A constructive manifestation of the Kleene-Kreisel continuous functionals

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Kleene–Kreisel continuous functionals (1950's)

Since then, many equivalent formulations, probably all of the form:

1. Start with a cartesian closed category with a natural numbers object.

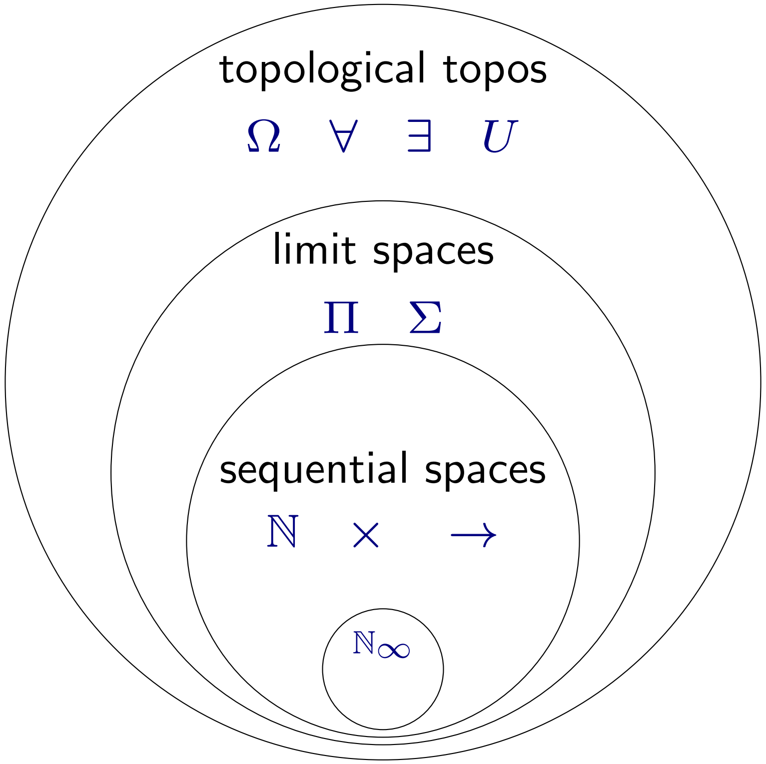
   $\times \qquad \to \qquad \mathbb{N}$

2. Start from $\mathbb{N}$ and close under exponentials $X \to Y$ (also written Y^X).
   (Automatically closed under products $X \times Y$.)

Such categories include (not in chronological order):

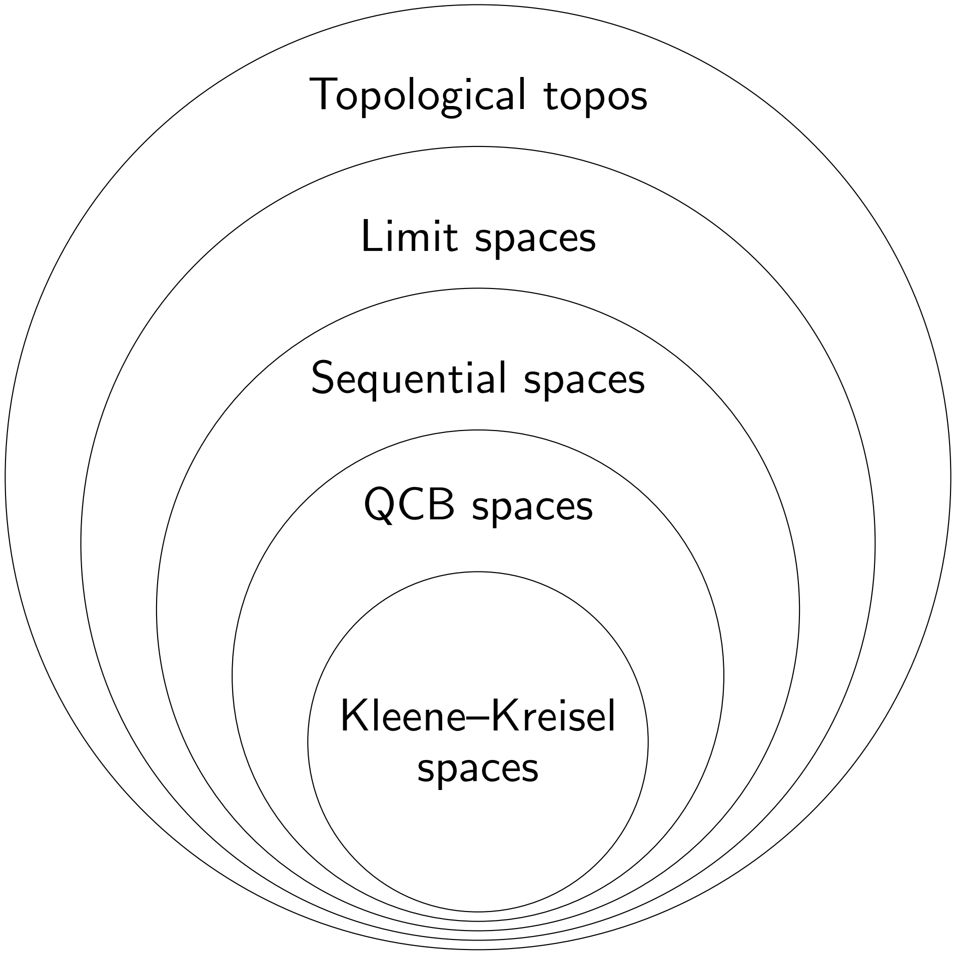
- Compactly generated topological spaces.
- Sequential topological spaces.
- QCB spaces.
- Scott domains with partial equivalence relations.
- Filter spaces.
- Kuratowski limit spaces.
- Johntone's topological topos.

Johnstone's topological topos (1979)



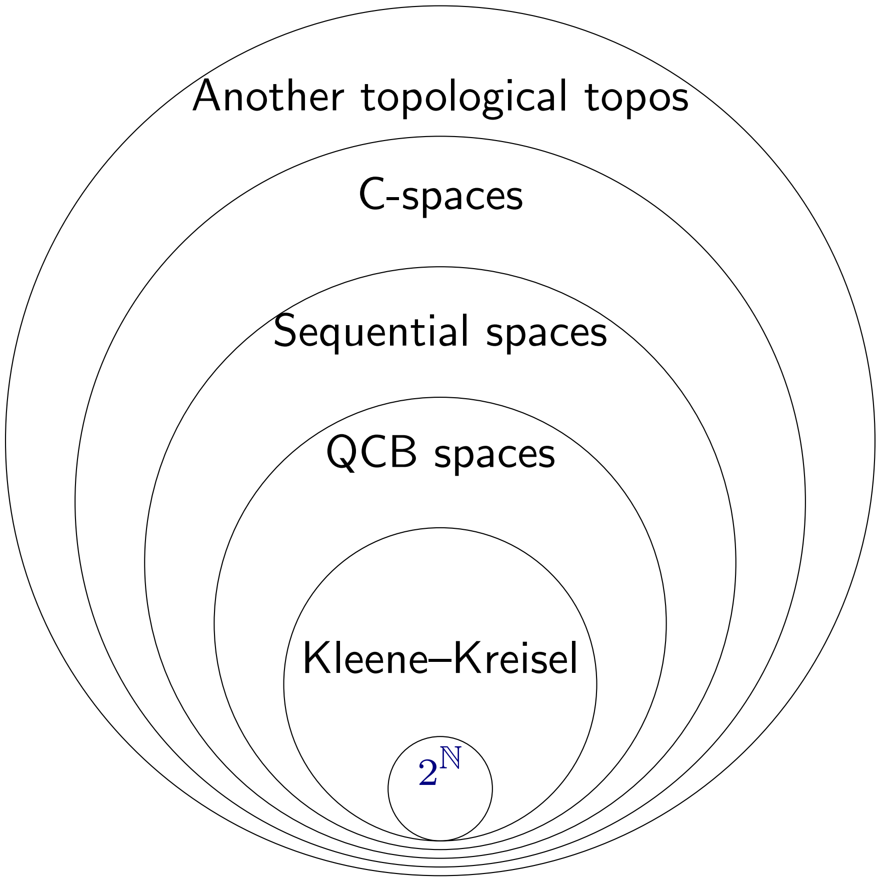
- $\mathbb{N}_\infty$ is the one-point compactification of the discrete natural numbers.
- Every object in the topos is a colimit of copies of $\mathbb{N}_\infty$.

Kleene–Kreisel continuous functionals



Another topological topos

Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$. Admits a constructive treatment.



Validates the Uniform Continuity axiom (UC)

$$\forall f\colon 2^{\mathbb{N}} \to \mathbb{N} \quad \exists n \in \mathbb{N} \quad \forall \alpha, \beta \in 2^{\mathbb{N}} \qquad \alpha =_n \beta \implies f(\alpha) = f(\beta).$$

All functions are uniformly continuous.

(This contradicts the principle of excluded middle.)

Uniform continuity axiom

$$\forall f\colon 2^{\mathbb{N}} \to \mathbb{N} \quad \exists n \in \mathbb{N} \quad \forall \alpha, \beta \in 2^{\mathbb{N}} \qquad \alpha =_n \beta \implies f(\alpha) = f(\beta).$$

1. **True** in Brouwerian intuitionistic mathematics (INT).

2. **Independent** of

   - Bishop's mathematics (BISH),
   - higher-type Heyting arithmetic (HA^{ω}),
   - Martin-Löf's type theory (MLTT), ...

   Becomes provably false if excluded middle is postulated.
   Becomes provably true if Brouwerian axioms are postulated.

3. **False** in Markov's constructive recursive mathematics (RUSS).

   (Kleene tree.)

We can postulate the uniform continuity axiom (UC)

But we lose the computational content of constructive proofs (e.g. in MLTT).

Can we give computational meaning to it?

Can we extract computational content from constructive proofs that use UC?

Want to do this without invoking any constructively contentious axiom.

E.g. Bar Induction, Fan Theorem, any form of choice, powerset, continuity, . . .

Goal

Constructively build a model of some forms of constructive mathematics, including BISH, HA^ω, MLTT (and of course excluding RUSS):

1. in which the uniform continuity axiom holds,

2. but without assuming any constructively contentious axiom in the meta-theory used to define the model.

Natural choice of a meta-theory

Martin-Löf type theory with a universe, in its intensional form.

1. Sufficiently powerful.
2. Has a computational interpretation.
3. Implemented as a subset of various systems such as Coq, Lego, Agda.

We have formalized our construction of the model and proofs in Agda.

> In this talk, however, I will use informal, rigorous mathematical language, trusting that the reader can recognize (non-)constructive arguments.

A point we want to emphasize is this

Because the meta-theory has a computational interpretation, we don't need to write an algorithm to extract computational content from proofs that use UC.

We don't need to consider notions of computability to build the model.

The capability of performing computations is implicitly built-in in our meta-theory.

Models of uniform continuity

Mike Fourman (1982) constructed sheaf models of uniform continuity.

Kripke–Joyal semantics for the quantifiers $\forall, \exists$.

Local truth.

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Local truth.

We instead want Brouwer–Heyting–Kolmogorov semantics for the quantifiers:

$$\prod_{f\,:\,2^{\mathbb{N}}\to\mathbb{N}} \ \sum_{n\,:\,\mathbb{N}} \ \prod_{\alpha,\beta\,:\,2^{\mathbb{N}}} \alpha =_n \beta \implies f(\alpha) = f(\beta).$$

Sort of realizability interpretation.

Precursors of our work include

1. Johnstone's paper On a topological topos (1979).
2. Fourman's papers Continuous truth and Notions of choice sequence (1982).
3. van der Hoeven and Moerdijk's paper Sheaf models for choice sequences (1984).
4. Bauer and Simpson's unpublished work Continuity begets continuity (2006).

Also related to

1. Spanier's quasi-topological spaces (1961).
   (Introduced for the purposes of homotopy theory.)
2. Hyland's description of the Kleene–Kreisel spaces as compactly generated spaces (1970's).

Remark

The model of Kleene–Kreisel continuous functionals validates UC.

(As is well known in the higher-type recursion theory literature.)

But its treatment is highly non-constructive.

We are working with a constructive manifestation of the KK-functionals.

Technical motivation: Spanier's quasi-topological spaces

Def. A quasi-topology on a set X assigns to each compact Hausdorff space K, a set $P(K, X)$ of *probes* $p\colon K \to X$, such that:

1. All constant maps are in $P(K, X)$.
2. If $t\colon K' \to K$ is continuous and $p \in P(K, X)$, then $p \circ t \in P(K', X)$.
   (Presheaf condition.)
3. If $(t_i\colon K_i \to K)_{i \in I}$ is a finite, jointly surjective family and $p\colon K \to X$ is a function with $p \circ t_i \in P(K_i, X)$ for every $i \in I$, then $p \in P(K, X)$.
   (Sheaf condition.)

Def. A function $f\colon X \to Y$ of quasi-topological spaces is continuous if $f \circ p \in P(K, Y)$ for every $p \in P(K, X)$.
(Naturality condition.)

Facts

1. Quasi-topological spaces form a cartesian closed category (Spanier 1961).

   They have topological spaces as a (non-cartesian closed) full subcategory.

2. Their topological co-reflection gives the compactly generated topological spaces.

   Which also form a cartesian closed category (Hurewicz 1958).

3. Quasi-topological spaces form a quasi-topos (Dubuc 1970's).

4. Quasi-topological spaces embed in a "gros" topos.

   The quasi-topological spaces arise as the "concrete" sheaves.

Variations

1. Rather than all compact Hausdorff spaces, consider only one.

   $\mathbb{N}_\infty$, the one-point compactification of discrete space $\mathbb{N}$.

   One gets the Kuratowski limit spaces.

   The topological reflection gives the sequential spaces.

   The ambient topos is Johnstone's topological topos.

   (Topos implicitly considered by Bauer and Simpson.)

2. Again consider only one space, but forget compactness.

   $\mathbb{N}^{\mathbb{N}}$, the Baire space.

   Topos considered by Fourman, and by van der Hoeven and Moerdijk.

Our variation

Consider only one compact Hausdorff space, the Cantor space $2^{\mathbb{N}}$.

We now describe the (concrete) sheaves in a way amenable for treatment in MLTT.

But informally, as discussed before.

Underlying category of the site

The monoid C of uniformly continuous maps $t\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Presheaves

A presheaf can be described as a set P equipped with an action

$$\begin{array}{rcl} P \times C & \to & P \\ (p, t) & \mapsto & p \cdot t \end{array}$$

satisfying

$$\begin{array}{rcl} p \cdot \mathrm{id} & = & p, \\ p \cdot (t \circ u) & = & (p \cdot t) \cdot u. \end{array}$$

Natural transformation

A natural transformation of presheaves $(P, \cdot)$ and $(Q, \cdot)$ is a function $f\colon P \to Q$ that preserves the action:

$$f(x \cdot t) = (fx) \cdot t.$$

The coverage

1. Let 2^n denote the set of binary strings of length n.

2. For $s \in 2^n$, let $\mathrm{cons}_s \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the concatenation map

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

For each natural number n we have the covering family $(\mathrm{cons}_s)_{s\in 2^n}$.

1. Jointly surjective.

2. Disjoint images.
   (Simplifies the amalgamation property in the definition of sheaf.)

The coverage axiom

We need to check that for every $t \in C$,

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall s \in 2^n.\ \exists t' \in C.\ \exists s' \in 2^m.\ t \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ t'.$$

But this is equivalent to the uniform continuity of every $t \in C$, and hence holds.

Sheaves

A presheaf $(P, \cdot)$ is a sheaf if and only if

For any n and any family $(p_s \in P)_{s \in 2^n}$, there is a unique $p \in P$ with

$$p \cdot \mathrm{cons}_s = p_s.$$

It suffices to check the case $n = 1$

A presheaf $(P, \cdot)$ is a sheaf if and only if

For any two $p_0, p_1 \in P$ there is a unique $p \in P$ with

$$p \cdot \mathrm{cons}_0 = p_0, \qquad p \cdot \mathrm{cons}_1 = p_1$$

This is good for *checking* that a presheaf is a sheaf.

The case for arbitrary n is good when we *use* sheaves.

Concrete sheaf

A sheaf $(P, \cdot)$ where the action $P \times C \to P$ is function composition.

Then P must be a set of functions $2^{\mathbb{N}} \to X$ for a suitable set X.

Concrete sheaves can be described as C-spaces

Def. A C-topology on a set X is a collection P of *probes* $2^{\mathbb{N}} \to X$ subject to the following conditions:

1. All constant maps are in P.
2. If $t\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is uniformly continuous and $p \in P$, then $p \circ t \in P$.
   (Presheaf condition.)
3. For any two maps $p_0, p_1 \in P$, the unique map $p\colon 2^{\mathbb{N}} \to X$ defined by $p(i * \alpha) = p_i(\alpha)$ is in P.
   (Sheaf condition.)

A C-space is a set X equipped with a C-topology.

A function $f\colon X \to Y$ of C-spaces is continuous if $f \circ p \in P_Y$ whenever $p \in P_X$.
(Naturality condition.)

Compare with Spanier's quasi-topological spaces

Def. A quasi-topology on a set X assigns to each compact Hausdorff space K, a set $P(K, X)$ of *probes* $p\colon K \to X$, such that:

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Def. A function $f\colon X \to Y$ of quasi-topological spaces is continuous if $f \circ p \in P(K, Y)$ for every $p \in P(K, X)$.
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Concrete sheaves are the same thing as C-spaces

The following two categories are equivalent:

1. Concrete sheaves with natural transformations.
2. C-spaces with continuous maps.

C-spaces form a locally cartesian closed category

The constructions are the same as in the category of sets, with suitable C-topologies.

For example,

1. to get products, we C-topologize cartesian products,
2. to get exponentials, we C-topologize the sets of continuous maps.

These constructions are *different* from those needed to get cartesian closedness of sheaves, but isomorphic.

They are simpler, which is good for our formalization purposes.

Discrete C-spaces

Def. A C-space X is discrete if for every C-space Y, all functions $X \to Y$ are continuous.

Def. A map $p \colon 2^{\mathbb{N}} \to X$ into a set X is called *locally constant* iff $\exists n \in \mathbb{N}.\ \forall \alpha, \beta \in 2^{\mathbb{N}}.\ \alpha =_n \beta \implies p(\alpha) = p(\beta)$.

Lemma

1. The locally constant functions from $2^{\mathbb{N}}$ into a set X form a C-topology on X.
2. For any C-topology P on X, every locally constant function $2^{\mathbb{N}} \to X$ is in P.

That is, the locally constant maps $2^{\mathbb{N}} \to X$ form the smallest C-topology on X.

Lemma

A C-space is discrete iff its probes are precisely the locally constant functions.

Def. We thus refer to the collection of locally constant maps $2^{\mathbb{N}} \to X$ as the *discrete* C-topology on X.

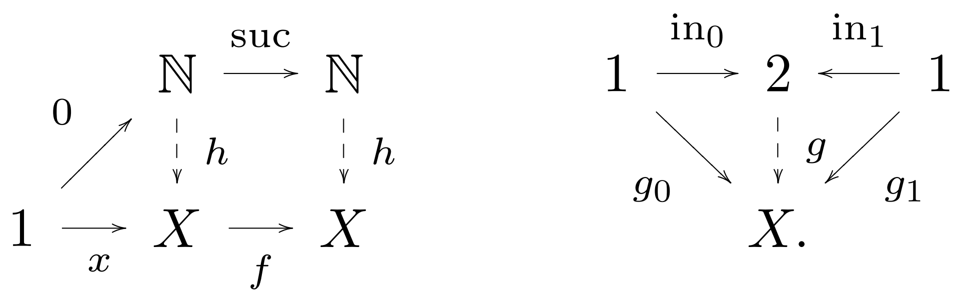
Natural numbers object

The discrete C-topology on 2 or $\mathbb{N}$ is the set of uniformly continuous maps.

Theorem In the category of C-spaces:

1. The coproduct of two copies of the terminal space 1 is the discrete space 2.
2. The discrete space $\mathbb{N}$ of natural numbers is the natural numbers object.

Proof The unique maps g and h in Set in the diagrams below are continuous by the discreteness of $\mathbb{N}$ and 2:



The Fan functional

The Yoneda embedding maps the monoid C to C-spaces.

$$y(\star) = 2^{\mathbb{N}} = \text{exponential of discrete spaces in the category } C\text{-space}.$$

The Yoneda Lemma says that, for any C-space X, a map $2^{\mathbb{N}} \to X$ is a probe iff it is continuous in the sense of the category C-Space.

Lemma (Slightly non-trivial)
The exponential $\mathbb{N}^{2^{\mathbb{N}}}$ is a discrete C-space.

Theorem There is a continuous *Fan functional* $\mathrm{fan}\colon \mathbb{N}^{2^{\mathbb{N}}} \to \mathbb{N}$ that calculates (minimal) moduli of uniform continuity.

Proof Because any $f \in \mathbb{N}^{2^{\mathbb{N}}}$ is uniformly continuous, we can let $\mathrm{fan}(f)$ be the least witness of this fact. By the lemma, fan is continuous.

Modelling uniform continuity

C-spaces provide a model of Gödel's system T and of Martin-Löf' dependent type theory:

1. Cartesian closed structure — simply typed λ-calculus.
2. Local cartesian closed structure — dependent typed λ-calculus with Π and Σ.
3. Natural numbers object — base type and primitive recursion principle.

Theorem (for HA^ω and MLTT)
The uniform continuity axiom is validated by the fan functional.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall \alpha, \beta \in 2^{\mathbb{N}}.\ (\alpha =_n \beta \implies f\alpha = f\beta).$$

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$$\Pi f\colon 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n \in \mathbb{N}.\ \Pi \alpha, \beta \in 2^{\mathbb{N}}.\ (\alpha =_n \beta \implies f\alpha = f\beta).$$

Kleene-Kreisel within C-spaces

Theorem
The Kleene–Kreisel spaces can be constructed within C-spaces.

The proof is non-constructive (and the only such proof in this work).

But the C-space manifestation of KK spaces admits a constructive treatment, which is the point we make in this work.

If UC already holds in the meta-theory

What does our model construction do? Short answer: nothing!

Definition. The collection of all maps $2^{\mathbb{N}} \to X$ is called the indiscrete C-topology on X.

It renders all maps $Y \to X$ continuous for any C-space Y.

Lemma

1. The category of indiscrete C-spaces is equivalent to that of sets.
2. Indiscrete C-spaces form an exponential ideal.
3. If UC holds in our meta-theory, then the discrete space $\mathbb{N}$ is also indiscrete.

The Kleene–Kreisel and full type hierarchies

Corollary
If UC holds in the meta-theory used to build the model, then
the Kleene–Kreisel hierarchy is equivalent to the full type hierarchy.

Recall that the full type hierarchy is the smallest collection of sets and maps obtained by starting from the natural numbers and iterating exponentials.

Summary

1. Constructive manifestation of the Kleene–Kreisel continuous functionals.

2. Validation of uniform continuity axiom with a weak constructive meta-theory.

3. Equivalence of the Kleene–Kreisel hierarchy and the full type hierarchy under the assumption that UC holds in the meta-theory.

4. Can be formulated in Martin-Löf type theory and is formalized in Agda.
   We can therefore run examples.

5. Automatic computational content from type-theoretic proofs that use UC.
   Without any need to explicitly consider computability theory.

End