The topology of the universe
and related continuity phenomena
in Martin-Löf Type Theory

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including joint and disjoint work with

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Continuous phenomena in Martin-Löf in type theory

1. Types are spaces, functions are continuous.
2. Searchable types are compact sets.

   $\mathbb{N}_\infty$. Closure under image, finite products, arbitrary sums, and products with at most one factor (and not countable products).
3. Injectivity of the universe.

   Gives universe indiscrete.
   Gives certain sums with a points at infinity,
   and (hence) searchable countable ordinals.
4. Continuity of all functions $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ implies $0 = 1$.

   Something wrong with the BHK interpretation of logic.
   How to fix it.
5. Related, but different, story for the uniform continuity of all functions $2^\mathbb{N} \to \mathbb{N}$.

This talk is a selection and simplification of some of the above.

Some type theories

1. Gödel's system T. Base type $\mathbb{N}$ and type former $\to$.

   Natural set-theoretic interpretation (but not fully abstract).
   Topological interpretation in Kleene–Kreisel spaces (fully abstract).

2. Platek–Scott-Plotkin PCF. Adds general recursion to system T.

   Domain-theoretic interpretation. Game interpretation.

3. Martin-Löf type theory (MLTT). $\emptyset$, 1, 2, $\times$, $\to$, $+$, Σ, Π, $=$, W, U.

   No general recursion.

   A programming language.
   A foundation of constructive mathematics.

   (But compatible with classical mathematics.)

   Set-theoretic interpretation. Types are sets.
   Topological interpretation. Types are spaces.
   Homotopical interpretation. Spaces up to homotopy equivalence.

A wish that can't be fulfilled literally

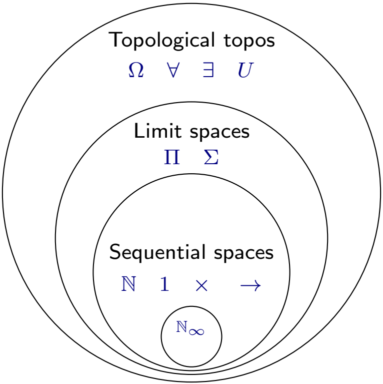
1. Types are interpreted as topological spaces.
2. Terms are interpreted as points of spaces.
3. Functions are interpreted as continuous maps.

The category of continuous maps of topological spaces is not even cartesian closed (it doesn't have exponentials (function spaces)).

Hence it can't interpret system T or PCF or MLTT.

However, there are natural continuous models of type theories.

Johnstone's topological topos (1979)



1. The site is the category of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ with the canonical coverage.
2. Taking colimits of $\mathbb{N}_\infty$ in topological spaces gives sequential spaces.
3. The limit spaces arise as the subobjects of sequential spaces.

Examples of MLTT-definable spaces

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5. The interpretation of the type

$$\mathbb{N}_\infty \stackrel{\mathsf{def}}{=} \left(\sum_{\alpha : \mathbb{N} \to 2} \quad \prod_{n : \mathbb{N}} \quad \alpha_n = 0 \to \alpha_{n+1} = 0 \right)$$

gives the one-point compactification of $\mathbb{N}$, with $\infty \stackrel{\mathsf{def}}{=} (\lambda i.1, -)$.

Here "$=$" is the identity type, interpreted as a sub-singleton.

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6. The interpretation of the type

$$\sum_{x:\mathbb{N}_\infty} x = \infty \to 2$$

is a T_1, non-Hausdorff, but compact, space with two points at infinity,

$$\infty_0 \stackrel{\text{def}}{=} (\infty, \lambda p.0), \quad \infty_1 \stackrel{\text{def}}{=} (\infty, \lambda p.1).$$

Question (Voevodsky, 17 October 2011)

What is the space in the topological topos that interprets the universe U?

(Using Streicher's interpretation universes in Grothendieck toposes.)

The (perhaps shocking) answer is that it is indiscrete.

More precisely, its *topological reflection* is indiscrete.

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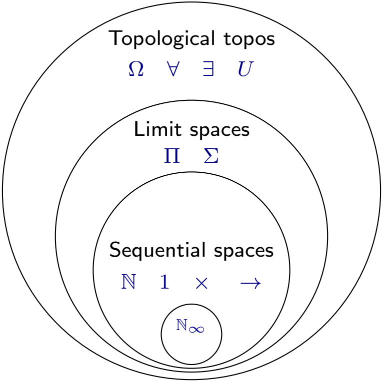

But there is a model-independent answer, too:

Any given sequence of types converges to any type we wish,
up to isomorphism.

This can be proved *internally* in MLTT, before we consider any model.

Convergence via $\mathbb{N}_\infty$

$\mathbb{N}_\infty$ is the generic convergent sequence.

A sequence $\mathbb{N} \to X$ converges to a limit $x_\infty \colon X$ if and only if it extends to a function

$$\mathbb{N}_\infty \to X$$

that maps ∞ to x_∞.

Any function of any two types is sequentially continuous

Without invoking continuity axioms.

Proof. Automatic.
Let $f\colon X \to Y$ be a function.
If $x_n \to x_\infty$ in X, then there is $\mathbb{N}_\infty \to X$ that maps n to x_n and ∞ to x_∞.
Now compose this function $\mathbb{N}_\infty \to X$ with your given function $f\colon X \to Y$.
The resulting function $\mathbb{N}_\infty \to Y$ maps n to $f(x_n)$ and ∞ to $f(x_\infty)$.
This means that $f(x_n) \to f(x_\infty)$.
Q.E.D.

Sensible to consider convergence up to isomorphism in U

Definition. We say that a sequence of types

$$X : \mathbb{N} \to U$$

converges to a limit $X_\infty : U$ if and only if it extends to a function

$$\hat{X} : \mathbb{N}_\infty \to U$$

with

$$\hat{X}_n \cong X_n, \qquad \hat{X}_\infty \cong X_\infty.$$

(For a univalent universe, this definition is equivalent to the previous one.)

Theorem. The universe is intrinsically indiscrete

Every sequence of types converges to any desired type.

A formal rendering in Agda of the proof to be give below is here:
`www.cs.bham.ac.uk/~mhe/agda/TheTopologyOfTheUniverse.html`

Lemma. Every sequence of types converges to the type $\mathbf{1}$

Given $X : \mathbb{N} \to U$, define $\hat{X} : \mathbb{N}_\infty \to U$ by

$$\hat{X}_u = \prod_{k\,:\,\mathbb{N}} (u = k \to X_k).$$

Then

$$\hat{X}_n = \prod_{k\,:\,\mathbb{N}} (n = k \to X_k) \cong X_n,$$

and

$$\hat{X}_\infty = \prod_{k\,:\,\mathbb{N}} (\infty = k \to X_k) \cong \prod_{k\,:\,\mathbb{N}} (\mathbf{0} \to X_k) \cong \prod_{k\,:\,\mathbb{N}} \mathbf{1} \cong \mathbf{1}.$$

Theorem. Every sequence of types converges to any type

Let $X : \mathbb{N} \to U$ and $Y : U$ be given.
(i) $X_n \longrightarrow \mathbf{1}$ Previous lemma.
(ii) $(\mathbf{0})_n \longrightarrow \mathbf{1}$ Special case of (i).
(iii) $(\mathbf{0})_n \longrightarrow Y$ Multiply (ii) by Y.
(iv) $(\mathbf{1})_n \longrightarrow \mathbf{0}$ Apply $(- \to \mathbf{0})$ to (ii).
(v) $X_n \longrightarrow \mathbf{0}$ Multiply (i) and (iv).
(vi) $X_n \longrightarrow Y$ Add (iii) and (v).

Here we are using the fact that all functions are sequentially continuous.
E.g. multiplication by Y.

(There is a more direct argument with two steps only. Exercise.)

Corollary. Rice's Theorem for the Universe

It is a constructive taboo to say that the universe has a non-trivial, extensional, decidable property.

For any extensional $P \colon U \to 2$ and $X, Y : U$,
if $P(X) \neq P(Y)$ then WLPO.

That $P \colon U \to 2$ is extensional means that $X \cong Y \implies P(X) = P(Y)$.
(For a univalent universe, all functions $U \to 2$ are extensional.)

WLPO says that every binary sequence is constantly 1 or it isn't.
It solves the Halting Problem.

$P(X) \neq P(Y) \implies$ WLPO

Assume w.l.o.g. that $P(X) = 0$ and $P(Y) = 1$.
By the Universe Indiscreteness Theorem, there is $Q\colon \mathbb{N}_\infty \to U$ with

$$\forall (n\colon \mathbb{N}).Q(n) \cong X, \qquad Q(\infty) \cong Y.$$

Let $p\colon \mathbb{N}_\infty \to 2$ be $P \circ Q$. By the extensionality of P, we have that

$$\forall (n\colon \mathbb{N}).p(n) = 0, \qquad p(\infty) = 1.$$

Hence for any given $x\colon \mathbb{N}_\infty$ we can decide whether $x = \infty$ by checking the decidable condition

$$p(x) = 1.$$

This amounts to WLPO.
It is decidable whether every binary sequence is constanly 1.

Sharper version of Rice's Theorem for the Universe

Found by Alex Simpson after I gave a similar talk some time ago.

T.F.A.E.

1. There is a non-trivial, extensional map $U \to 2$.
2. $\Pi(X : U).\neg X + \neg\neg X$.

One writes $\neg X$ to mean $X \to \emptyset$.

Corollary. (By taking contra-positives.) T.F.A.E.

1. All extensional maps extensional maps $U \to 2$ are constant.
2. $\neg\Pi(X : U).\neg X + \neg\neg X$.

The topological topos validates continuity axioms

Continuity axiom (Cont)

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\forall f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha : \mathbb{N}^{\mathbb{N}}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Uniform continuity axiom (UC)

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n : \mathbb{N}.\ \forall \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

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- This assumes a non-constructive meta-theory.
- Towards the end I discuss another topological topos developed within a constructive meta-theory by Chuangjie Xu and myself.

  (Also formalized in Agda by Chuangjie.)
- For the moment ignore constructivity issues until further notice.

Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has Π and Σ.

If we apply the BHKCH interpretation:

Continuity axiom (Cont):

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Continuity axiom (Cont): ✕ Moreover, no topos can validate this.

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Theorem of intensional Martin-Löf type theory

If all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous then $0 = 1$.

$$\left(\prod_{f \,:\, \mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \; \sum_{n : \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

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I could instead say "not all functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous", but:

1. This would give the false impression that there might exist a non-continuous function to be found by looking hard enough.
   (In the topological topos all functions are continuous, and yet this holds.)
2. It is $0 = 1$ that our proof actually does give from the assumption.
   (A technicality that leads to the next item.)
3. We would need a universe to map the type $0 = 1$ to the type $\emptyset$, and our proof doesn't require universes.
   (So we are more general.)

Theorem of intensional Martin-Löf type theory

$$\left(\prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\alpha:\mathbb{N}^{\mathbb{N}}}\ \sum_{n:\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta\right) \to 0 = 1.$$

Proof sketch. Let

$$\phi : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\alpha:\mathbb{N}^{\mathbb{N}}}\ \sum_{n:\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta.$$

Using ϕ and the projections and choosing $\alpha = 0^\omega$, we get

$$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$$

and

$$\gamma : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} 0^\omega =_{Mf} \beta \to f0^\omega = f\beta.$$

Now define $m = M(\lambda\alpha.0)$ and consider

$$f\beta = M(\lambda\alpha.\beta(\alpha_m)).$$

Then argue $Mf = 0$ and $Mf > 0$ give $0 = 1$, using $f0^\omega = m$.
(Induction on Mf not needed). Q.E.D.

Proof discussion

This is an adaptation of a well known argument (due to Kreisel?).

1. Continuity, choice and extensionality are together impossible.
2. No *extensional* modulus-of-continuity functional M.
3. But here we are working in *intensional* Martin-Löf type theory.
4. No *continuous* modulus-of-continuity functional M.
5. We used our hypothetical M to define a non-continuous function f and hence prove M wrong.
6. And this is exactly what is happening in the topological topos:
   - All functions are continuous.
   - But there is no continuous way of finding moduli of continuity.
   - No finite amount of information about f suffices to determine its modulus.

Σ versus $\exists$

Fix an object X.

1. Σ is understood in slices $\mathcal{E}/X$.
   If we have an object classifier U (universe), we can understand it as

   $$\Sigma : (X \to U) \to U.$$

   Given a family of objects we get an object.

2. $\exists$ is understood as a function

   $$\exists : (X \to \Omega) \to \Omega.$$

3. They are related via a reflection of U into Ω:

   $$U \underset{\hookleftarrow}{\overset{\|-\|}{\longrightarrow}} \Omega.$$

   $$(\exists x : X.P(x)) = \|\Sigma x : X.P(x)\|.$$

   (Used in Homotopy Type Theory to define $\exists$ from Σ.)

Continuity in type theory extended with $\| - \|$

Add a universal map $| - | : X \to \|X\|$ into types with at most one element.

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type P with at most one element.

(We are quotienting X by the relation that identifies any two points.)

$$\prod_{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \left\| \sum_{n : \mathbb{N}} \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right\|.$$

- In a sheaf topos, this means we can find n locally but not globally.
- In a realizability topos, we can find n intensionally but not extensionally.
- In other toposes this of course acquires other meanings.
- In type theory, it seems difficult to give a direct *meaning-explanation*.

Another well-known example

If you try to say that $f : X \to Y$ is a surjection by saying

$$\prod_{y:Y} \sum_{x:X} fx = y,$$

you are actually saying that f has a section $Y \to X$.

You should instead say

$$\prod_{y:Y} \left\| \sum_{x:X} fx = y \right\|.$$

A similar distinction arises in the definition of the image of a function, and many other definitions and theorems and proofs.

Disclosing secrets

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type P with at most one element.

We can disclose a secret $\|X\|$ to P provided we have a map $X \to P$.

Example. If $A(n)$ is decidable then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Proof sketch. If we have any n with $A(n)$, we can find the minimal n, using the decidability of $A(n)$, but "having a minimal n such that $A(n)$" is a type with at most one element.

More general lemma

(With Chuangjie Xu.)

Assume that $A(n)$ has at most one element for every $n : \mathbb{N}$.

If for any given n we have that $A(n)$ implies that $A(m)$ is decidable for all $m < n$, then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Theorem of MLTT extended with $\| - \|$

$$\Pi f\colon 2^{\mathbb{N}} \to \mathbb{N}.\ \big\|\Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big\|$$
$$\to \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Proof. Set $A(n) = \big(\Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big)$ in the lemma.

Corollary. The topological topos validates the uniform-continuity axiom

$$\Pi f\colon 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Because the premise of the theorem is validated.

(In the topological topos, the theorem can be seen as getting global existence from local existence by compactness.)

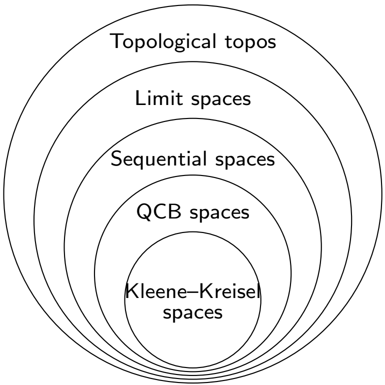
Getting constructive

1. Kleene–Kreisel functionals constructively.
2. Another topological topos for that.
3. If all functions $2^{\mathbb{N}} \to \mathbb{N}$ are continuous, then the Kleene–Kreisel hierarchy agrees with the full-type hierarchy.
4. A model of type theory that constructively validates the uniform-continuity axiom.
5. Implemented in Agda.

Kleene–Kreisel continuous functionals

Identified in the 1950's as

- Kleene's countable functionals.
- Kreisel's continuous functionals.



Start from $\mathbb{N}$ and close under exponentiation.

This is automatically closed under finite products, excluding the empty product 1.

Fully abstract model of Gödel's system T. (By the Kleene-Kreisel density theorem.)

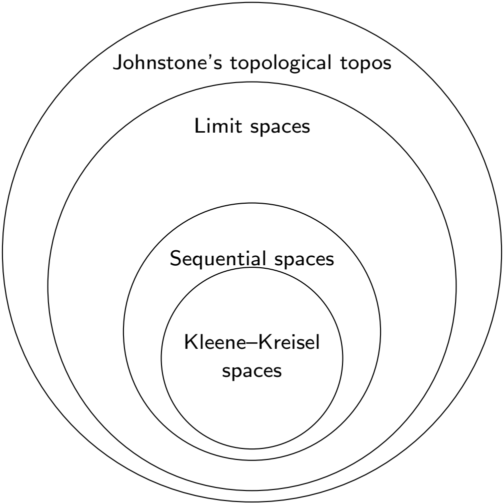
The set-theoretical full type hierarchy is not fully abstract (Kreisel).

Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^\mathbb{N}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.



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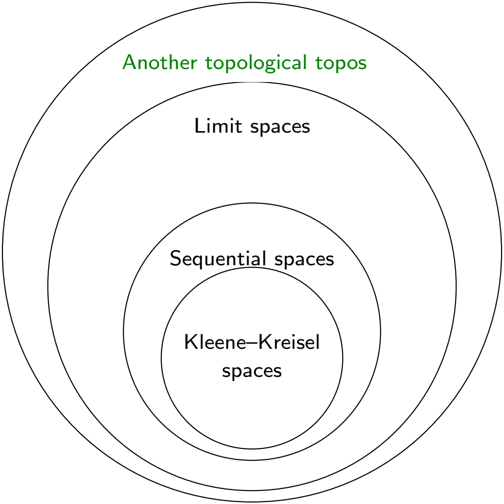

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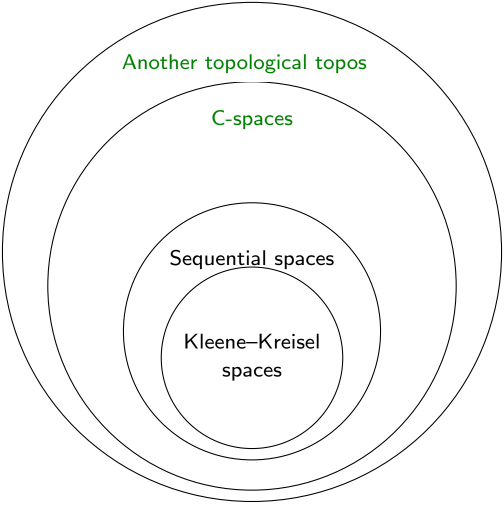

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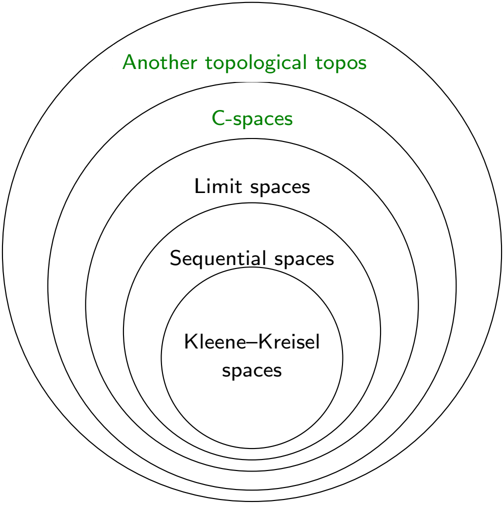

The uniform-continuity coverage

1. Let 2^n denote the set of binary strings of length n.

2. For $s \in 2^n$, let $\mathrm{cons}_s \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the concatenation map

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

3. A function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is uniformly continuous iff

$$\forall m : \mathbb{N}. \exists n : \mathbb{N}. \forall s : 2^n. \exists f' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}. \exists s' : 2^m. f \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ f'.$$

4. This shows that the countable collection $\{(\mathrm{cons}_s)_{s:2^n} \mid n : \mathbb{N}\}$ satisfies the coverage axiom.

5. This coverage is subcanonical.

6. Moreover, crucially: $y\left(2^{\mathbb{N}}\right)$ has the universal property of the exponential $2^{\mathbb{N}}$ in the resulting topos, where of course 2 is $1+1$ and $\mathbb{N}$ is the natural numbers object of the topos.

What we get

1. A constructive treatment of sheaves and C-spaces suitable for development in Martin–Löf type theory.

   Definitions, theorems and proofs implemented in Agda.

   We don't need $\| - \|$.

   We need $\neg\neg$(function extensionality).

2. C-Spaces give a constructive model of dependent types with the uniform continuity axiom.

   At the moment we haven't modelled the universe.

   The amalgamation property for the "naive" version of the Hofmann–Streicher universe holds only up to isomorphism.

   We want to avoid sheafification.

3. If we assume that all functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then we can show constructively that the full type hierarchy is equivalent to the Kleene–Kreisel continuous hierarchy.

End