A Hausdorff compactification of the Samborski function space

(and also of $C_{co}(X, \mathbb{R})$)

locally compact Hausdorff

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Based on

- Jimmie Lawson's talk on Monday
- My 2002 paper "Function-space compactifications of function spaces". Top App.
- Samborski's 2002 paper "A new function space and extension of partial differential operators in it" Univ de Caen Tech Rep CNRS UMR 6139.
- Some interactions with Andrej Bauer & Klaus Keimel last night.

The Samborski function space

Let X be a locally compact, Hausdorff topological space.

~~Second countable for psychological simplicity:~~

For any $f : X \longrightarrow [-\infty, \infty]$, define

$$f_*(x) = \sup \{ \inf f(U) \mid x \in U,\ U \text{ is open} \},$$

$$f^*(x) = \inf \{ \sup f(U) \mid x \in U,\ U \text{ is open} \}.$$

Then f is lower semicontinuous iff

$$f = f_*$$

and it is upper semicontinuous iff

$$f = f^*$$

and it is continuous iff

$$f = f_* = f^*.$$

Moreover, $f_* \leq f \leq f^*$

and they all agree at points of continuity of f.

$$F \overset{\text{def}}{=} \left\{ (\underline{f}, \bar{f}) \;\middle|\; \underline{f}, \bar{f} : X \to [-\infty, \infty],\ \underline{f} = \bar{f}_*,\ \bar{f} = \underline{f}^* \right\}.$$

With a suitable topology, this is the Samborski function space.

Proposition (Samborski)

$$\underset{\text{with compact-open topology}}{C(X, [-\infty, \infty])} \xhookrightarrow{\text{homeomorphic embedding}} \underset{\text{with Samborski topology.}}{F}$$

$$f \longmapsto (f, f)$$

And that's why, among other things, the Samborski topology is interesting.

A theorem of Jimmie Lawson (Monday), formulated later, gives a domain-theoretic description of this topology.

Moreover, it gives further credibility to it.

Another view of the Samborski function space F.

$\mathcal{R} = \mathrm{I}[-\infty, \infty]$, interval domain of compact non-empty intervals under Scott topology of reverse-inclusion order.

$$x \longmapsto \overline{x} \stackrel{\text{def}}{=} \sup x$$

$$\mathcal{R} \underset{lsc}{\overset{usc}{\rightrightarrows}} [-\infty, \infty]$$

$$x \longmapsto \underline{x} \stackrel{\text{def}}{=} \inf x$$

$$f \longmapsto \left(\underline{f}(x) \stackrel{\text{def}}{=} \underline{f(x)},\ \overline{f}(x) \stackrel{\text{def}}{=} \overline{f(x)} \right)$$

$$C(X, \mathcal{R}) \quad \text{bijection} \quad \mathcal{F} \stackrel{\text{def}}{=} \{ (\underline{f}, \overline{f}) \mid \underset{lsc}{\underline{f}} \leq \underset{usc}{\overline{f}} \}$$

$$\mathcal{F} \supseteq F$$

$$f(x) \stackrel{\text{def}}{=} [\underline{f}(x), \overline{f}(x)] \longleftarrow (\underline{f}, \overline{f})$$

Take Scott (= compact open topology) on $C(X, \mathcal{R})$.

Write $(X \to \mathcal{R})$ for the resulting space.

(Tranfer the topology to $\mathcal{F}$ via the bijection.)

Proposition (Lawson, this meeting)

$$F \cong \mathrm{Max}\,(X \to \mathcal{R}) \quad \text{homeomorphically.}$$

i.e the Samborski topology is the relative Scott topology.

By general domain theory, we conclude that also

$$\boxed{\text{The Samborski topology agrees with the Lawson topology.}}$$

(Because X is locally compact and $\mathcal{R}$ is a bounded complete continuous dcpo, so is $(X \to \mathcal{R})$, and because the Scott and Lawson topologies agree on the set of maximal elements of such a domain.)

By the blue argument, we also conclude that

$$\boxed{\text{The Lawson topology of } (X \to \mathcal{R}) \text{ is compact Hausdorff.}}$$

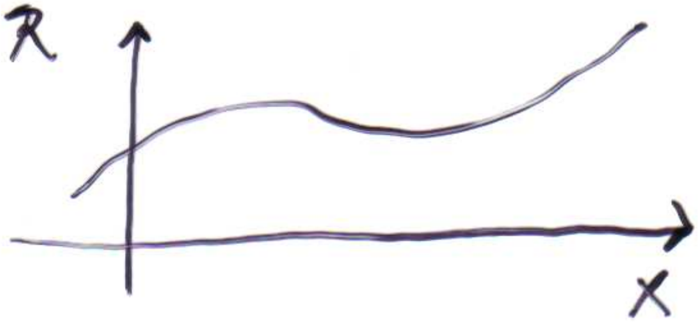
Hence

$$\boxed{\text{The Lawson closure of } F \text{ is a Hausdorff compactification}}$$

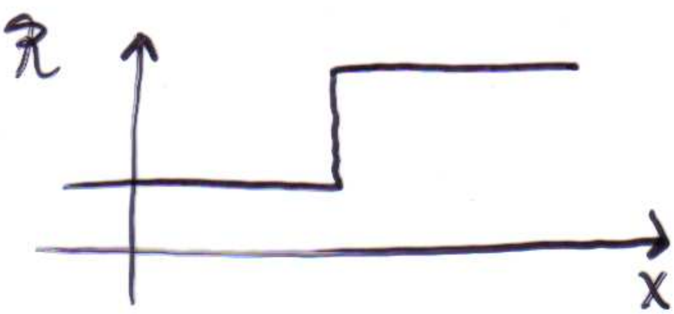
Examples to keep in mind.

Take $X = [0,1]$ with Euclidean topology

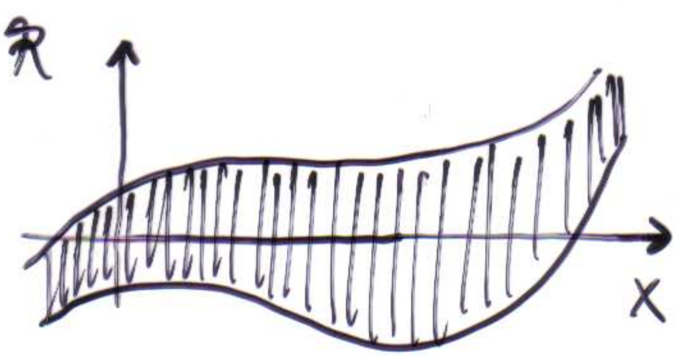
The following are examples of <u>continuous</u> maps $\boxed{[0,1] \to \mathbb{R}.}$



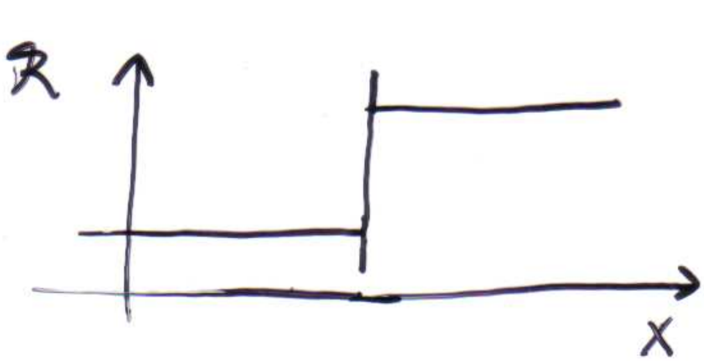
maximally valued



maximal



neither



neither

Unfortunately, the conjecture fails as badly as it can. (Tuesday).

Consider $X = [0,1]$ or $X = [0,1]^2$.

Then the Lawson closure of the maximally valued functions in $(X \to \mathcal{R})$ is the whole of $(X \to \mathcal{R})$.

Wednesday, with Klaus Keimel and Andrej Bauer:

In fact, the same conclusion holds for any locally compact Hausdorff space X.

(We have 2.5 proofs.)

In any case, $(X \to \mathcal{R})$ under the Lawson topology is ~~both~~ a compactification of both $C_0(X, [-\infty, \infty])$ (and hence $C_0(X, \mathbb{R})$) and the Samborski function space.

Hence the title of the talk.

Conjecture (Monday)

The Samborski function space F is already Lawson closed.

(And hence Samborski compact.)

Moreover, F is the Lawson closure of the set of maximally valued functions $(X \to \mathcal{R})$.

N.B.

$$C(X, [-\infty, \infty]) \overset{\text{homeomorphically}}{\cong} \text{maximally valued functions } (X \to \mathcal{R})$$

with compact-open topology

with relative

Scott
= Lawson
= Compact-open

topology

So, if the conjecture were true, then the Samborski function space would be the Hausdorff compactification of $C_{co}(X, [-\infty, \infty])$ taken as the running example in my function-space compactifications paper.

$\boxed{\text{That would be really nice.}}$

II. Even worse, from the point of view of Monday's conjecture.

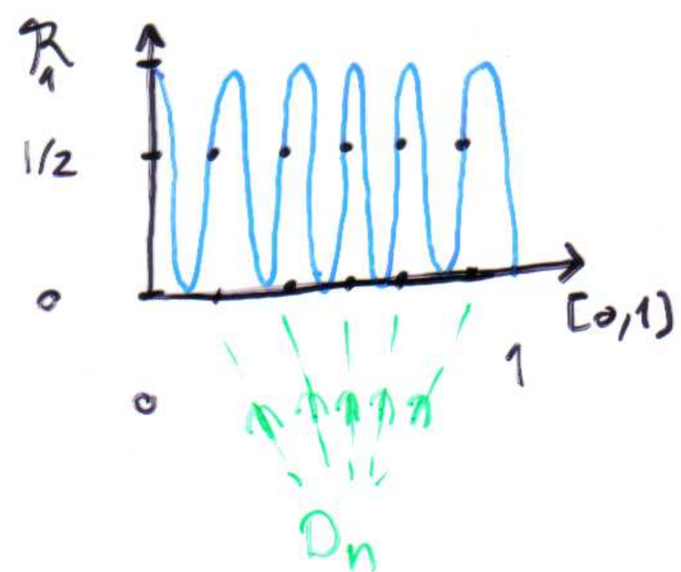
(which, I remind you, was:

Lawson closure of the set of the maximally valued functions $=$ Samborski function space,

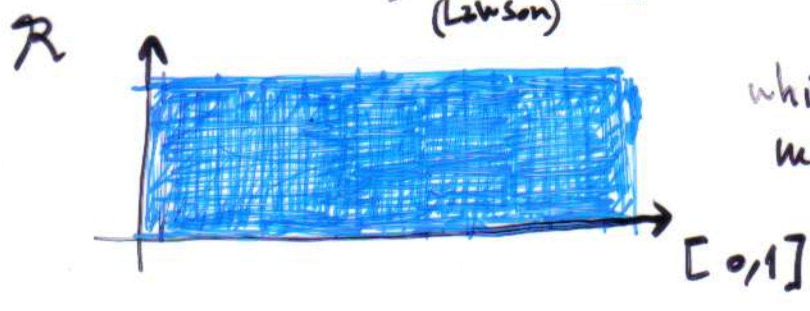
and hence the Samborski function space would be compact.)

Let $D_n \subseteq [0,1]$ be finite sets s.t. $\bigcup_n D_n$ is dense.

Define $f_n : [0,1] \longrightarrow \mathbb{R}$ by



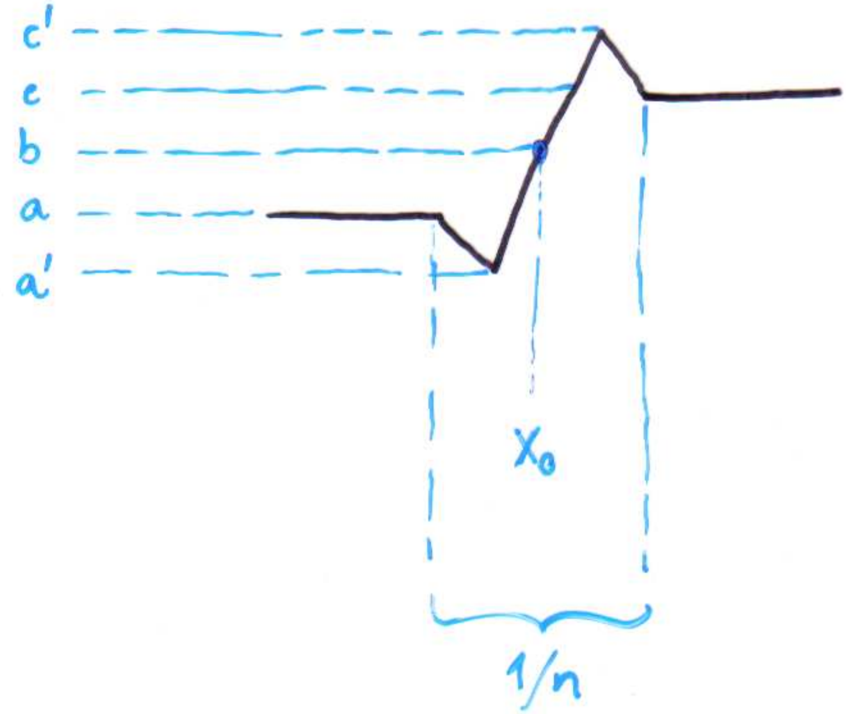
Then, by an argument of the type of the previous construction, $\lim\limits_{(\text{Lawson})} f_n$ exists and is:



which is far from maximal; it is $\perp$!

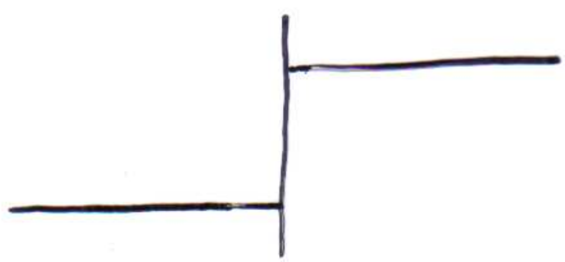
Tuesday's Constructions

I. Let $f_n : [0,1] \longrightarrow \mathcal{R}$ be



Because $([0,1] \to \mathcal{R})$ is Lawson compact, f_n has a convergent subsequence in the Lawson topology.

Whatever it is, its limit is



because this is the order-theoretic lim-sup, which is what the Lawson limit has to be. ∎

But this is not a Samborski function. (not maximal)

III An improvement of the argument of II

shows that any (Scott) continuous $f : [0,1] \to \mathcal{R}$

is the limit of a sequence of maximally valued

(and hence Sambarski) functions in the Lawson

topology.

(i.e the maximally valued Scott continuous functions

are Lawson dense.)

Last evening:

IV A further improvement (joint work with

Andrej and simultaneous but independent joint

work with Klaus) allows us to replace $[0,1]$

by any locally compact Hausdorff space in III.

Corollary Monday's conjecture fails as badly as it can.

On the positive side:

We have calculated the function-space compactification

of the running example, $C_{co}(X, \mathbb{R})$, of my

function-space compactifications paper.

Theorem It is $(X \to \mathcal{R})$ under the Lawson topology.

In other words:

Theorem For any locally compact ~~space~~ Hausdorff space X, $C_{co}(X, \mathbb{R})$ is a dense subspace of the compact Hausdorff space consisting of the Scott continuous functions $X \to \bar{\mathbb{R}}$ under the Lawson topology.

Conclusion

Sometimes it is good to formulate wrong conjectures.

In fact

My function-space compactifications paper originated with a wrong conjecture and a derived positive result in the first of these (so far) three series of topology meetings in Dagstuhl.

I thank the organizers and I hope we'll have many more such meetings in this nice environment.