Computability of continuous solutions of higher-type equations

Martín Escardó

School of Computer Science, Birmingham University, UK

CiE 2009

Heidelberg, 19-24 July 2009

Summary

1. Brief background.

   (a) Kleene–Kreisel spaces.
   (b) Their exhaustible subspaces ($\approx$ compact subspaces).

2. Computability of solutions of equations over Kleene–Kreisel spaces.

   Enough/necessary to assume that:
   (a) The solution is unique.
   (b) The unknown belongs to an exhaustible space.

3. Beyond Kleene–Kreisel spaces.

4. Interesting examples of exhaustible ranges for the unknowns.

   Certain sets $K \subseteq \mathbb{R}^{[-\epsilon,\epsilon]}$ of analytic functions.

Kleene–Kreisel spaces

Characterization. Least collection of objects

1. containing the (discrete) natural-number object $\mathbb{N}$,
2. closed under finite products $X \times Y$ and exponentials Y^X,

in a suitable cartesian closed category of space-like objects.

The following categories work, among others:

Super-categories of Top: filter spaces, limit spaces, equilogical spaces.

Sub-categories of Top: sequential spaces, k-spaces, QCB spaces.

Definition. A kk-space is a computable retract of a Kleene–Kreisel space.

Exhaustible spaces

Idea. Can algorithmically check all points in finite time.

Definition. A space X is exhaustible iff the functional

$$\forall\colon 2^X \to 2$$

defined by

$$\forall(p) = 1 \iff p(x) = 1 \text{ for all } x \in X$$

is computable.

NB. Any $p \in 2^X$ is the characteristic function of a clopen set.

Main topological tool

Lemma. For any kk-space X,

the functional $\forall\colon 2^X \to 2$ is continuous $\iff$ X is compact.

Corollary
Exhaustible kk-spaces are compact.

Proof. Computable functions are continuous.

Theorems [Escardó 2007 (LICS) & 2008 (LMCS)]

1. Finite kk-spaces are exhaustible (of course).

2. Exhaustible kk-spaces are closed under finite and countable products.

3. Hence e.g. the Cantor space $2^{\mathbb{N}}$ is exhastible (previously Berger).

4. Computable images of exhaustible spaces are exhaustible.

5. Any non-empty exhaustible kk-space is a computable image of $2^{\mathbb{N}}$.

6. Any exhaustible kk-space is computably homeomorphic to an exhaustible subspace of the Baire space $\mathbb{N}^{\mathbb{N}}$ (and hence is a Stone space).

7. Any exhaustible non-empty subspace of a Kleene-Kreisel space is a computable retract (and hence a kk-space).

8. Arzela–Ascoli type characterization.

Computability of solutions of higher-type equations

Theorem. Assume:

(a) X and Y are kk-spaces,
(b) X is exhaustible,
(c) $f: X \to Y$ is computable,
(d) $y \in Y$ is computable.

Then, uniformly in the above data,

1. If $f(x) = y$ has a unique solution $x \in X$, then it is computable.

2. The non-solvability of the equation $f(x) = y$ is semi-decidable.

Corollary.

If $f: X \to Y$ is a computable bijection of exhaustible kk-spaces, then it has a computable inverse.

Cf: a continuous bijection of compact Hausdorff spaces is a homeomorphism.

Subsummed by the above theorem

1. Equations of the form $g(x) = h(x)$, even with parameters.

   (Easy group-theoretical trick.)

2. Finite systems of equations with finitely many variables.

   (Because kk-spaces are closed under finite products.)

3. Certain countable systems with countably many variables.

   (Because kk-spaces are closed under countable cartesian powers.)

Applied to compute unique solutions:

Lemma. Assume:

(a) X is a kk-space.
(b) $K_n \subseteq X$ is a sequence of sets that are exhaustible uniformly in n.
(c) $K_n \supseteq K_{n+1}$.

If $\bigcap_n K_n$ is a singleton $\{x\}$, then x is computable, uniformly in the data.

Applied to semi-decide non-existence of solutions:

Lemma. Assume:

(a) X is an exhaustible kk-space.
(b) $K_n \subseteq X$ is a sequence of sets that are decidable uniformly in n.
(c) $K_n \supseteq K_{n+1}$.

Emptiness of $\bigcap_n K_n$ is semi-decidable, uniformly in the data.

Used to build sets K_n suitable for the application of the above two lemmas:

Lemma. For every kk-space X there is a family $(=_n)$ of equivalence relations that are decidable uniformly in n and satisfy

$$\begin{aligned} x = y \quad &\Longleftrightarrow \quad \forall n.\, x =_n y, \\ x =_{n+1} y \quad &\Longrightarrow \quad x =_n y. \end{aligned}$$

The proof uses the Kleene–Kreisel density theorem.

Proof of the theorem

The set $K_n = \{x \in X \mid f(x) =_n y\}$ is exhaustible, because it is a decidable subset of an exhaustible space.

Therefore the result follows from the above lemmas, because

$$x \in \bigcap_n K_n \iff \forall n. f(x) =_n y \iff f(x) = y.$$

To go beyond kk-spaces, can use representations

E.g. The compact interval $[-1, 1]$ has an exhaustible set 3^ω of representatives.

Binary representation with digit set $3 = \{-1, 0, 1\}$.

Definition. A represented space is *exhaustible* if it has an exhaustible set of representatives.

Computability of solutions of higher-type equations II

Theorem. Assume:

(a) X and Y are computational metric spaces,
(b) X is computationally complete and exhaustible,
(c) $f\colon X \to Y$ is computable,
(d) $y \in Y$ is computable.

Then, uniformly in the above data,

1. If $f(x) = y$ has a unique solution $x \in X$, then it is computable.

2. The non-solvability of the equation $f(x) = y$ is semi-decidable.

Example: Exhaustible spaces of analytic functions

Theorem. Let $\epsilon \in (0, 1)$ and $b > 0$ be computable.

The space $A = A(\epsilon, b)$ of analytic functions $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$ of the form

$$f(x) = \sum_n a_n x^n$$

with $a_n \in [-b, b]$ is exhaustible.

Corollary.

1. The Taylor coefficients of any $f \in A$ can be computed from f.
2. For $f \in \mathbb{R}^{[-\epsilon,\epsilon]}$, it is semi-decidable whether $f \notin A$.

References and advertisement

1. Normann. Computing with functionals – computability theory or computer science? Bulletin of Symbolic logic, 12 (2006), no. 1, 43–59.

2. Normann. 50 years of continuous functionals. Slides of CiE 08 invited talk.

3. Escardó. Infinite sets that admit fast exhaustive search. LICS, IEEE, pp.443-452, 2007.

4. Escardó. Exhaustible sets in higher-type computation. LMCS-4 (3:3) 2008.

5. Escardó and Oliva. Selection functions, bar recursion, and backward induction. Preprint, 2009.

Thank you.