# Applications of the $\lambda$-calculus to topology

(Proofs that fit in the margin of Fermat's book.)

Martín Escardó

University of Birmingham, UK

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Ten proofs in topology that together fit in a single page of Fermat's book

(i) $(x \notin Q) = (\forall y \in Q.\ x \neq y)$ (ii) $(\forall x \in C.\ p(x)) = \forall x \in X.\ x \notin C \vee p(x)$ (iii) $(\forall y \in f(Q).\ p(y)) = \forall x \in Q.\ p(f(x))$

(iv) $(\forall z \in X \times Y.\ p(z)) = \forall x \in X.\ \forall y \in Y.\ p(x,y)$ (v) $(f \neq g) = \exists x \in X.\ f(x) \neq g(x)$ (vi-ix) Exercise

(x) $X$ is compact iff $\forall Y.\ X \times Y \xrightarrow{\pi_2} Y$ is closed. „⇒" Let $Y$ be any space and $C \subseteq X \times Y$ closed. $y \in \pi_2(C)$ iff $\exists x \in X.\ (x, y) \in C$. $y \notin \pi_2(C)$ iff $\forall x \in X.\ (x, y) \notin C$.

„⇐" Choose $Y = S^X$ ($S$ Sierpinski) and $C \subseteq X \times S^X$ as $\{(x, p) \mid \neg p(x)\}$ (consider $ev: S^X \times X \to S$). Now $p \in \pi_2(C)$ iff $\exists x.\ \neg p(x)$; $p \notin \pi_2(C)$ iff $\forall x.\ p(x)$. Hence $\forall x.\ p(x)$ is continuous in $p$, whence $X$ is compact.

Consider the following topological facts:

(i) A compact subspace of a Hausdorff space is closed.

(ii) A closed subspace of a compact space is compact.

(iii) A continuous image of a compact space is compact.

(iv) A product of two compact spaces is compact.

(v) If $X$ is exponentiable and $Y$ is Hausdorff, then $Y^X$ is Hausdorff.

(vi) If $X$ is exponentiable and compact then

$$\sup : \mathbb{R}^X \longrightarrow \mathbb{R} \quad \text{is continuous.}$$

(vii) $\int_0^1 : \mathbb{R}^{[0,1]} \longrightarrow \mathbb{R}$ is continuous.

(viii) $2^{2^{\mathbb{N}}} \cong \mathbb{N}$ ($2$ and $\mathbb{N}$ discrete)

(ix) If $X$ is exponentiable, $Q \subseteq X$ is compact and $V \subseteq Y$ is open, then $\{f \in Y^X \mid f(Q) \subseteq V\} \subseteq Y^X$ is open.

(x) $X$ is compact iff $\forall Y.\ X \times Y \xrightarrow{\pi_2} Y$ is closed

(xi) $X \xrightarrow{f} Z$ is proper

(i.e. closed with compact fibers)

iff $\forall\, Y \xrightarrow{g} Z$ $\quad g^*f$ is closed

where

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f^*g"] \arrow[d, "g^*f"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Summary of proofs

(i) $(x \notin Q) = (\forall y \in Q .\ x \neq y)$

(ii) $(\forall x \in C .\ p(x)) = \forall x \in X .\ x \notin C \vee p(x)$

(iii) $(\forall y \in f(Q) .\ p(y)) = \forall x \in Q .\ p(f(x))$

(iv) $(\forall z \in X \times Y .\ p(z)) = \forall x \in X .\ \forall y \in Y\ p(x, y)$

(v) $(f \neq g) = \exists x \in X .\ f(x) \neq g(x)$

(vi-ix) Exercise. (Actually solved in this talk)

(x) See below

(xi) Similar to (x)

# Tool: Church's $\lambda$-calculus

A calculus of functions

(0) $3 + y : \mathbb{R}$

(i) $\lambda y.\ 3+y : \mathbb{R} \to \mathbb{R}$

(ii) $\lambda x \lambda y.\ x+y : \mathbb{R} \to \mathbb{R}^{\mathbb{R}}$

(iii) $(\lambda x. \lambda y\ x+y)(3) = \lambda y.\ 3+y$

(iv) $\lambda f. \int_0^1 f : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$

(v)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$g \circ f = \lambda x.\ g(f(x))$$

$R \to (R \to R)$

Let $X$ and $Y$ be topological spaces.

Can we topologize $C(X,Y)$ in such a way that, for all spaces $A$,

$$A \xrightarrow{f} C(X,Y) \quad \text{is continuous}$$

iff

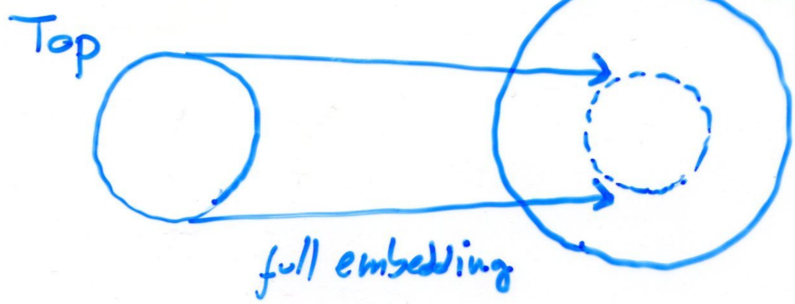
$$\begin{aligned} A \times X &\xrightarrow{\bar{f}} Y \\ (a,x) &\longmapsto f(a)(x) \end{aligned} \quad \text{is continuous ?}$$

No.

We can do that if and only if $X$ is core-compact.

$\Rightarrow$ Top is not cartesian closed

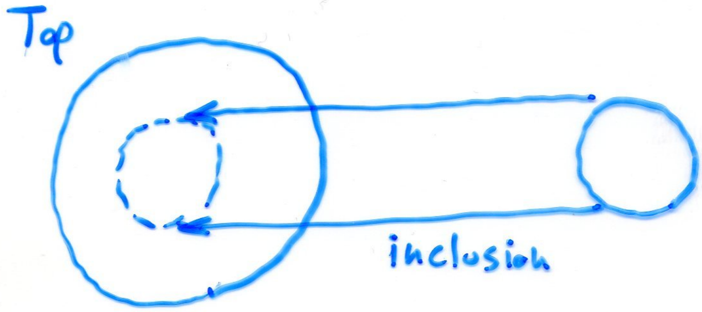
1) Enlarge




bigger category

e.g.
- filter spaces
- limit spaces
- equilogial spaces
- quasitopological spaces
- presheaves

2) shrink




smaller category

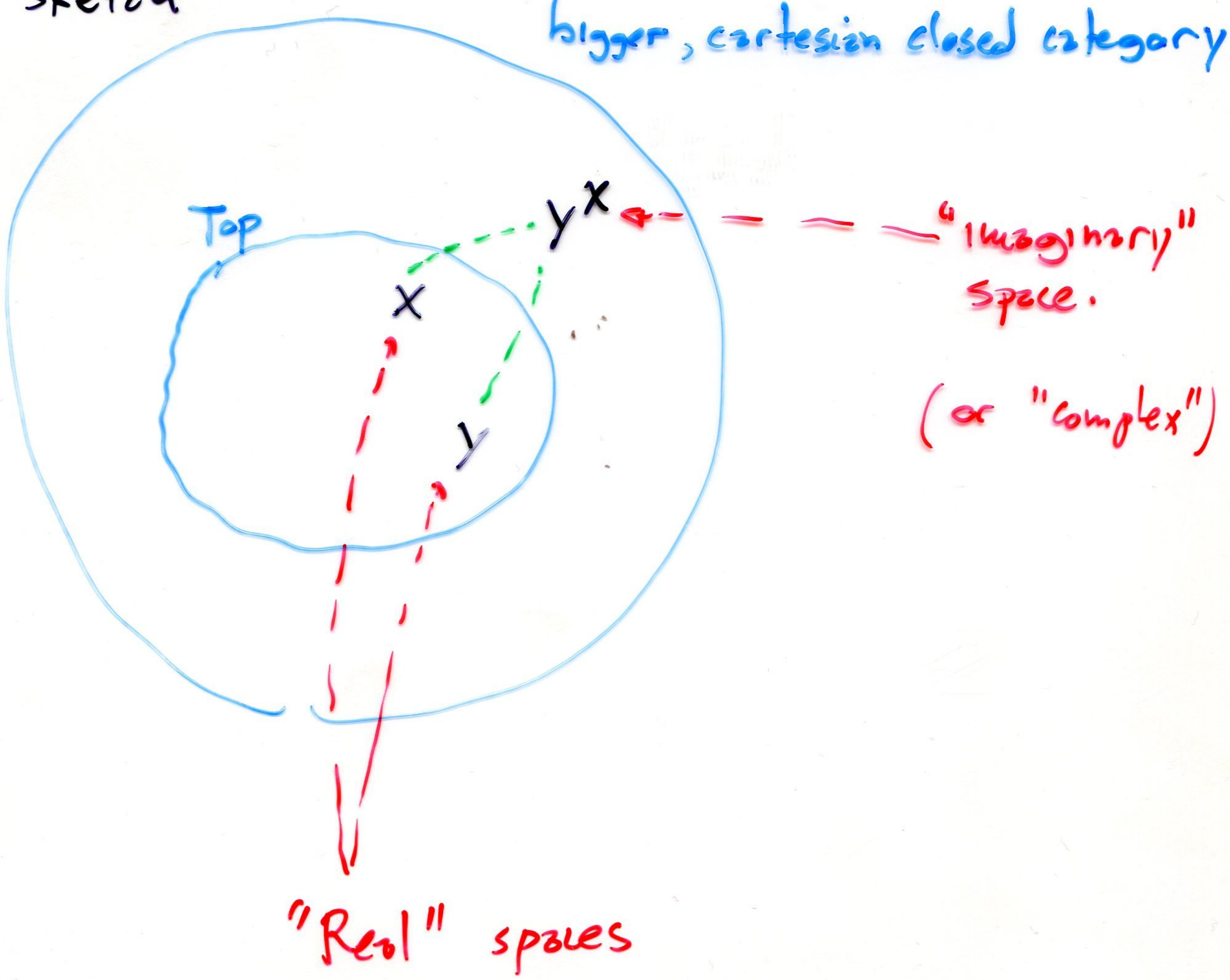
e.g.
- sequential spaces
- → compactly generated spaces
- quotients of loc compact
- quotients of $2^{nd}$ countable

The $\lambda$-definibility lemma

Let $X$ and $Y$ be topological spaces.

If $f : X \longrightarrow Y$ is $\lambda$-definable from continuous functions, then it is itself continuous.

Proof sketch



# Real versus imaginary spaces

$\boxed{\text{Lemma}}$ In many situations, such as the ones considered in this talk, "imaginary" spaces obey the rules of "real" spaces.

$S =$ , the wonderful Sierpinski space

(i) $p : X \longrightarrow S$ continuous $\iff$ $p^{-1}(1)$ open.

(ii) $U \subseteq X$ open iff $\chi_U : X \longrightarrow S$ continuous.

$\boxed{\text{Lemma}}$ Let $Q$ be a subspace of a space $X$.

Define $\forall_Q : S^X \longrightarrow S$

$\forall_Q(p) = 1$ iff for all $x \in Q$. $p(x) = 1$.

Then $Q$ is compact iff $\forall_Q$ is continuous.

$\boxed{\text{Lemma}}$ $(\vee), (\wedge) : S^2 \longrightarrow S$ are continuous.

A compact subspace of a Hausdorff space is closed.

Proof

Let $X$ be Hausdorff and $Q \subseteq X$ be compact.

Then $(\neq) : X \times X \longrightarrow S$

and $\forall_Q : S^X \longrightarrow S$

are continuous.

$$(x \notin Q) = (\forall y \in Q.\ x \neq y) = \forall_Q (\lambda y.\ x \neq y)$$

I.e. we can continuously detect that $x$ is not in $Q$, and hence that $Q$ is closed, by continuously checking that it is distinct from all points in $Q$.

If we can continuously tell points of $X$ apart and we can continuously quantify over $Q$, then we can continuously detect membership in the complement of $Q$.

(Also, can replace "continuously" by "computably".)

Translation of the $\lambda$-expression $\boxed{\lambda x.\ \forall y \in Q.\ x \neq y}$

From

$$(\neq) : X \times X \longrightarrow S$$

$$\forall_Q : S^X \longrightarrow S$$

get

$$\overline{(\neq)} : X \longrightarrow S^X$$

and then

$$\begin{tikzcd} X \arrow[r, "\overline{(\neq)}"] \arrow[rr, bend right, dashed, "\chi_{X \setminus Q}"'] & S^X \arrow[r, "\forall_Q"] & S \end{tikzcd}$$

A closed subspace of a compact space is compact.

Let $X$ be compact and $C \subseteq X$ be closed.

$$\left(\forall x \in \cancel{X}.\ p(x)\right) = \left(\forall x \in X.\ x \notin C \vee p(x)\right)$$

- To show that $C$ is compact, show that its universal quantifier is continous.

- To do this, $\lambda$-define it from known continuous functions.

A continuous image of a compact space is compact.

Let $f : X \longrightarrow Y$ be continuous and $Q \subseteq X$ be compact.

$$\left( \forall y \in f(Q) .\ p(y) \right) = \left( \forall x \in Q .\ p(f(x)) \right)$$

- To show that $f(Q)$ is compact, show that its universal quantifier is continuous
- To do this, $\lambda$-define it from known continuous functions, in this case the quantifier of $Q$.

# Binary Tychonoff's theorem.

A product of two compact spaces is compact

Let $X$ and $Y$ be compact.

$$\left(\forall z \in X \times Y \quad p(z)\right) = \forall x \in X \; \forall y \in Y \quad p(x,y)$$

Magic ?

$$\forall_X : S^X \longrightarrow S \qquad\qquad \forall_Y : S^Y \longrightarrow S$$

$$(\forall_X)^Y : (S^X)^Y \longrightarrow S^Y$$

$$\begin{tikzcd} S^{X \times Y} \arrow[r] \arrow[rrr, dashed, bend right, "\forall_{X \times Y}"'] & (S^X)^Y \arrow[r, "(\forall_X)^Y"] & S^Y \arrow[r, "\forall_Y"] & Y \end{tikzcd}$$

"Fubini's rule"

If $X$ is exponentiable and $Y$ is Hausdorff then $Y^X$ is Hausdorff

start with $\quad (\neq) : Y \times Y \longrightarrow S$

$$\exists_X : S^X \longrightarrow S$$

then "define" $\quad (\neq) : Y^X \times Y^X \longrightarrow S$

by

$$(f \neq g) = \left(\exists x.\ f(x) \neq g(x)\right)$$

We don't need to know what the topology of $Y^X$ is!

(In any case, we'll look at it shortly.)

## Tychonoff

$Y$ compact, $X$ discrete $\Rightarrow$ $Y^X$ compact

## "co-Tychonoff"

$Y$ discrete, $X$ compact $\Rightarrow$ $Y^X$ discrete

Discrete = "co-Hausdorff"

$$Y \times Y \longrightarrow S \qquad \text{continuous}$$
$$(x, y) \longmapsto (x = y)$$

$\rightsquigarrow$

$$(=) : Y^X \times Y^X \longrightarrow S$$
$$(f = g) = \forall x \in X : f(x) = g(x)$$

**Corollary** $2^{2^{\mathbb{N}}} \cong \mathbb{N}$ $\qquad$ $2$ & $\mathbb{N}$ discrete.

**Proof** $2^{\mathbb{N}}$ compact by Tychonoff

$2^{2^{\mathbb{N}}}$ discrete by co-Tychonoff

$2^{2^{\mathbb{N}}} \cong$ clopens of $2^{\mathbb{N}}$ $\rightsquigarrow$ There are countably many.

# Topology of $Y^X$

If $Q \subseteq X$ is compact and $V \subseteq Y$ is open then

$$N(Q,V) \stackrel{\text{def}}{=} \{ f \in Y^X \mid f(Q) \subseteq V \} \text{ is open.}$$

$$\big(f \in N(Q,V)\big) = \big(\forall x \in Q.\ f(x) \in V\big)$$

- The characteristic function of $N(Q,V)$ is $\lambda$-definable from continuous functions, and hence it is continuous.

The End