Maybe locales are made of points after all

. . . or maybe not

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Plan

1. Related work.
2. Background.
3. Topology via the lambda-calculus.
4. Compactly generated locales.

Related work

1. Taylor (abstract Stone duality).
2. Vickers (geometric logic).
3. Vickers–Townsend (double powerlocale and exponentiation).
4. Escardó, and Bauer–Lesnik (synthetic topology in topos with a dominance).

But here I am interested in a more down-to-earth approach.

What is a locale?

A locale is a certain kind of space.

I will in fact often use the word *space* to refer to locales.

A locale is *like* a topological space.

The notion of open is primary.

But an open is not made out of points.

We have *open subspaces* rather than *open sets*.

Difference with topological space:

The subspaces are determined by the opens rather than by points.

E.g. the subspaces of the real line form very different totalities in topology and in locale theory.

Usual mathematical formulation of the notion of a locale

1. A *frame* is a complete lattice in which finite meets distribute over arbitrary joins:
$$U \wedge \bigvee_i V_i = \bigvee_i U \wedge V_i.$$

2. A *frame homomorphism* is a function that preserves finite meets and arbitrary joins.

3. The category of *continuous maps of locales* is the opposite of the category of *frame homomorphisms*.

If you take this definition, then a locale *is* a frame.

But is this what a locale "really is"?

Isbell's view

From *First Steps in Descriptive Theory of Locales* (1991):

> "The needed background is almost all in [Johnstone's *Stone Spaces*] if you can stand the point of view: Johnstone's locales keep intruding their frames into innocent conversation, rather as if people were continually showing you their skeletons-
>
> > 'Webster was much possessed by death
> > And saw the skull beneath the skin.' "

(Thanks to Alex Simpson for suggesting this appropriate quotation for this talk.)

Isbell (1991)

"For me, a locale A has a frame $T(A)$ of open parts (sublocales) and a lattice $S(A)$ of all sublocales. $S(A)$ upside down, is the frame $T(A_d)$ of the dissolution locale A_d, which has a distinguished monomorphism to A representing the sublocales of A by their pullbacks (intersections) in A_d, which are precisely the closed sublocales of A_d."

In other words, $S(A)$ is freely generated as a coframe by formally adding boolean complements to the coframe of closed sublocales.

Where I come from

The idea that open sets are observable properties of computational processes.

1. Brouwer (1920's): fundamental insights leading to the ideas below.

2. Scott (1960's): property of finite character.

3. Smyth (1980's): semi-decidable property.

4. Plotkin (1980's): physically feasible property.

5. Abramsky (1990's): observable property.

6. Vickers (1990's): affirmable property.

Points

1. Points are fictions.

   Rather useful, bonafide, ficticious entities, in fact, if you can get hold of them.

2. They are not the substance spaces are made of.

3. Rather, points are made out of opens (or of what we can observe).

4. More generally, (sub)spaces are made out of opens.

5. The notion of open in locale theory is even more primary than in topology.

   Because everything, including points, are made out of opens.

If taken at face value, the assertion of the title is false

Even the real line $\mathbb{R}$ and the Cantor space 2^ω may lack enough points when constructed in the category of locales.

If classical mathematics is available, these spaces do have enough points.

If only constructive mathematics is available, 2^ω having enough points is equivalent to an axiom, accepted by Brouwer, but by no means generally accepted (the Fan Theorem).

Interesting locales with no points at all (among many others)

Even if classical mathematics is available:

1. Isbell's smallest dense sublocale.

   Cf. Baire category theorem.

2. Alex Simpson's locale of random sequences.

   There is no sequence satisfying all the proposed axioms for random sequences.

Synthetic topology

Reason about spaces as if they were made out of points.

A *space* can be (i) a topological space, (ii) a locale, (iii) a type in a programming language, and (iv) many other things.

Price: give up classical axioms (excluded middle, axiom of choice).

Methodology: use λ-calculus, and more generally category theory, going all the way up to topos theory.

Moreover, postulate anti-classical axioms when convenient.

Pay-off

1. Get clean, short, rather general proofs, with strong computational content.

   Even in situations where computational content was not what one was looking for *a priori*.

2. Get surprising results in computability theory and its frontiers.

   But this is not what I want to emphasize in this talk.

Synthetic topology by example

A closed sublocale of a compact locale is itself compact.

Approach using bones:

1. Define sublocale, typically a nucleus on the frame.

2. Define closed nuclei.

3. Define compact nuclei.

4. Do some lattice theory.

5. Get a proof that looks very different from the proof in topology.

Synthetic proof

If X is compact and if $F \subseteq X$ is closed, then F is compact.

Proof.

$$\forall x \in F . x \in U \iff \forall x \in X . x \notin F \vee x \in U .$$

Explanation.

Compact: ability to universally quantify over open-set memberships.

$$\forall_X : \mathcal{S}^X \to \mathcal{S}.$$

Closed: ability to deny membership.

$$\chi_{X \setminus F} : X \to \mathcal{S}.$$

Define $\forall_F : \mathcal{S}^F \to \mathcal{S}$ from $\forall_X$ and $\chi_{X \setminus F}$ and $(\vee) : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ using lambda-calculus.

Synthetic topology by example

A compact sublocale of a Hausdorff locale is closed.

Approach using bones:

1. Further define Hausdorff sublocale (closed diagonal).

2. Characterize this in frame theoretic terms.

   (Complicated as concrete characterizations of localic products are non-trivial.)

3. Do some lattice theory.

4. Get a proof that looks very different from the proof in topology.

Synthetic proof

A compact sublocale K of a Hausdorff locale X is closed.

Proof.

$$x \notin K \iff \forall y \in K. x \neq y.$$

Explanation.

Hausdorff: ability to tell points apart.

$$(\neq) \colon X \times X \to \mathcal{S}$$

Define $\chi_{X \setminus K}$ from $\forall_K$ and $\neq$ using lambda-calculus.

Formalization

Needs cartesian closedness, or exponentiability as the next best thing.

Not available in geneneral.

Needs theorems justifying the logical explanations of topological concepts.

This step is fine!

Synthetic topology in general

We are willing to get proofs that are different from proofs in topology and in locale theory.

But we want the same proof for both.

Moreover, we want a proof that also applies to a computational context.

But this is the reverse what actually happened in practice:

1. Proofs in a computational context were obtained.

2. They turn out to be to be proofs in topology and locale theory!

3. And they happen to be simpler than the proofs in topology and locale theory.

We stumble into a limitation of topology and locale theory

Their categories of spaces are not cartesian closed.

Topologists stumbled into this fact much earlier, starting in the 1930's.

(Before categories were around, and for different reasons (homotopy).)

In computation, the categories involved were cartesian closed from the start.

(Even if it was not trivial to get them.)

Solutions in topology

Identify a cartesian closed subcategory.

Compactly generated spaces.

Identify a cartesian closed supercategory.

Quasi-topological spaces.

Solutions in locale theory

Non-existing.

Identify a cartesian closed subcategory.

Partial work I want to report here.

Identify a cartesian closed supercategory.

Work performed by Heckmann.

Work performed by Vickers–Townsend.

Compactly generated spaces

In topology, they form a cartesian closed subcategory.

In locale theory,

1. It is not even clear what they should be and how they behave once we identify them.

2. Once this is elucidated, there are serious obstacles in the way of obtaining cartesian closeness.

Compactly generated Hausdorff space

For a Hausdorff space, one defines

$$\mathcal{K}\,X = \text{colimit of the compact subspaces of } X \text{ connected by inclusions.}$$

Just X with a finer topology.

Hence Hausdorff too.

X is compactly generated iff this is the same topology as that of X.

Just the identity on points.

Cartesian closedness

Key facts:

1. Compact Hausdorff spaces are exponentiable.

2. $\mathcal{K}$ is a coreflection.

3. Hausdorff spaces form an exponential ideal closed under limits.

$$Y^X = \mathcal{K}\, Y^{\operatorname{colim}_{Q \subseteq X} Q} = \mathcal{K} \lim_{Q \subseteq X} Y^Q.$$

A routine verification shows that this has the universal property of an exponential, with binary products $\mathcal{K}(X \times Y)$.

Compactly generated spaces without separation

It is not appropriate to consider colimit of compact subspaces.

Will not discuss this in this talk.

Compactly generated Hausdorff locales

For a Hausdorff locale X, we again define

$\mathcal{K}\,X =$ colimit of the compact sublocales of X connected by inclusions.

The universal property of colimit gives a canonical map

$$\varepsilon_X \colon \mathcal{K}\,X \to X.$$

We say that X is *compactly generated* if this is a homeomorphism.

Lawson duality

A preframe is a poset with finite meets and directed joins such that the former distribute over the latter.

For a preframe L, one has a preframe

$$L^{\wedge} = \text{Lawson dual of } L = \text{Scott open filters of } L.$$

For any preframe homomorphism $h\colon L \to M$, one has a preframe homomorphism $h^{\wedge}\colon M^{\wedge} \to L^{\wedge}$ defined by

$$h^{\wedge}(\gamma) = \{u \in L \mid h(u) \in \gamma\}.$$

This makes Lawson dualization into a contravariant endofunctor.

Lawson duality

$$\begin{aligned} e_L\colon L &\to L^{\wedge\wedge} \\ u &\mapsto \{\phi \in L^{\wedge} \mid u \in \phi\} \end{aligned}$$

is a natural transformation.

Preframe of compact fitted sublocales

For any locale X, let

$\mathcal{Q}\,X =$ compact fitted sublocales under *reverse* sublocale inclusion,

where a sublocale is called fitted if it is the meet of its neighbourhoods.

The Hofmann–Mislove–Johnstone theorem

The assignment

$$Q \mapsto \{U \in \mathcal{O}\, X \mid Q \leq U\}$$

is an order (and hence preframe) isomorphism

$$\mathcal{Q}\, X \cong (\mathcal{O}\, X)^{\wedge}.$$

Here $\mathcal{O}\, X$ is the frame of opens (topology of the locale X).

Main theorem for compactly generated Hausdorff locales

If X is Hausdorff then all compact sublocales are fitted and

$$\mathcal{O}\,\mathcal{K}\,X \cong (\mathcal{Q}\,X)^{\wedge}.$$

Therefore X is compactly generated if and only if the opens are determined by the compacts via Lawson dualization:

$$X \text{ is compactly generated } \iff \mathcal{O}\,X \cong (\mathcal{Q}\,X)^{\wedge}.$$

Main theorem for compactly generated Hausdorff locales

By the HMJ theorem, it follows that

$$\mathcal{O}\,\mathcal{K}\,X \cong (\mathcal{O}\,X)^{\wedge\wedge}.$$

From this we conclude that

$$X \text{ is compactly generated } \iff \mathcal{O}\,X \cong (\mathcal{O}\,X)^{\wedge\wedge} \text{ naturally.}$$

Hence compactly generated Hausdorff locales seem to be very nice and well behaved objects.

Obstacles towards cartesian closeness

We don't know whether the canonical map $\mathcal{K}\,X \to X$ is a monomorphism.

Hence we don't know whether we have a coreflective subcategory.

Compact Hausdorff spaces are exponentiable (Hyland). Good.

But Y^X is Hausdorff if and only if Y is Hausdorff and X has open domain.

Moreover, what about limits of Hausdorff locales?

Summary of talk

1. Synthetic topology works well for topological spaces and for program types.

2. There are difficulties making it work for locales.

   It does work in some interesting cases, but not in general.

   No known large enough ccc of locales.

   No known locally small super ccc of locales.

Appendix

The patch construction

A striking connection with the patch construction.

This construction coreflectively transforms a locally stably compact locale X into a locally compact Hausdorff locale, denoted by $\mathrm{Patch}\, X$ and given by

$$\mathcal{O}\,\mathrm{Patch}\, X = \text{frame of Scott continuous nuclei on } \mathcal{O}\, X.$$

The patch construction

If X is additionally compact, the preframe $\mathcal{Q}\,X$ is a frame. Moreover, this is the topology of another compact, stably locally compact locale, here denoted by X^{op}:

$$\mathcal{O}\,X^{\mathrm{op}} = \mathcal{Q}\,X.$$

Then $X \cong X^{\mathrm{op}\ \mathrm{op}}$, which shows that $\mathcal{Q}\,X^{\mathrm{op}} \cong \mathcal{O}\,X$,

$$\mathrm{Patch}\,X^{\mathrm{op}} \cong \mathrm{Patch}\,X,$$

the locale X is Hausdorff if and only if $X \cong X^{\mathrm{op}}$, if and only if $X \cong \mathrm{Patch}\,X$.

Oddity

For any Hausdorff locale X, the preframe $\mathcal{Q}\,X$ is a frame if and only if X is compact.

Hence if the locale X^{op} exists then it is homeomorphic to X and both are compact Hausdorff.

However, for any preframe, the Scott continuous nuclei form a frame.

As a first step towards the main theorem, we show that, for X Hausdorff,

$$\mathcal{O}\,\mathcal{K}\,X \cong \text{frame of Scott continuous nuclei on } \mathcal{Q}\,X.$$

Oddity

We can legitimately imagine $\mathcal{K}\,X$ as the patch of the non-existent locale X^{op}.

Moreover, a nucleus j on $\mathcal{Q}\,X$ is Scott continuous if and only if the filter $j^{-1}(1)$ is Scott open, and that such nuclei are fitted.

Hence

$$\mathcal{O}\,\mathcal{K}\,X \cong \text{frame of fitted nuclei } j \text{ on } \mathcal{Q}\,X \text{ with } j^{-1}(1) \text{ Scott open.}$$

This brings us back to the HMJ theorem.

A generalized HMJ theorem.

In terms of frames and nuclei, the HMJ theorem says that, for any frame L, the assignment $j \mapsto j^{-1}(1)$ is an order isomorphism from compact fitted nuclei on L to the preframe $L^{\wedge}$.

Moreover, a nucleus j is compact if and only if the filter $j^{-1}(1)$ is Scott open.

This holds, more generally, if L is a Heyting preframe.

Theorem

For any Heyting preframe L, every $\phi \in L^{\wedge}$ is of the form $j^{-1}(1)$ for a unique compact fitted nucleus j on L, given by $j = \bigsqcup\{u^{\circ} \mid u \in \phi\}$.

Here a u° is the "open" nucleus

$$u^{\circ}(v) = (u \Rightarrow v),$$

and a nucleus is said to be fitted if it is a join of open nuclei.

In other words, the theorem says that there is an isomorphism

$$L^{\wedge} \cong \text{preframe of fitted nuclei } j \text{ on } L \text{ with } j^{-1}(1) \text{ Scott open}$$

given by

$$\Delta(\phi) = \bigsqcup\{u^{\circ} \mid u \in \phi\}, \qquad \nabla(j) = j^{-1}(1).$$

Now, a sufficient condition for $\mathcal{Q}\,X$ being a Heyting preframe is that the meet of any two compact fitted sublocales, calculated in the lattice of sublocales, be compact, because then $\mathcal{Q}\,X$ has all non-empty joins, which are enough to construct Heyting implication.

Because this condition holds if the locale X is Hausdorff, the main result $\mathcal{O}\,\mathcal{K}\,X \cong (\mathcal{Q}\,X)^{\wedge}$ is obtained by considering $L = \mathcal{Q}\,X$ in the above theorem.

What makes the above theorem difficult is that, in general, such joins are not computed pointwise.