Compact, totally separated and well-ordered types in univalent mathematics

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Progression of this work

1. This work took a long time since its gestation in 2011 until its completetion.

2. I reported preliminary results at various stages, including here in Stockholm.

3. Because the work is constructive, I wanted to implement it in the computer.

4. I chose type theory as the mathematical foundation / programming language.

5. However, I was disatisfied with the identity type, despite its clear elegance.

6. I wanted to reason extensionally, but so-called extensional type theory didn't feel like the right answer. I just don't reason extensionally in this way.

7. And I was not willing to deploy the setoid bureaucracy to emulate extensionality.

8. Then I met univalent type theory, which offered what I was wishing for.

9. So I took a break to understand univalent type theory in detail.

10. I intermitently came back to the 2011 programme and eventually completed it last year, with a full implementation in Agda.

Some distinctive characteristics of univalent type theory

1. The notion of proposition is taken to mean subsingleton type.

2. We have a distinction between three notions:

    Designated existence (data, given by Σ).

    Unspecified existence (property, given by truncated Σ).

    Classical existence (also property, given by doubly negated Σ).

3. The axiom of choice is understood in terms of unspecified existence, rather than designated existence.

    It implies that unspecified and classical existence agree.

    Moreover, their agreement is equivalent to excluded middle.

4. The notion of unique existence is taken to mean that a Σ type is a singleton.

    This generalizes the categorical notion of uniqueness up to unique isomorphism.

5. The principle of unique choice holds.

Constructive mathematics, typically

▶ Reformulate classical notions and theorems so that they admit constructive proofs.

▶ Redevelop classical mathematics in a constructive way.

Constructive mathematics here

▶ Prove things that are axioms in classical mathematics,
   but that are dubious from a constructive point of view.

   Instances of excluded middle.

   Instances of (global) choice.

Even more difficult to sell than the typical practice of constructive mathematics.

But in computation they correspond to decidability questions and to
selection problems over infinite search spaces.

Brief and incomplete summary of results

Work in a spartan univalent type theory.

(Development in Agda at github (`martinescardo/TypeTopology`).)

1. For a wide range of *infinite* types X, we have that for every $p : X \to \mathbb{2}$, the type

$$\Sigma(x : X), p\, x = 0$$

of roots of p is decidable, where $\mathbb{2} := \mathbb{1} + \mathbb{1}$ and $0, 1$ are its two points.

We can either choose a root of p or else tell that there is none.

2. The simplest counter-example is the type of natural numbers.

(When $X = \mathbb{N}$, this is Bishop's Limited Principle of Omniscience (LPO).)

The simplest example is the the type of conatural numbers.

3. The examples here turn out to be ordinals, and maybe this is not a coincidence.

Main improvement of the results since the last time we presented them

▶ We worked with ordinals in a weak sense.

▶ We could only prove that transfinite induction is satisfied for decidable predicates.

▶ We now know that it is satisfied for arbitrary predicates, so we get ordinals in a proper sense.

A spartan univalent type theory

A spartan Martin-Löf type theory.

$\mathbb{0}$, $\mathbb{1}$, $\mathbb{N}$, $+$, Π, Σ, W, Id.   (With η-rules for Π, Σ, W.)

Hierarchy of open-ended universes, ranged over by $\mathcal{U}, \mathcal{V}$.

Intensional.

Axioms. (We can use cubical type theory and cubical Agda instead of axioms.)

Existence of propositional truncations.

Univalence.

Some foundational consequences of the axioms.

Function extensionality, propositional extensionality, availability of set quotients.

Consistent classical axioms for univalent type theory

1. Excluded middle.
2. Choice.
3. Propositional resizing and impredicativity.

However, *global choice* is inconsistent with univalence.

(So, although we can say that every vector space has an unspecified basis using choice, we don't get an explicit function that assigns a basis to every vector space.)

We are neutral in the following sense:

▶ We don't assume or reject the above consistent classical axioms.

▶ Moreover, we don't assume or reject countable choice, dependent choice, Markov principle, Brouwerian axioms, Church's Thesis.

Compact types

We consider three notions of exhaustively searchable type.

▶ We say that a type X is *compact* if the type $\Sigma(x : X), p\,x = 0$ is decidable for every $p : X \to \mathbb{2}$. (It is decidable whether p has a designated root.)

(Sometimes Σ-*compact* for emphasis.)

We also consider two successively weaker notions, namely

▶ $\exists$-*compactness* (it is decidable whether there is an unspecified root) and

▶ Π-*compactness* (it is decidable whether all points are roots),

obtained by replacing Σ by $\exists$ and Π in the definition of compactness.

E.g. for $X := \mathbb{N}$, we have that

Σ- and $\exists$-compactness turn out to agree and amount to LPO.

Π-compactness amounts to WLPO.

Justification of the topological terminology

For the model of simple types consisting of Kleene–Kreisel spaces, these notions of compactness agree and coincide with topological compactness under classical logic.

But we reason constructively here.

The type of $\mathbb{N}_\infty$ conatural numbers is compact

- ▶ This type is coinductively specified by $0 : \mathbb{N}_\infty$ and $\mathrm{succ} : \mathbb{N}_\infty \to \mathbb{N}_\infty$.
- ▶ It can be concretely constructed as the type of decreasing binary sequences:

$$\mathbb{N}_\infty := \Sigma_{\alpha : \mathbb{N}_\infty \to 2} \quad \Pi_{n : \mathbb{N}} \quad \alpha_n = 0 \to \alpha_{n+1} = 0.$$

and so is available in our spartan univalent type theory.

- ▶ Embedding with empty complement:

$$\begin{array}{ccc}
\mathbb{N} \;+\; \mathbb{1} & \to & \mathbb{N}_\infty \\
n & \mapsto & 1^n 0^\omega \\
* & \mapsto & 1^\omega
\end{array}$$

This is a bijection if and only if LPO holds.

- ▶ In Johntone's Topological Topos *model* of MLTT, the interpretation of $\mathbb{N}_\infty$ is the one-point compactification of the discrete natural numbers.

Ordinals

An ordinal is a type X equipped with a proposition-valued binary relation
$- < - : X \to X \to \mathcal{U}$ which is

- transitive,
- well-founded (satisfies transfinite induction), and
- extensional (any two elements with the same predecessors are identified).

The HoTT book additionally requires the type X to be a set, but we show that this follows automatically from the extensionality requirement.

E.g. the types of natural and conatural numbers are ordinals.

By univalence, the type of ordinals in a universe is itself an ordinal in the next universe, and in particular is a (big) set.

Ordinal arithmetic

- Addition is implemented by the type former $+$, and
- multiplication by the type former $\times$ with the lexicographic order.

The compact ordinals we construct are, moreover, *order-compact* in the sense that a minimal element of $\Sigma(x : X), p\,x = 0$ is found, or else we are told that this type is empty.

Additionally, we have a selection function of type $(X \to 2) \to X$ which gives the infimum of the set of roots of any $p : X \to 2$.

In particular our compact ordinals have a top element by considering $p = \lambda x.1$.

Lexicographic order on the type $\Sigma_{x:X} \, A \, x$

$(x, a) < (y, b) := (x < y) + \Sigma_{p:x=y} \text{ transport } A \, p \, a < b$

Induced preorder:

$u \leq v := \neg(v < u)$

Fact. The induced preorder of the lexigraphic order is characterized from the induced preorders as

$(x, a) \leq (y, b) \Longleftrightarrow (x \leq y) \times \Pi_{p:x=y} \text{ transport } A \, p \, a \leq b$

Discrete types

We say that a type is discrete if it has decidable equality.

Again this corresponds to the topological notion with the same name.

Totally separated type

It may happen that a non-trivial type has no nonconstant function into the type $\mathbb{2}$ of booleans so that it is trivially compact.

We again borrow terminology from topology, for spaces whose clopens separate the points.

We say that a type is totally separated if the functions into the booleans separate the points, in the sense that

▶ any two points that satisfy the same boolean-valued predicates are equal.

This is a boolean Leibniz principle.

Totally separated reflection

- We construct a totally separated reflection for any type, and

- show that a type is compact, in any of the three senses, if and only if its totally separated reflection is compact in the same sense.

Interplay between the notions

We show that compact types, totally separated types, discrete types and function types interact in very much the same way as their topological counterparts, where

- arbitrary functions in type theory play the role of continuous maps in topology, and
- *without* assuming Brouwerian continuity axioms.

For instance,

1. if the types $X \to Y$ and Y are discrete then X is Π-compact, and
2. if $X \to Y$ is Π-compact, and X is totally separated and Y is discrete, then X is discrete, too.
3. The simple types are all totally separated, which agrees with the situation with Kleene–Kreisel spaces, but
4. it is easy to construct types which fail to be totally separated,
   - e.g. the homotopical circle,
   or whose total separatedness gives a constructive taboo
   - e.g. $\Sigma(x : \mathbb{N}_\infty), x = \infty \to 2$, where we get two copies of the point ∞.

Notation for discrete and compact ordinals

We define infinitary ordinal codes, or expression trees, similar to the so-called Brouwer ordinals, including

- one,
- addition,
- multiplication, and
- countable sum with an added top point.

Two Interpretations of the notation

We interpret these trees in two ways, getting
- ▶ discrete and
- ▶ compact ordinals respectively.

In both cases, addition and multiplication nodes are interpreted as ordinal addition and multiplication.

But in the countable sum with a top point, the top point is added with
- ▶ $- + \mathbb{1}$ in one case, and so is isolated, and
- ▶ by a limit-point construction in the other case.

Given our sequence $\mathbb{N} \to \mathcal{U}$ of types, we extend it to a family $\mathbb{N}_\infty \to \mathcal{U}$ so that it maps ∞ to a singleton type, by a certain universe injectivity construction, and then take its sum.

Two interpretations of ordinal notations

We denote the above interpretations of ordinal notations ν by

- Δ_ν,
- K_ν.

We have that

1. Δ_ν is discrete and a retract of $\mathbb{N}$.
2. K_ν is compact, totally separated and a retract of $\mathbb{N} \to 2$.
3. $\Delta_\nu \hookrightarrow K_\nu$.

   E.g. $\mathbb{N} + 1 \hookrightarrow \mathbb{N}_\infty$.
4. This embedding is a bijection $\iff$ LPO holds.
5. But it always has empty complement.
6. It is order preserving and reflecting.
7. By transfinite iteration of the countable sum, one can get rather large compact ordinals using Setzer's work.

Advertisement

If you would like to learn univalent mathematics and its expression in the computer system Agda, I invite you to check my lecture notes

https://www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/