Playing rationally against irrational players with monads

Martín H. Escardó
School of Computer Science
University of Birmingham, UK

Joint work with
Paulo Oliva
Queen Mary University of London

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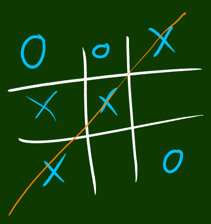
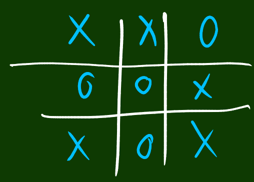
Example: Tic-Tac-Toe (X plays first)



X wins



draw

- If both players play rationally then the optimal outcome of the game is a draw.
  } use monads J & K to explain this

- But what if one of them, or both, play irrationally?
  } use monads T, J_T & K

Example: T = probability distribution monad

1. Both players are rational:

   The game is a draw with probability 100 %

2. X is rational and O plays randomly with uniform distribution:

   The game is a draw with probability 0.5 %
   and an X win with probability 99.5 %

3. X plays uniformly randomly and O plays rationally:

   draw 8.4 %
   O win 91.6 %

4. Both play randomly:

   draw 12.7 %
   X win 58.5 %
   O win 28.8 %

Example: T = non-empty powerset monad

This accounts for one or more players playing erratically (with no assigned probabilities)

1. Both erratic:
   game can be any of X win, draw, O win.

2. One erratic:
   game can be any of the other wins or draw.

3. Both rational:
   draw

Example: T = identity monad

This amounts to our previous work, with all players rational.

What is presented here generalizes previous work.

Plan of the talk

1. • How to specify combinatorial games of perfect information using a certain monad K (double dualization) and use it to calculate optimal outcomes.

   • How to use a certain monad J (selection) and a monad morphism $J \to K$ to calculate optimal strategies.

2. How to add irrationality with an affine strong monad T, an algebra $\alpha : TR \to R$, and a generalization J_T of J.
   (when $T = \mathrm{Id}$, we have $J_T = J$)

$\eta_1 : 1 \to T1$ is iso

Assumption

We work in a cartesian closed category.

Examples we have used so far applications:

- Category of sets
- Kleene-Kreisel spaces
- Scott domains
- (∞-)toposes
- compactly generated spaces
- Alex Simpson's QCB spaces
- Types

game theory
proof theory
searchable sets

The monads K & J : motivation

X set of goods.

R set of prices.

$p : X \to R$ table of prices.

$\varepsilon : (X \to R) \to X$ selects a cheapest good in a given table.

$\phi : (X \to R) \to R$ selects the lowest price in a given table.

Fundamental equation: $\underbrace{p(\varepsilon p)}_{\text{price of cheapest good}} = \underbrace{\phi p}_{\text{lowest price in table}}$

$\phi = \min$

$\varepsilon = \operatorname{argmin}$

$p(\operatorname{argmin} p) = \min p$

The monads K & J : motivation

X set of possible plays in a game

R set of possible outcomes

$p : X \to R$ the outcome of a play

$\varepsilon : (X \to R) \to X$ selects an optimal play

$\phi : (X \to R) \to R$ selects the optimal outcome

Fundamental equation: $\underbrace{p(\varepsilon p)}_{\text{outcome of optimal play}} = \underbrace{\phi p}_{\text{optimal outcome of the game}}$

The monads K & J: motivation

X set of things

R set of truth values

$p : X \to R$ predicate

$\varepsilon : (X \to R) \to X$ Hilbert's ε

$\phi : (X \to R) \to R$ existential quantifier

$0 < 1$

false true

$\exists = \max$

Fundamental equation: $p(\varepsilon p) = \underbrace{\phi p}_{\exists x.\, p x}$

Hilbert's definition of $\exists$ in his ε-calculus.

$\phi = \exists$

$\varepsilon =$ Hilbert's ε

The monads K & J : motivation

$0 < 1$

false true

$\forall = \min$

X set of things

R set of truth values

$p : X \to R$ predicate

$\varepsilon : (X \to R) \to X$

$\phi : (X \to R) \to R$ universal quantifier

Fundamental equation : $p(\underbrace{\varepsilon p}_{a}) = \underbrace{\phi p}_{\forall x.\, p x}$

$\phi = \forall$

$\varepsilon =$ in every pub p there is a person x such that if x drinks then everybody drinks.

The monads K & J : definitions

double dualization (aka continuation) monad:

- $KX = \underbrace{\overbrace{(X \to R)}^{p} \to R}_{\phi}$
- $\eta_X : X \to KX$
  $x \mapsto \lambda p . p x$
- Given $f : X \to Y$ we get
  $Kf : KX \to KY$
  $\phi \mapsto \lambda p . \phi(\lambda x . p(f x))$
- $\mu_X : K(KX) \to KX$
  $\Phi \mapsto \lambda p . \Phi(\lambda \phi . \phi p)$

selection monad:

- $JX = \underbrace{\overbrace{(X \to R)}^{p} \to X}_{\varepsilon}$ selection monad
- $\eta_X : X \to JX$
  $x \mapsto \lambda p . x$
- $Jf : JX \to JY$
  $\varepsilon \mapsto \lambda p . f \, \varepsilon((\lambda x . p(f x)))$
- $\mu_X : J(JX) \to JX$
  $E \mapsto \lambda p . E(\lambda \varepsilon . p(\varepsilon p))\, p$

Monad morphism $J \to K$

$$\theta_X : JX \longrightarrow KX$$
$$\varepsilon \longmapsto \lambda p.\, p(\varepsilon p)$$

We often write $\overline{\varepsilon}$ for $\theta\,\varepsilon$.

$$\overline{\varepsilon}\, p = p(\varepsilon p)$$

ϕ c.f. fundamental equation

E.g. $\overline{\mathrm{argmax}} = \max$

"argmax is a selection function for max"

The two monads are strong

$$X \times KY \xrightarrow{t_{X,Y}} K(X \times Y)$$

$$(x, \phi) \longmapsto \lambda p.\ \phi(\lambda y.\ p(x,y))$$

$$X \times JY \xrightarrow{t_{X,Y}} J(X \times Y)$$

$$(x, \varepsilon) \longmapsto \lambda p.\ (x, \varepsilon(\lambda y.\ p(x,y)))$$

We have monoidal monad structures

Because we have strong monads $T = J$ and $T = K$ on a c.c.c.

$$TX \times TY \xrightarrow{\otimes} T(X \times Y)$$

we want this left-to-right $\longrightarrow$ $(\mu, \nu) \longmapsto (T(\lambda x. t_{X,Y}(x, \nu)))\,\mu$

not this right-to-left $\longrightarrow$ $(\mu, \nu) \longmapsto (T(\lambda y. t_{Y,X}(\mu, x))\,\nu$

The monads are not commutative.

We have monoidal monad structures

Because we have strong monads $T = J$ and $T = K$ on a c.c.c.

We use this to calculate optimal outcomes

$$
\begin{tikzcd}
KX \times KY \arrow[r, "\otimes"] & K(X \times Y) \\
JX \times JX \arrow[u, "\theta_X \times \theta_Y"] \arrow[r, "\otimes"'] & J(X \times Y) \arrow[u, "\theta_X"']
\end{tikzcd}
$$

We use this to calculate optimal strategies

Examples

$$KX \times KY \xrightarrow{\otimes} K(X \times Y)$$

$$(\min, \max) \longmapsto \lambda p.\ \min_x \max_y p(x,y)$$

$$JX \times JY \xrightarrow{\otimes} J(X \times Y)$$

$$(\mathrm{argmin}, \mathrm{argmax}) \longmapsto \text{long formula omitted}$$

We have

$$\overline{\mathrm{argmax} \otimes \mathrm{argmin}} = \min \otimes \max$$

Examples

$$KX \times KY \xrightarrow{\otimes} K(X \times Y)$$

$$(\forall\ ,\ \exists) \longmapsto \lambda p.\ \forall x. \exists y.\ p(x,y)$$

$$JX \times JY \xmapsto{\otimes} J(X \times Y)$$

$$(\text{arg-}\forall\ ,\ \text{arg-}\exists) \longmapsto \cdots$$

We have

$$\overline{\text{arg-}\forall \otimes \text{arg-}\exists} = \forall \otimes \exists$$

Fundamental theorem (with easy proof)

For any $\varepsilon : JX$, $\delta : JY$ we have

$$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}.$$

In other words, if

ε is a selection function for $\phi : KX$

δ is a selection function for $\psi : KY$

then $\varepsilon \otimes \delta$ is a selection function for $\phi \otimes \psi$.

Example

In every school there is a child x_0 and a teacher y_0 such that if x_0 gives an apple to y_0 then every child gives an apple to some teacher.

X = set of children $\qquad p(x,y)$ = x gives an apple to y

Y = set of teachers

$$(x_0, y_0) := (\text{arg-}\forall \otimes \text{arg-}\exists)(p)$$

Definition of history free game
of fixed length n for simplicity

1. Sets of moves $X_0, X_1, \dots, X_{n-1}$
2. Quantifiers $\phi_0 : K X_0, \dots, \phi_{n-1} : K X_{n-1}$
3. Outcome function $p : X_0 \times X_1 \times \dots \times X_{n-1} \to R$

- Then the optimal outcome is $(\phi_0 \otimes \phi_1 \otimes \dots \otimes \phi_{n-1})(p)$
- If the quantifiers have selection functions $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ then an optimal play is given by $(\varepsilon_0 \otimes \varepsilon_1 \otimes \dots \otimes \varepsilon_{n-1})(p)$

Calculation of optimal strategy

If the moves $x_0, x_1, \ldots, x_k$ have been played, then the next optimal move to play is

$$x_{k+1} = \text{first move of} \left(\left(\varepsilon_{k+1} \otimes \varepsilon_{k+2} \otimes \cdots \otimes \varepsilon_{n-1} \right) \left(\lambda x_{k+1}, x_{k+2}, \ldots, x_{n-1} . p(x_0, \ldots, x_{n-1}) \right) \right)$$

History dependent (as e.g. Tic-Tac-Toe)

In brief summary

$$\otimes^K : KX \times (X \to KY) \to K(X \times Y)$$

$$\otimes^J : JX \times (X \to JY) \to J(X \times Y)$$

previous move

Irrational players

consider an additional monad T
(e.g. of probability distributions)
with a monad algebra $\alpha : TR \to R$

$$J_T X = (X \to R) \to TX$$

Examples

1. Select an argument of $p : X \to R$ with uniform distribution among maximal values (rational player)
2. Select all elements of X with uniform distribution (irrational player)

How do we compute optimal outcomes and optimal strategies?

As before:

- The monad J_T is also strong.
- We use $\otimes_{J_T}$ to compute strategies in equilibrium (this generalizes Nash equilibrium)

→ We implemented them in Haskell for the purposes of computation.

→ In Agda for the purposes of verification.