Compact totally separated types

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A search problem

Given a type X and a function $p : X \to \mathbf{2}$,

- either exhibit $x : X$ with $p\,x = 0$, a root of p,
- or else determine that p has no root.

$$\prod_{p:X\to\mathbf{2}} \Big(\Big(\sum_{x:X} p\,x = 0\Big) + \Big(\prod_{x:X} p\,x = 1\Big)\Big)$$

Trivial for finite X.

Perhaps surprisingly, some infinite types X admit this.

Three readings

- Computation. Exhaustive search of an infinite type in finite time.
- Topology. Compactness.
- Logic. Infinite instances of choice, proved rather than postulated.

We call such types compact.

But perhaps we should call them boolean compact.

Requiring this for every type of a universe amounts to global choice

We can find a point of every non-empty type,

$$\prod_{X:\mathcal{U}} (\neg\neg X \to X),$$

which is stronger than the axiom of choice (and contradicts univalence).

The principle $\prod_{X:\mathcal{U}} (X \text{ is a proposition} \to \neg\neg X \to X)$ is consistent with univalence.

So we ask which types are compact, and how to build them

- The natural numbers $\mathbb{N}$ fail to be compact constructively: their compactness is Bishop's LPO.
- But there are plenty of infinite compact types.
- All the ones we can build come with well-orders, and the ordinals so obtained measure their logical complexity.

Constructive taboos used below:

$$\mathrm{LPO} :\equiv \prod_{p:\mathbf{2}^{\mathbb{N}}} \mathrm{decidable}\Big(\sum_n p\, n = 0\Big), \qquad \mathrm{WLPO} :\equiv \prod_{p:\mathbf{2}^{\mathbb{N}}} \mathrm{decidable}\Big(\prod_n p\, n = 1\Big).$$

Setting

Intensional Martin-Löf type theory with the univalence axiom and set quotients.

Univalence is needed from the ordinals onwards, and before that only function and propositional extensionality. (We could work with setoids instead.)

- We reason constructively, and assume no continuity principle.
  In particular, $p : X \to \mathbf{2}$ is not assumed to be continuous.
- So the results hold in all topos models, and are consistent with excluded middle and choice.
- Johnstone's topological topos serves as a guiding model: types are spaces, and functions into $\mathbf{2}$ classify clopen subspaces.
- For a closed subtype of a simple type, compactness in our sense agrees with topological compactness there.

Perhaps the simplest infinite compact type

The decreasing binary sequences, or conatural numbers:

$$\mathbb{N}_\infty :\equiv \sum_{\alpha:\mathbf{2}^{\mathbb{N}}} \prod_{i:\mathbb{N}} \alpha\,(i{+}1) \leq \alpha\, i, \qquad \underline{n} = 1^n 0^\omega, \quad \infty = 1^\omega.$$

Topologically, the one-point compactification of $\mathbb{N}$.

Theorem. The type $\mathbb{N}_\infty$ is compact. (Already in Gödel's system T, JSL 2013.)
Given $p : \mathbb{N}_\infty \to \mathbf{2}$, define

$$(\varepsilon\, p)_n = \min\{p\,\underline{k} \mid k \leq n\}.$$

- The sequence $\varepsilon\, p$ is decreasing, so a point of $\mathbb{N}_\infty$.
- If $p\,(\varepsilon\, p) = 1$, then p is constantly 1.
- In fact $\varepsilon\, p$ is the infimum of the roots of p, and a root if p has one.

Used by Normann and Tait (fan functional) and by Pradic and Brown (Cantor–Bernstein implies excluded middle).

Totally separated types

Compactness says little about a type with few functions into $\mathbf{2}$.
(E.g. it is consistent that every $p : \mathbb{R} \to \mathbf{2}$ is constant,
making $\mathbb{R}$ trivially (boolean) compact.)

A type X is totally separated if the functions into $\mathbf{2}$ separate its points:

$$\prod_{x,y:X} \Big(\prod_{p:X\to\mathbf{2}} p\,x = p\,y \Big) \to x = y.$$

This is a boolean Leibniz principle.
Topologically, the clopens separate the points.

- Totally separated types are sets in the sense of HoTT/UF.
- They are closed under Π, $+$, $\times$ and retracts, and include the discrete types.
- They fail to be closed under Σ constructively.
  E.g. the total separatedness of $\sum_{u:\mathbb{N}_\infty} \big((u = \infty) \to \mathbf{2}\big)$ implies $\neg\neg$WLPO.

Building compact types

Closure. The types $\mathbf{0}$ and $\mathbf{1}$ are compact, and compact types are closed under $+$, Σ (hence $\times$), retracts and images.

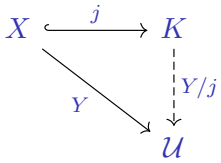
Lemma. A type X is compact and pointed if and only if every $p : X \to \mathbf{2}$ has a universal witness x_0:

$$p\,x_0 = 1 \;\to\; \forall x.\; p\,x = 1.$$

Micro-Tychonoff theorem. If P is a proposition and $Y\,x$ is compact pointed for every $x : P$, then $\prod_{x:P} Y\,x$ is compact pointed.

Full Tychonoff is not provable: compactness of the Cantor type $\mathbf{2}^{\mathbb{N}}$ is independent, true in the topological topos and false in the effective topos.

Extending families along embeddings



$$(Y \,/\, j)\, k :\equiv \prod_{(x,-) : \mathrm{fib}_j(k)} Y\, x$$

The fiber $\mathrm{fib}_j(k) :\equiv \sum_{x:X} j\, x = k$ is a proposition, because j is an embedding.

- $(Y \,/\, j)(j\, x) \simeq Y\, x$, so this is an extension.
- Off the image, $(Y \,/\, j)\, k \simeq \mathbf{1}$.
- By micro-Tychonoff, if every $Y\, x$ is compact pointed, so is every $(Y \,/\, j)\, k$.
- Hence, if K is compact pointed, so is $\sum_{k:K} (Y \,/\, j)\, k$.
- Case of interest: $\mathbb{N} \hookrightarrow \mathbb{N}_\infty$.

Under univalence, ordinals and totally separated ordinals are also injective in this sense.

Ordinals

An ordinal is a type with a proposition valued relation $<$ that is transitive, extensional and well-founded. (HoTT Book.)

Two classical properties of ordinals:

- Trichotomy: $x < y$ or $x = y$ or $x > y$.
  - For all ordinals, it is equivalent to excluded middle.
  - Many trichotomous ordinals exist constructively, and they are discrete.
  - In the topological topos, discrete compact objects are finite.
  - So trichotomy and compactness together give only finite examples.
- Least element property for non-empty subsets.
  - For all non-empty subsets, it fails constructively. (We get excluded middle)
    For all non-empty complemented subsets, it holds for some (compact) ordinals.
  - Classically every subset is complemented.
    Topologically, the complemented subsets are the clopens.

Two manifestations of $\omega + 1$

$$\begin{aligned} \mathbb{N} + \mathbf{1} &\hookrightarrow \mathbb{N}_\infty \\ \operatorname{inl} n &\mapsto 1^n 0^\omega \\ \operatorname{inr} \star &\mapsto 1^\omega \end{aligned}$$

$\mathbb{N} + \mathbf{1}$ is

- trichotomous
- discrete
- compact iff LPO
- with infima of complemented subsets iff LPO

the inclusion is

- an order embedding
- extremely dense: its image has empty complement
- a bijection iff LPO

$\mathbb{N}_\infty$ is

- compact, totally separated
- with infima of complemented subsets
- discrete iff WLPO

Every decreasing sequence is of the form $1^n 0^\omega$ or 1^ω: equivalent to LPO.
There is no decreasing sequence other than these: holds.

Classically the same ordinal. In the topological topos and in the effective topos, different.

Generalizing $\omega + 1$

We want a system of ordinal codes with two interpretations

$$\text{discrete ordinal} \hookrightarrow \text{compact ordinal},$$

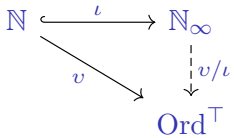
with an embedding whose image has empty complement, as in $\mathbb{N} + \mathbf{1} \hookrightarrow \mathbb{N}_\infty$.

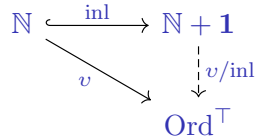
For $\upsilon : \mathbb{N} \to \mathrm{Ord}^\top$ (topped ordinals), we define

$$\textstyle\sum^1 \upsilon :\equiv \sum_{u:\mathbb{N}_\infty} (\upsilon \,/\, \iota)\, u \qquad \text{extended sum, over } \mathbb{N}_\infty$$

$$\textstyle\sum_1 \upsilon :\equiv \sum_{u:\mathbb{N}+\mathbf{1}} (\upsilon \,/\, \mathrm{inl})\, u \qquad \text{successor sum, over } \mathbb{N} + \mathbf{1}$$

ordered lexicographically.





Generalizing $\omega + 1$

$$\textstyle\sum^1 v :\equiv \sum_{u:\mathbb{N}_\infty} (v \,/\, \iota)\, u \qquad \text{extended sum, over } \mathbb{N}_\infty$$

$$\textstyle\sum_1 v :\equiv \sum_{u:\mathbb{N}+\mathbf{1}} (v \,/\, \mathrm{inl})\, u \qquad \text{successor sum, over } \mathbb{N} + \mathbf{1}$$

- There is an extremely dense, order-preserving and order-reflecting embedding $\sum_1 v \hookrightarrow \sum^1 v$, induced by $\mathbb{N} + \mathbf{1} \hookrightarrow \mathbb{N}_\infty$.
- Topped fibers make lexicographic sums extensional.
  In general, extensionality of lexicographic sums implies excluded middle.
- The extended sum generalizes the squashed sum of JSL 2013.

Brouwer codes and five interpretations

$$\mathsf{Z} : \mathrm{B}, \qquad \mathsf{S} : \mathrm{B} \to \mathrm{B}, \qquad \mathsf{L} : (\mathbb{N} \to \mathrm{B}) \to \mathrm{B}.$$

	Z	S	L	name
$[\![-]\!]_{\mathrm{sup}}$	$\mathbf{0}$	$-+\mathbf{1}$	sup	standard
$[\![-]\!]_{\Sigma}$	$\mathbf{0}$	$-+\mathbf{1}$	$\sum$ over ω	trichotomous
$[\![-]\!]_{\Sigma_1}$	$\mathbf{1}$	$-+\mathbf{1}$	$\sum_1$	discrete
$[\![-]\!]_{\mathrm{sup}^1}$	$\mathbf{1}$	$-+\mathbf{1}$	sup of the extension	compact
$[\![-]\!]_{\Sigma^1}$	$\mathbf{1}$	$-+\mathbf{1}$	$\sum^1$	compact totally separated

- Suprema exist constructively, assuming univalence and set quotients, and a supremum of compact ordinals indexed by a compact type is compact.
- Z goes to $\mathbf{1}$ in the compact ones, so that every denoted ordinal is pointed and hence micro-Tychonoff is applicable, and in the discrete one, to match the compact totally separated one.

Properties of the various interpretations

	holds for every code	fails constructively
$[\![b]\!]_{\mathrm{sup}}$		trichotomy, compactness, least element property (LPO), discreteness (WLPO)
$[\![b]\!]_{\Sigma}$	trichotomy, discreteness	compactness, least element property (LPO)
$[\![b]\!]_{\Sigma_1}$	trichotomy, discreteness, retract of $\mathbb{N}$	compactness, least element property (each $\iff$ LPO)
$[\![b]\!]_{\mathrm{sup}^1}$	compactness	total separatedness ($\neg\neg$WLPO)
$[\![b]\!]_{\Sigma^1}$	compactness, total separatedness, retract of $\mathbf{2}^{\mathbb{N}}$, least element property	trichotomy, discreteness (WLPO)

The failure for $[\![-]\!]_{\mathrm{sup}^1}$ comes from a Sierpiński ordinal: compact totally separated types are not closed under suprema constructively.

Comparing the interpretations

Theorem. Assuming excluded middle, for every Brouwer code b,

standard	$[\![b]\!]_{\mathrm{sup}}$	$\leq$	$[\![b]\!]_{\Sigma}$	trichotomous
	$\mid\wedge$		$\mid\wedge$	
compact	$[\![b]\!]_{\mathrm{sup}^1}$	$\leq$	$[\![b]\!]_{\Sigma^1}$	compact totally separated

Moreover, $[\![b]\!]_{\Sigma_1} = [\![b]\!]_{\Sigma^1}$.

So the compact totally separated interpretation is the largest.

- Excluded middle is used for the monotonicity of $- + \mathbf{1}$, and to turn order-preserving maps into initial segments.
- The top row for every b gives LPO, and the bottom row gives $\neg\neg$WLPO.

Comparing $[\![-]\!]_{\Sigma_1}$ and $[\![-]\!]_{\Sigma^1}$ constructively

They differ only at L: successor sum versus extended sum, that is, $\mathbb{N} + \mathbf{1}$ versus $\mathbb{N}_\infty$.

Theorem. For every Brouwer code b there is a map

$$\iota_b : [\![b]\!]_{\Sigma_1} \to [\![b]\!]_{\Sigma^1},$$

defined by induction on b, that is

- an embedding,
- extremely dense,
- order-preserving and order-reflecting.

It is an equivalence for every b if and only if LPO holds.

The case $b = \mathsf{L}(n \mapsto \mathsf{Z})$ is $\mathbb{N} + \mathbf{1} \hookrightarrow \mathbb{N}_\infty$ again.

The least element property

A topped ordinal X has infima of complemented subsets if every $p : X \to \mathbf{2}$ has a point x_0 that is

- a conditional root: a root if p has a root, and
- the infimum of the roots, for $x \leq y :\equiv \neg(y < x)$.

This strengthens compact pointedness, and gives the least element property for non-empty complemented subsets.

Theorem. For every Brouwer code b, the ordinal $[\![b]\!]_{\Sigma^1}$ has infima of complemented subsets, so every non-empty complemented subset has a least element.

By induction on b, using the selection function of $\mathbb{N}_\infty$ and a strengthening of the micro-Tychonoff theorem for infima.

For $[\![b]\!]_{\Sigma_1}$, the least element property for every b is equivalent to LPO.

Beyond ω-branching

In Brouwer codes, L is always indexed by $\mathbb{N}$. We let the branching be indexed by any discrete ordinal built earlier.

By induction-recursion, a type E of codes together with the discrete interpretation $\Delta : \mathrm{E} \to \mathrm{Ord}^{\top}$:

$$\ulcorner \mathbf{1} \urcorner,\ \ulcorner \omega{+}1 \urcorner : \mathrm{E}, \qquad \ulcorner + \urcorner,\ \ulcorner \times \urcorner : \mathrm{E} \to \mathrm{E} \to \mathrm{E}, \qquad \ulcorner \Sigma \urcorner : \prod_{\nu : \mathrm{E}} \big((\Delta\nu \to \mathrm{E}) \to \mathrm{E}\big),$$

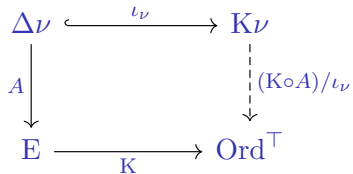
$$\Delta \ulcorner \mathbf{1} \urcorner = \mathbf{1}, \qquad \Delta \ulcorner \omega{+}1 \urcorner = \omega + 1, \qquad \Delta(\ulcorner \Sigma \urcorner\, \nu\, A) = \textstyle\sum_{x : \Delta\nu} \Delta(A\, x),$$
$$\Delta(\nu_0 \ulcorner + \urcorner \nu_1) = \Delta\nu_0 + \Delta\nu_1, \qquad \Delta(\nu_0 \ulcorner \times \urcorner \nu_1) = \Delta\nu_0 \times \Delta\nu_1.$$

Theorem. Every $\Delta\nu$ is trichotomous and a retract of $\mathbb{N}$, hence discrete. Its compactness for every ν is equivalent to LPO.

The compact interpretation of E

Simultaneously, by recursion, $\mathrm{K} : \mathrm{E} \to \mathrm{Ord}^\top$ and embeddings $\iota_\nu : \Delta\nu \hookrightarrow \mathrm{K}\nu$, with $\mathrm{K}^\ulcorner\omega{+}1^\urcorner = \mathbb{N}_\infty$ and

$$\mathrm{K}(^\ulcorner\Sigma^\urcorner\, \nu\, A) \;=\; \sum_{y:\mathrm{K}\nu} \big((\mathrm{K} \circ A) \,/\, \iota_\nu\big)\, y.$$



by injectivity of $\mathrm{Ord}^\top$

Theorem. For every $\nu : \mathrm{E}$, the ordinal $\mathrm{K}\nu$ has infima of complemented subsets, hence is compact, and ι_ν is extremely dense, order-preserving and order-reflecting.

- The Brouwer codes embed into E, preserving both interpretations.
- But total separatedness of $\mathrm{K}\nu$ for every ν gives $\neg\neg$WLPO.

Isolated points and limit points

A point x is isolated if $x = y$ is decidable for every y.

Under excluded middle every point of a set is isolated, so "not isolated" is no good as a definition of limit point.

Instead we call x a limit point if

$$x \text{ is isolated } \to \text{ WLPO.}$$

In $\mathbb{N}_\infty$, the finite points are isolated, and ∞ is a limit point.

Every point is isolated or a limit point

Theorem. There is a limit point indicator $\ell_\nu : \Delta\nu \to \mathbf{2}$, defined by induction on ν, such that for every $\nu : \mathrm{E}$ and $x : \Delta\nu$,

- if $\ell_\nu\, x = 0$, then $\iota_\nu\, x$ is isolated in $\mathrm{K}\nu$,
- if $\ell_\nu\, x = 1$, then $\iota_\nu\, x$ is a limit point of $\mathrm{K}\nu$.

Reaching high compact ordinals

- In Gödel's T, iterated squashed sums give compact totally separated ordinals of every rank below ε_0 (JSL 2013), and Normann proved that T can do no better.
- Coquand, Hancock and Setzer extend T with the type B of Brouwer codes, with its constructors but no elimination principle.
  The closed terms of type B denote every ordinal below ε_0, and, with one universe added, every ordinal below $\varphi_{\varepsilon_0}0$.
- Evaluating these codes by $[\![-]\!]_{\Sigma^1}$ gives compact totally separated ordinals that classically dominate the ordinals the codes denote, so dominate every ordinal below $\varphi_{\varepsilon_0}0$.
- The inductive-recursive codes E denote strictly more compact ordinals than the Brouwer codes.

Related work

Cherubini, Coquand, Geerligs and Moeneclaey, A foundation for synthetic Stone duality. (TYPES 2024)

Summary

- Compact types in Martin-Löf type theory, constructively, with no continuity axioms. Univalence is needed only for the ordinals.
- Total separatedness makes compactness informative.
- All compact types we can build are well-ordered.
- Two notation systems, each with a discrete and a compact interpretation, and an extremely dense order embedding between them.
- The discrete side gets trichotomy. The compact side gets the least element property for complemented subsets. Not both, constructively.
- The compact interpretation $[\![-]\!]_{\Sigma^1}$ of Brouwer codes is also totally separated, a Stone space in the topological topos.

Formalized in Agda, in the `TypeTopology` repository, with a companion file following the paper item by item.

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