Equality of mathematical structures

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Equality of mathematical structures independently of foundations

For all mathematical purposes,

1. Two groups are regarded to be the same if they are isomorphic.

   One e.g. says that the additive integers are "the" free group on one generator.

2. Two topological spaces are the same if they are homeomorphic.

3. Two metric spaces are the same if they are isometric.

4. Two categories are the same if they are equivalent.

5. Two rings, graphs, posets, functor algebras, . . . are the same if . . .

Seemingly philosophical question

- Do we choose the above notions of sameness, motivated by particular mathematical applications?

- Or are these notions of sameness imposed upon us, independently of what we want to do with groups, topological spaces, metric spaces and categories?

A mathematical answer

If

- we adopt Martin-Löf type theory as our mathematical foundation,
- take the notion of equality to be the identity type, and
- assume Voevodsky's univalence axiom,

then

- we can prove that equality of groups is isomorphism, equality of topological spaces is homeomorphism, equality of metric spaces is isometry, equality of categories is equivalence etc.

Mathematical answer continued

Sometimes we do regard two groups to be the same iff they have the same elements.

- Example. The subgroups of a group form an algebraic lattice under inclusion, with the finitely generated subgroups as the compact elements.
- If we identify isomorphic subgroups, this doesn't work.

In univalent type theory, we have that:

1. In the type of groups, equality is isomorphism.
2. But in the type of subgroups of a given ambient group, equality is "having the same elements" of the ambient group.

- This is possible because equality in MLTT is not global.
- Equality is uniformly defined for all types, but each type has its own equality type.

Credits

- For a large class of algebraic structures, this was first proved by Coquand and Danielsson (2013).
- For objects of categories, this was formulated by Aczel and proved in the HoTT book, and for categories themselves this was proved by Ahrens, Kapulkin and Shulman (2015), and is also in the HoTT book.
- Here we provide more examples and some general techniques to describe equality.
- This talk will be mainly expository, based on our lecture notes:

  Introduction to Univalent Foundations of Mathematics with Agda

  https://www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/

  A minor novelty is a general theorem and various specializations that can be used to attack all the above characterizations of equality, and more, in a uniform and modular way.

Martin-Löf type theory

- We expect most members of this audience to be only vaguely familiar, if at all familiar, with MLTT.

- But instead of first explaining MLTT and the univalence axiom, and then using them to discuss equality of mathematical structures, I will start right away with the subject matter, introducing MLTT and univalence on the fly informally.

The type of magmas

1. A magma is a set X equipped with a binary operation "$\cdot$" subject to no laws.
2. In set theory, a magma is a pair $(X, \cdot)$.
3. In univalent type theory, not all types are sets, and so this information needs to be explicitly provided.
4. A magma is a triple $(X, i, \cdot)$ where X is a type, i is the information that X is a set, and $\cdot : X \times X \to X$ is a binary operation on X.
5. The collection of all (small) magmas forms a (large) type, the type Magma.
6. We have $(X, i, \cdot) : \mathsf{Magma}$.

The type of magmas

1. The small types form a large type $\mathcal{U}$, called a universe.

   It is important here that the universe is a genuine type.

2. A magma is a triple $(X, i, \cdot)$ where $X : \mathcal{U}$ and $i : \text{is set}\, X$ and $\cdot : X \times X \to X$.

3. We define $\text{Magma} = \sum_{X:\mathcal{U}} \text{is set}\, X \times (X \times X \to X)$.

   - Here "$\times$" is the cartesian product.
   - And the sum $\sum$ generalizes the cartesian product, in that the second factor

     $$\text{is set}\, X \times (X \times X \to X)$$

     depends on the element X of the first factor $\mathcal{U}$.

     Some people write $(X : \mathcal{U}) \times \text{is set}\, X \times (X \times X \to X)$ to emphasize this.

Small vs. large, and set vs. not set

- The "set vs. not set" distinction is about the nature of the equality of the type, not about size. A type is a set if its equality relation is truth valued – more later.
- The four combinations $\{\text{small}, \text{large}\} \times \{\text{set}, \text{non-set}\}$ are possible, e.g.:

$\{\text{small}, \text{set}\}$	:	the type $\mathbb{N}$ of natural numbers, the type of functions $\mathbb{N} \to \mathbb{N}$,
$\{\text{small}, \text{not set}\}$	:	constructed using so-called higher inductive types, (we won't discuss them in this lecture)
$\{\text{large}, \text{set}\}$	:	the type of small ordinals, the type of submagmas of a given small magma,
$\{\text{large}, \text{not set}\}$	:	the type of small magmas, the type of small sets, the type of small types.

Equality in the type of magmas

1. In telegraphic summary, two magmas are equal if and only if they are isomorphic, assuming univalence.
   - What is rather imprecise, but misleadingly true, in this summary is "if and only if".
   - We can also say "isomorphic magmas are equal", which is equally misleading.
2. In the absence of univalence, we can't say much about equality of magmas.

   But the above is consistent, because univalence is consistent.

To remove the above imprecision, we need to pause to discuss equality in MLTT.

Identity types in MLTT

1. For each type X and any two points $x, y : X$, we stipulate that there is a type

$$\mathsf{Id}_X(x, y)$$

   that collects the ways in which the points x and y are identified.

2. For each point $x : X$, we stipulate that there is an element

$$\mathsf{refl}(x) : \mathsf{Id}_X(x, x).$$

3. Finally, we require some sort of initiality of the above data, formulated as an induction principle.

A particular case of the induction principle for the identity type (recursion)

1. $\mathsf{Id}_X : X \times X \to \mathcal{U}$.
2. refl : $\prod_{x:X} \mathsf{Id}_X(x,x)$.

3'. For any given

$$\begin{aligned} A &: X \times X \to \mathcal{U} \\ a &: \prod_{x:X} A(x,x) \end{aligned}$$

we get a function

$$h : \prod_{x,y:X} \mathsf{Id}_X(x,y) \to A(x,y)$$

that maps $(x, x, \mathsf{refl}(x))$ to $a(x)$.

Using the full induction principle, h is the unique such function up to pointwise identity, and, assuming univalence, the unique such function up to plain identity.

The full induction principle for the identity type

1. $\mathsf{Id}_X : X \times X \to \mathcal{U}$.
2. $\mathsf{refl} : \prod_{x:X} \mathsf{Id}_X(x,x)$.
3. For any given

$$A \ : \ \prod_{x,y:X} \mathsf{Id}_X(x,y) \to \mathcal{U},$$

$$a \ : \ \prod_{x:X} A(x,x,\mathsf{refl}(x)),$$

   we stipulate that there is a function

$$h : \prod_{x,y:X} \ \prod_{p:\mathsf{Id}_X(x,y)} A(x,y,p)$$

   that maps $(x, x, \mathsf{refl}(x))$ to $a(x)$.

   The function h is also written $\mathsf{J}(A, a)$.

Identity types in MLTT

Specified by

1. Id (type valued equality relation),
2. refl (its reflexivity),
3. J (its induction principle).

- Using J we can prove that the identity type satisfies the usual properties of equality, such as symmetry, transitivity, substitution of equals for equals, etc.
- Why don't we axiomatize equality as in set theory?
    - Because logic in type theory is type valued rather than truth valued.
    - And, more importantly, because elements of types don't have elements themselves in general, and hence the axiom of extensionality of set theory can't be formulated.

Characterization of magma identity assuming univalence

1. The type

$$\mathsf{Iso}(\mathcal{A},\mathcal{B}) := \sum_{f:A\to B} \text{is isomorphism } f$$

collects all the isomorphisms of two magmas $\mathcal{A} = (A, i_A, \cdot_A)$ and $\mathcal{B} = (B, i_B, \cdot_B)$.

This type has zero, one, or more elements, depending on what $\mathcal{A}$ and $\mathcal{B}$ are.

2. The univalence axiom implies that there is a canonical bijection

$$\mathsf{Id}_{\mathsf{Magma}}(\mathcal{A},\mathcal{B}) \simeq \mathsf{Iso}(\mathcal{A},\mathcal{B})$$

that maps the reflexive identification to the identity equivalence.

We will deliberately postpone the formulation of the univalence axiom until we have gained enough experience to be able to appreciate its formulation and meaning.

Discussion

1. The definition of the type of magmas doesn't say what magma homomorphism is.

   Somehow, "magically", MLTT with univalence "knows", without we having to tell it, what a magma isomorphism has to be.

2. From a categorical point of view, we are defining the type of objects only, and then univalence is telling us what their isomorphisms are.

   I emphasize strongly that univalence doesn't tell us what their arrows are.

   - Different notions of arrow on the same objects making them into a category can give rise to the same notion of isomorphism.

     Example: Topological spaces with continuous and open maps respectively.

   - Univalence automatically determines the isomorphisms, but not the homomorphisms.

Discussion

3. The above “magic” should not be attributed to the univalence axiom alone.

   Without Martin-Löf’s notion of identity type, the univalence axiom can’t even be formulated.

   In particular, it can’t be formulated in ZFC for its standard notion of equality.

Next we want to formulate the notion of magma isomorphism and discuss its perils.

Magma isomorphisms

- For magmas $\mathcal{A} = (A, i_A, \cdot_A)$ and $\mathcal{B} = (B, i_B, \cdot_B)$, a function $f : A \to B$ is an isomorphism iff it preserves the multiplication operation and has an inverse (which automatically also preserves the multiplication).

- This definition works, for the above characterization of magma identity, only because we have required the underlying types of magmas to be sets.

This is a good time to pause to discuss what we mean when we say that a type is a set.

Types that are sets

1. A type X is called a set if for any two $x, y : X$, the identity type $\mathsf{Id}_X(x, y)$ has at most one element. Equality is a truth value.

   - Example (in MLTT with or without univalence). The type of natural numbers.
   - Counter-example (in MLTT with univalence). The type of magmas.

2. More generally, types may be

   - -2-groupoids (singletons),
   - -1-groupoids (subsingletons),
   - 0-groupoids (sets),
   - 1-groupoids (their identity types are sets), ...,
   - $(n+1)$-groupoids (their identity types are n-groupoids), ...,
   - ∞-groupoids (no restriction).

Equality notation for the identity type

With the strong warning that the identity type

$$\mathsf{Id}_X(x, y)$$

in general has multiple elements and hence cannot be regarded as a truth value, we write

$$x = y$$

to denote it, leaving X implicit.

Invertible maps

1. We say that a map $f : X \to Y$ is invertible if there is $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

2. If we collect all the ways in which f is invertible, this becomes

$$\sum_{g : Y \to X} (g \circ f = \mathrm{id}_X) \times (f \circ g = \mathrm{id}_X).$$

3. If equality were truth valued, we could say "and" instead of "$\times$", but because it is not, we use the cartesian product to take into account all the possible combinations.

Invertibility data

1. Invertibility data for a function $f : X \to Y$ is collected in the type

$$\sum_{g:Y\to X} (g \circ f = \mathrm{id}_X) \times (f \circ g = \mathrm{id}_Y).$$

2. In the presence of univalence, this type often has multiple elements.

   While the function g is unique when it exists, in the sense that any two such g are identified in at least one way, the invertibility equations can, and do, hold in multiple ways in examples of interest.

∞-Magmas

To appreciate the importance of such considerations, we next

- discuss magmas without the requirement that their underlying types are sets,
- observe that the above characterization of equality fails for them,
- explain what needs to be refined to recover the characterization in the absence of the sethood requirement.

∞-Magmas

We remove the requirement that the underlying type be a set.

1. $\infty\,\mathsf{Magma} = \sum_{X:\mathcal{U}} X \times X \to X$.
2. The above characterization of magma equality doesn't work anymore.
3. To fix it, we need a refinement of the notion of invertibility that is property rather than data.
4. In the case of magmas, the requirement that the underlying type is a set made invertibility to become property.
5. We want it to become property without the sethood requirement.

Voevodsky's notion of type equivalence

Let $f : X \to Y$ be a function.

1. The fiber of a point $y : Y$ collects the points $x : X$ that are mapped to a point identified with y, together with the identification datum $p : f(x) = y$:

$$f^{-1}(y) := \sum_{x:X} f(x) = y.$$

2. f is called an equivalence if its fibers are all singletons.

   For each $y : Y$ there is a unique $x : X$ up to a unique identification $p : f(x) = y$.

   Cf. the categorical notion of unique existence up to a unique isomorphism.

3. Here a type A is called a singleton if we have a point a_0 and an identification $a = a_0$ for any given $a : A$:

$$\sum_{a_0:A} \prod_{a:A} a = a_0.$$

Voevodsky's notion of type equivalence

$$\begin{array}{rcll}
\text{invertible}\, f & := & \sum_{g:Y\to X} (g\circ f = \mathrm{id}_X)\times(f\circ g = \mathrm{id}_X) & \text{(data)}\\
f^{-1}(y) & := & \sum_{x:X}\ f(x) = y & \text{(data)}\\
\text{is subsingleton}\, A & := & \prod_{a,b:A}\ a = b & \text{(property)}\\
\text{is singleton}\, A & := & \sum_{a_0:A}\ \prod_{a:A}\ a = a_0 & \text{(property)}\\
\text{is equivalence}\, f & := & \prod_{y:Y}\ \text{is singleton}(f^{-1}(y)) & \text{(property)}
\end{array}$$

- The type is equivalence f is a retract of the type invertible f:

$$\text{is equivalence}\, f \underset{\twoheadleftarrow}{\hookrightarrow} \text{invertible}\, f.$$

Invertibility data can be improved to the equivalence property.

(Using a procedure similar to that to improve an equivalence to an adjoint equivalence.)

Characterization of ∞-magma identity

1. For any two ∞-magmas $\mathcal{A} = (A, \cdot_A)$ and $\mathcal{B} = (B, \cdot_A)$, we have a canonical equivalence

$$(\mathcal{A} = \mathcal{B}) \simeq \sum_{f : A \to B} \text{is equivalence}\, f \times \prod_{x,y:A} f(x \cdot y) = f(x) \cdot f(y).$$

2. Hence the type ∞ Magma, like the type Magma, is not a set, as its identity types $\mathcal{A} = \mathcal{B}$ have multiple elements in general.

3. The homomorphism condition, in the absence of the requirement that the underlying types are sets, is data rather than property.

   The same equivalence can be a homomorphism in more than one way.
   This is crucially needed for this characterization of ∞-magma identity to work.

Voevodsky's univalence axiom

1. We collect all the equivalences of two types X and Y in a type $X \simeq Y$:

$$X \simeq Y := \sum_{f:X \to Y} \text{is equivalence } f.$$

2. Martin-Löf's recursion principle for the identity type gives a canonical function

$$\prod_{X,Y:\mathcal{U}} X = Y \;\to\; X \simeq Y$$

   that maps $(X, X, \mathrm{refl}(X))$ to (id_X, i), where i is the unique datum exhibiting the identity function $\mathrm{id}_X : X \to X$ as an equivalence.

3. Voevodsky's univalence axiom says that this canonical map is itself an equivalence.

   There is a canonical one-to-one correspondence between Martin-Löf identifications of X and Y and Voevodsky equivalences from X to Y.

Monoids

A monoid is a magma with a neutral element and associative multiplication.

1. The type of monoids is

$$\sum_{(X,i,\cdot):\mathrm{Magma}} \sum_{1:X} \left(\prod_{x:X} (x \cdot 1 = x) \times (1 \cdot x = x) \right) \times \prod_{x,y,z:X} x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

   Martin-Löf identities of the type of monoids are in canonical bijection with magma isomorphisms that preserve the neutral element.

2. What about ∞-monoids? Don't work by just removing the sethood requirement.
   We need coherence data.

A modular way to get the characterizations of identity types

1. The type of ∞-magmas.
2. The type $\sum_{X:\mathcal{U}} X$ of pointed types.
   - The Martin-Löf identifications of two pointed types $(X, 1_X)$ and $(Y, 1_Y)$ are in canonical bijection with the equivalences $X \to Y$ that map 1_X to a point identified with 1_Y.
3. Combine two mathematical structures on the same type.
   - E.g. Combine the above to get pointed ∞-magmas.

   Combine the two notions of homomorphism.
   - E.g. Magma homomorphism and preservation of the base point, to get monoid homomorphism.

A modular way to get the characterizations of identity types

4. Given a mathematical structure on a type, consider axioms for it.
   - Required to be property rather than data.
   - E.g. From pointed ∞-magmas get monoids with the axioms
     - the underlying type is a set,
     - multiplication is associative,
     - the base point is a neutral element of multiplication.

     Then the characterization of monoid identity is inherited from that of pointed ∞-magma identity.
   - E.g. Adding the further axiom that every element has an inverse we get the type of groups, which then inherits the characterization of equality from monoids, and hence coincides with the equality their underlying pointed ∞-magmas.

Structure identity principle

- All the examples of this talk, and many more, can be derived from a general structure identity principle for the characterization of equality of mathematical structures.
- Various forms of such a principle are formulated and proved in the above references.
- This talk is based on the SIP of our lecture notes mentioned above.

Standard notion of structure

1. The above examples have the form $\sum_{X:\mathcal{U}} S\,X$ where $S : \mathcal{U} \to \mathcal{U}$ defines structure and properties on X.
   - E.g. For magmas, $S\,X = \text{is set}\,X \times (X \times X \to X)$
   - E.g. For monoids we additionally require a neutral element and associativity.

Standard notion of structure $S : \mathcal{U} \to \mathcal{U}$

Given by data (ι, ρ, θ) with

$$\iota \quad : \quad \prod_{A,B:\sum_{X:\mathcal{U}} S\,X} A \simeq B \to \mathcal{U}$$

(notion of homomorphic equivalence)

$$\rho \quad : \quad \prod_{A:\sum_{X:\mathcal{U}} S\,X} \iota(A, A, \mathrm{id}, i)$$

(identities are homomorphic)

$$\theta \quad : \quad \prod_{X:\mathcal{U}} \prod_{s,t:SX} \text{is equivalence } \kappa_{(\iota,\rho,s,t)}$$

(two structures s, t on X making the identity $X \to X$ into a homomorphism are the same)

where

- i is the unique witnesses that id is an equivalence.
- $\kappa_{(\iota,\rho,s,t)}$ is the canonical map $s = t \to \iota((X, s), (X, t), \mathrm{id}, i)$ that sends $\mathrm{refl}(s)$ to $\rho(X, s)$.

Characterization of equality for standard structures

The above data (ι, ρ, θ) is fairly easy to provide in common examples of interest.

- Guess what the notion of homomorphic equivalence ι should be.
- Give data ρ that the identity function is a homomorphic equivalence.
- Prove that two structures on the same type making the identity function into homomorphism are canonically the same.

Theorem. The identity type of mathematical objects equipped with standard structures is in canonical bijection with the type of homomorphic equivalences specified by ι.

The powerset of a type

Used to define the type of subgroups of a group and the type of of topological spaces.

1. The (large) type of truth values is $\Omega := \sum_{P:\mathcal{U}}$ is subsingleton P.

   Subtype classifier, where a type embedding is a map with subsingleton fibers.

2. The powerset of a type X is $\mathcal{P}\,X := X \to \Omega$.

   It is a set even if X isn't, because Ω is a set and sets forms an exponential ideal.

3. For $A : \mathcal{P}\,X$ and $x : X$, let $x \in A$ stand for $A(x)$.

4. Subsets $\{x : X \mid P(x)\}$ amount to functions $(x : X) \mapsto P(x)$.

5. We reuse the notation f^{-1} to denote the inverse image of a function $f : X \to Y$.

   For $V : \mathcal{P}\,Y$, we define $f^{-1}(V) := \{x : X \mid f(x) \in V\}$ as usual.

Equality in the powerset

Univalence implies the extensionality axiom for the powerset: for $A, B : \mathcal{P}\,X$, we have

$$(A = B) = \prod_{x:X} x \in A \iff x \in B.$$

- We don't need to discuss which identification we get, because the powerset of any type is a set.
- The type $A = B$ has at most one element.
- The type $(A = B) = \prod_{x:X} x \in A \iff x \in B$ has precisely one element.

Equality of subgroups of a given group

Let $\mathcal{G} = (G, \cdot, 1, a)$ be a group, where a witnesses the group axioms.

1. The type of subgroups of $\mathcal{G}$ is

$$\mathsf{Subgroups}\,\mathcal{G} := \sum_{A:\mathcal{P}\,G} A \text{ is closed under the group operations.}$$

2. This type is a set.

3. The type of subgroups inherits the characterization of equality from the powerset.

   Two subgroups are equal if and only if they have the same elements of the ambient group $\mathcal{G}$.

   This is because closure under the group operations is property rather than data.

Equality of topological spaces

1. Consider a given function axioms : $\mathcal{P}(\mathcal{P}\,X) \to \Omega$ that gives axioms for a collection τ of subsets of X, e.g. those for topological spaces, or for Hausdorff spaces, etc.

2. Define Space $:= \sum_{X:\mathcal{U}} \sum_{\tau:\mathcal{P}(\mathcal{P}\,X)}$ axioms τ

3. For two raw spaces $(X,\tau),(Y,\sigma) : \sum_{X:\mathcal{U}} \mathcal{P}(\mathcal{P}\,X)$ subject to no axioms, we have a canonical bijection

$$((X,\tau) = (Y,\sigma)) \simeq \sum_{f:X\to Y} \text{is equivalence}\, f \times \left(\{V : \mathcal{P}\,Y | f^{-1}(V) \in \tau\} = \sigma\right).$$

This says that the open sets of Y are precisely those whose inverse images are open in X

Equality of topological spaces

4. The equality of two spaces (X, τ, i) and (Y, σ, j) is the same as that of their underlying raw spaces (X, τ) and (Y, σ).

    - The axioms don't contribute to the characterization of equality.
    - For this to be the case, it is essential that the axioms are property rather than data.

Equality of metric spaces, posets and graphs

1. The type of raw metric spaces is $\sum_{X:\mathcal{U}} X \times X \to R$, where R is an arbitrary type.
2. The identity type of two raw metric spaces X and A, with distance functions denoted both by d, is in canonical bijection with the type

$$\sum_{f:X\to A} \text{is equivalence } f \times \prod_{x,y:X} d(x,y) = d(f(x), f(y)).$$

3. Metric spaces are obtained by taking R to be a type of real numbers and adding the metric-space axioms, so that their characterization of equality is inherited from that of raw metric spaces.
4. When $R = \Omega$ and we choose suitable axioms, this also accounts for various kinds of posets and graphs.

Equality of categories

1. The type of raw categories is

$$\sum_{X:\mathcal{U}} \sum_{\mathrm{hom}:X\times X\to\mathcal{U}} \left(\prod_{x:X} \mathrm{hom}(x,x)\right) \times \prod_{x,y,z:X} hom(x,y) \times \mathrm{hom}(y,z) \to \mathrm{hom}(x,z).$$

- We don't have an axiom saying that the hom-types are sets.
- We don't have axioms for identity and composition.

Equality of categories

1. The type of raw categories is

$$\sum_{X:\mathcal{U}} \sum_{\mathrm{hom}:X\times X\to\mathcal{U}} \left(\prod_{x:X} \mathrm{hom}(x,x)\right) \times \prod_{x,y,z:X} hom(y,z) \times \mathrm{hom}(x,y) \to \mathrm{hom}(x,z).$$

2. The identity type of raw categories with underlying types of objects X and A is in canonical bijection with the type

$$\textstyle\sum_{F:X\to A} \text{ is equivalence } F \times$$

$$\textstyle\sum_{\mathcal{F}:\prod_{x,y:X} hom(x,y)\to \mathrm{hom}(F(x),F(y))} \left(\prod_{x,y:X} \text{ is equivalence } \mathcal{F}_{x,y}\right) \times$$

$$F, \mathcal{F} \text{ preserve identities and composition.}$$

What is crucial is that we have an equivalence F of objects with a nested equivalence $\mathcal{F}$ of hom-types which jointly respect the categorical structure.

Equality of categories

3. We get categories by requiring that:
   - The hom types are sets.
   - The identities are left and right neutral.
   - Composition is associative.
   - Optionally, the univalence axiom for categories is satisfied.

   These axioms together form property of the categorical structure, and hence categories (univalent or not) inherit their equality from raw categories.

   That is, equality of categories depends on the structure and not on the axioms for categories, as in the previous examples.