO

$\Uparrow$

2. $\prod p : \mathbb{N} \to 2$, is-decidable$(\exists x : \mathbb{N}, px = 0)$ $\iff$ Bishop's LPO

$\Updownarrow$

3. $\prod p : \mathbb{N} \to 2$, is-decidable$(\Sigma x : \mathbb{N}, px = 0)$ $\iff$ Bishop's LPO

- So the type $\mathbb{N}$ is not compact in general.

  As we would expect.

- Under excluded middle, all types are $\exists$- and (hence) Π-compact.
- The Σ-compactness of all types is equivalent to global choice.

(strong) $\boxed{\text{Density}}$

We say that an embedding $X' \hookrightarrow X$ is dense if its image has empty complement.

Example. The embedding $2 \hookrightarrow \Omega$ is dense, where

$$\Omega = \Sigma\, P : \mathrm{Type},\ (\Pi\, x, y : P,\ x = y).$$

This means there is no truth value other than true or false.

Density theorem

Suppose $X' \hookrightarrow X$ is dense, Y is $\neg\neg$-separated.
If two functions $f, g : X \to Y$ agree on X', then they agree everywhere.

Example. Two functions $f, g : \mathbb{R} \to \mathbb{R}$ that agree on $x < 0$, $x = 0$ and $x > 0$, agree everywhere.

- There are no reals other than negative, zero or positive ones.
- It is a constructive taboo that every real would be of one of these three kinds.

Examples of Stone types

1. Of course the finite types $\mathrm{Fin}\ n$ are totally separated and compact in the strongest sense.

   Just perform exhaustive search.

2. Fix $k : \mathbb{N}$. Then the type of non-increasing sequences $\alpha : \mathbb{N} \to \mathbb{N}$ bounded by $k : \mathbb{N}$ is totally separated and compact in the strongest sense.

$\boxed{\text{The case } k := 1}$

1. $\mathbb{N}_\infty := \Sigma\, \alpha : \mathbb{N} \to 2\,,\ \Pi\, n : \mathbb{N}\,,\ \alpha_n \geq \alpha_{n+1}$.

   In Johnstone's topological topos, this gets interpreted as the one-point compactification of $\mathbb{N}$.

2. Theorem. $\mathbb{N}_\infty$ is Σ-compact.

   Now, an exhaustive-search technique doesn't apply.

proof outline

1. Given $p : \mathbb{N}_\infty \to 2$, we want to decide whether or not $\Sigma\, x : \mathbb{N}_\infty \,.\, p\,x = 0$. (whether p has a root.)

2. First let's understand $\mathbb{N}_\infty$ better

   (i) we have $1^n 0^\omega : \mathbb{N}_\infty$ corresponding to $n : \mathbb{N}$.

   (ii) we have $1^\omega : \mathbb{N}_\infty$, which is a new point at infinity.

   - These two possibilities don't* exhaust $\mathbb{N}_\infty$, but they are dense.
   - There is no element of $\mathbb{N}_\infty$ which is not of the forms (i) or (ii).

* More precisely, they do iff LPO holds.

proof outline

1. Given $p : \mathbb{N}_\infty \to 2$, we want to decide whether or not

$$\Sigma\, x : \mathbb{N}_\infty \,.\, p\,x = 0.$$

write this as $\underline{n}$

2. First let's understand $\mathbb{N}_\infty$ better.

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proof outline

1. Given $p : \mathbb{N}_\infty \to 2$, we want to decide whether or not

$$\Sigma\, x : \mathbb{N}_\infty \,.\, p\, x = 0 .$$

3. Now construct $x_0 : \mathbb{N}_\infty$ as follows:

$$x_0\,(n) = \min(p\,\underline{0}, \ldots, p\,\underline{n}).$$

4. We can show

   (i) $p\ x_0 = 1$ implies $p\ \underline{n} = 1$ for all $n : \mathbb{N}$,

   (ii) $p\ x_0 = 1$ implies $p\ \infty = 1$.

5. By density, $p\, x_0 = 1$ implies $p\, x = 1$ for all $x : \mathbb{N}_\infty$.

proof outline

1. Given $p : \mathbb{N}_\infty \to 2$, we want to decide whether or not

$$\Sigma\, x : \mathbb{N}_\infty \,.\, p\, x = 0.$$

5. At this point we have $x_0 : \mathbb{N}_\infty$ such that (Drinker's Paradox.)

$p\, x_0 = 1$ implies $p\, x = 1$ for all $x : \mathbb{N}_\infty$.

6. Finally, we reason by cases on $p\, x_0$:

- If $p\, x_0 = 0$, then $\Sigma\, x : \mathbb{N}_\infty \,.\, p\, x = 0$, namely $x := x_0$.
- If $p\, x_0 = 1$, then $\neg \Sigma\, x : \mathbb{N}_\infty \,.\, p\, x = 0$ by (5).

∎

Is the Cantor type $2^{\mathbb{N}}$ compact?

1. Johnstone's Topological Topos &
Relative realizability topos over K_2: Yes

2. Hyland's Effective Topos
(realizability over K_1) No
(Because of the Kleene Tree)

3. In our type theory Undecided

The delay monad

1. $\mathbb{N}_\infty$ is the final coalgebra of the functor $1+(-)$.

   So we may call $\mathbb{N}_\infty$ the type of conatural numbers.

2. Define $\mathbb{D}X := \sum_{x:\mathbb{N}_\infty} (\text{is-finite } x \to X)$

   where $\text{is-finite } x := \sum_{n:\mathbb{N}} \underline{n} = x$.

- Then $\mathbb{D}X$ is the final coalgebra of the functor $X+(-)$.
- We will show that X stone implies $\mathbb{D}X$ stone in a number of steps.

1. Micro-Tychonoff. If P has at most one element and X is a family of pointed compact types indexed by P, then $\prod_{p:P} X_p$ is pointed and compact.

2. Binary Tychonoff. If X and Y are compact, so is $X \times Y$.

3. Pointed compact types are closed under Σ.

4. $\Sigma\, x : \mathbb{N}_\infty,\ A\ x$ is totally separated if each $A\ x$ is and $A\ \infty$ has at most one element.

1. Micro-Tychonoff. If P has at most one element and X is a family of pointed compact types indexed by P, then $\prod_{p:P} X_p$ is pointed and compact.

Funny consequence. The function type $LPO \to \mathbb{N}$ is compact and totally separated.

5. Example. If X is a Stone type then so is

$$\mathbb{D}\, X := \Sigma\, x : \mathbb{N}_\infty, (\text{is-finite}\, x \to X).$$

In particular,

$$\mathbb{D}\, 1 \simeq \mathbb{N}_\infty$$

$\mathbb{D}^k\, 1 \simeq$ decreasing sequences of natural numbers bounded by k.

6. counterexample. If $\Sigma\, x : \mathbb{N}_\infty, (x = \infty \to 2)$ is totally separated then $\neg\neg$ WLPO holds:

$$\neg\neg\, (\Pi\, \alpha : \mathbb{N} \to 2,\ \text{is-decidable}\, (\Pi\, n : \mathbb{N},\ \alpha_n = 1))$$

Totally separated types are not closed under Σ in general.

$$\overbrace{\Sigma x : \mathbb{N}_\infty, (x = \infty \to 2)}^{\text{call this type } \mathbb{N}_{\infty_2}}$$

6. counterexample. If $\Sigma x : \mathbb{N}_\infty, (x = \infty \to 2)$ is totally separated, then $\neg\neg$ WLPO holds.

$$\neg\neg(\Pi \alpha : \mathbb{N} \to \mathbb{N}, \text{is-decidable}(\Pi n : \mathbb{N}, \alpha_n = 1))$$

Define

$$\infty_0 := (\infty, \lambda -, 0) : \mathbb{N}_{\infty_2},$$

$$\infty_1 := (\infty, \lambda -, 1) : \mathbb{N}_{\infty_2}.$$

Then $\infty_0 \neq \infty_1$, and from any given $p : \mathbb{N}_{\infty_2} \to 2$ with $p\,\infty_0 \neq p\,\infty_1$ we can deduce WLPO.

7. Co-Tychonoff. $\prod_{x:X} Y_x$ is discrete if X is compact and each Y_x is discrete.

Example. The type $\mathbb{N}_\infty \to \mathbb{N}$ is discrete.

8. Say that A is disconnected if there are pointed types A_0, A_1, with $A \simeq A_0 + A_1$.

If the type $X \to Y$ is discrete and Y is disconnected, then the type X is compact.

7. Co-Tychonoff. $\prod_{x:X} Y_x$ is discrete if X is compact and each Y_x is discrete.

8. If the type $X \to Y$ is discrete and Y is disconnected, then the type X is compact.

corollary of (7) & (8). If Y is a discrete type with two distinct points, then $X \to Y$ is discrete iff X is compact.

9. For any family of decidable types A_x indexed by $x : \mathbb{N}_\infty$, the type $\prod_{n : \mathbb{N}} A_{\underline{n}}$ is decidable.

This is an instance of WLPO that just holds.

10. For any $f : \mathbb{N}_\infty \to \mathbb{N}$, it is decidable whether it is not continuous.

11. WLPO holds iff there some non-continuous $f : \mathbb{N}_\infty \to \mathbb{N}$.

There is a more general result by Ishihara using countable choice.

12. A type is compact iff its totally separated reflection is.

13. The (large) type Ω of types with at most one element is compact, but <u>not</u> in general totally separated.

Concluding discussion

1. All compact types we are able to construct are well-ordered
2. For every ordinal that can be denoted by a Brouwer tree, we can construct a Stone type of that order type.
3. All compact types we can construct have a countable strongly dense subset.
4. In the above synthetic topology we didn't use any Brouwerian or topological axiom.

It is classically/constructively neutral synthetic topology.