Selection functions everywhere

Martín Escardó

Joint work with Paulo Oliva from Queen Mary, London.

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Selection

Paulo and I have published many papers about selection funcions.

This talk is a selection of the material.

Selection functions everywhere

1. Game theory. Optimal strategies

2. Proof theory. Program extraction from classical proofs with choice.

3. Topology. Tychonoff theorem

4. Logic and higher-type computability. Bar recursion.

5. Fixed points. Bekic's Lemma.

Selection functions briefly

X set of things.

Goods in a store; possible moves of a game; proofs of a proposition; point of a space.

R set of values.

Prices; outcomes win, lose, draw; how much money you win; true or false; proofs again.

$X \xrightarrow{p} R$ value judgement.

How you value it; how much it costs you; pay-off of a move; propositional function.

$(X \to R) \xrightarrow{\varepsilon} X$ selects something according to some criterion.

The best, the cheapest, any, something odd.

Example 1

X set of goods.

R set of prices.

$X \xrightarrow{p} R$ table of prices.

$(X \to R) \xrightarrow{\varepsilon} X$ selects a cheapest good in a given table.

$(X \to R) \xrightarrow{\phi} R$ determines the lowest price in a given table.

Fundamental equation:

$$p(\varepsilon(p)) = \phi(p).$$

The price of a cheapest good is the lowest in the table, of course.

$$\phi = \inf \qquad \varepsilon = \operatorname{arginf},$$

$$p(\operatorname{arginf}(p)) = \inf(p).$$

Example 2

X set of individuals.

R set of booleans $\text{false} = 0 < 1 = \text{true}$.

$X \xrightarrow{p} R$ property.

$(X \to R) \xrightarrow{\varepsilon} X$ selects an individual with the highest truth value.

$(X \to R) \xrightarrow{\phi} R$ determines the highest value of a given property.

Fundamental equation:

$$p(\varepsilon(p)) = \phi(p)$$

$$\begin{aligned} \phi &= \sup = \exists \\ \varepsilon &= \operatorname{argsup} = \text{arg-}\exists = \text{Hilbert's choice operator} \end{aligned}$$

$p(\varepsilon(p)) = \exists(p)$ Hilbert's definition of $\exists$ in his ε-calculus

Maximum-Value Theorem

Let X be a compact non-empty topological space.

Any continuous function $p\colon X \to \mathbb{R}$ attains its maximum value.

This means that there is $a \in X$ such that

$$\sup p = p(a).$$

However, the proof is non-constructive when e.g. $X = [0,1]$.

A maximizing argument a cannot be algorithmically calculated from p.

Of course, there is a Minimum-Value Theorem too.

Mean-Value Theorem

Any continuous function $p\colon [0,1] \to \mathbb{R}$ attains its mean value.

There is $a \in [0,1]$ such that

$$\int p = p(a).$$

Again this a cannot be found from p using an algorithm.

Universal-Value Theorem

Let X be a non-empty set and $2 = \{0, 1\}$ be the set of booleans.

> Any $p \colon X \to 2$ attains its universal value.

There is $a \in X$ such that

$$\forall p = p(a).$$

This is again a classical statement.

This is usually formulated as the Drinker Paradox:

> In any inhabited pub there is a person a s.t. if a drinks then everybody drinks.

We've also met the Existential-Value Theorem.

General situation

With ϕ among $\exists, \forall, \sup, \inf, \int, \ldots$,

$$\boxed{\phi(p) = p(a)}$$

for some a depending on p.

In favourable circumstances a can be calculated as

$$a = \varepsilon(p),$$

so that

$$\boxed{\phi(p) = p(\varepsilon(p))}$$

Selection function

Definition.

A selection function for a (logical, arithmetical, . . .) quantifier

$$\phi\colon (X \to R) \to R$$

is a functional

$$\varepsilon\colon (X \to R) \to X$$

such that

$$\phi(p) = p(\varepsilon(p)).$$

Monad morphism

Every $\varepsilon\colon (X \to R) \to X$ is the selection function of some $\phi\colon (X \to R) \to R$.

Namely ϕ defined by

$$\phi(p) = p(\varepsilon(p)).$$

Also written $\phi = \overline{\varepsilon}$.

This construction defines a monad morphism $\theta\colon J \to K$:

$$\begin{array}{ccc} \overbrace{(X \to R) \to X}^{JX} & \xrightarrow{\Theta} & \overbrace{(X \to R) \to R}^{KX} \\ \varepsilon & \longmapsto & \overline{\varepsilon} \end{array}$$

This is a morphism from the selection monad to the quantifier monad.

Oh, I mean to the continuation monad.

Units of the monads

$$\begin{array}{ccc} X & \xrightarrow{\eta} & KX \\ x & \longmapsto & \lambda p.p(x). \end{array}$$

(Universally and existentially) quantifies over the singleton $\{x\} \subseteq X$.

$$\eta(x) = \exists_{\{x\}} = \forall_{\{x\}}.$$

$$\begin{array}{ccc} X & \xrightarrow{\eta} & JX \\ x & \longmapsto & \lambda p.x. \end{array}$$

Produces a selection function for the singleton quantifier.

Functors of the monads

Let $f\colon X \to Y$.

$$\begin{array}{ccc} KX & \xrightarrow{Kf} & KY \\ \phi & \longmapsto & \lambda p.\phi(\lambda x.p(f(x))). \end{array}$$

If ϕ quantifies over a set $S \subseteq X$, then $Kf(\phi)$ quantifies over the set $f(S) \subseteq Y$.

$$\begin{array}{ccc} JX & \xrightarrow{Jf} & JY \\ \varepsilon & \longmapsto & \lambda p.f(\varepsilon(\lambda x.p(f(x)))). \end{array}$$

If ε is a selection function for ϕ, then $Jf(\varepsilon)$ is a selection function for $Kf(\phi)$.

Multiplication of the quantifier monad

Involves the perhaps unfamiliar notion of quantification over quantifiers.

$$\begin{array}{rcl} KKX & \xrightarrow{\mu} & KX \\ \Phi & \longmapsto & \lambda p.\Phi(\lambda\phi.\phi(p)). \end{array}$$

Suppose $A \subseteq KX$ is a set such that each $\phi \in A$ existentially quantifies over a set $B_\phi \subseteq X$, i.e.

$$\phi = \exists_{B_\phi}$$

Then the universal quantifier $\forall_A \in KKX$ of the set $A \subseteq KX$ satisfies

$$\mu(\forall_A)(p) = \forall\phi \in A \exists x \in B_\phi(p(x)).$$

Multiplication of the selection monad

Now we have selection functions that select selection functions.

$$\begin{array}{rcl} JJX & \xrightarrow{\mu} & JX \\ E & \longmapsto & \lambda p.E(\lambda\varepsilon.p(\varepsilon(p)))(p). \end{array}$$

Use the selection function E to find a selection function ε such that $p(\varepsilon(p))$, and apply this resulting selection function to p to find an element of X.

Algebras

$KA \to A$.

$((A \to R) \to R) \to A$.

Double-negation elimination.

$JA \to A$.

$((A \to R) \to A) \to A$.

Peirce's Law.

Get different proof translation of classical logic into intuitionistic logic.

BTW, I think this gives a better explanation of call/cc. (Blackboard.)

Strengths

$$\begin{array}{rcl} X \times KY & \xrightarrow{t} & K(X \times Y) \\ (x, \phi) & \longmapsto & \lambda p.\phi(\lambda y.p(x,y)). \end{array}$$

If ϕ quantifies over $S \subseteq Y$, then the resulting quantifier quantifies over $\{x\} \times S \subseteq X \times Y$.

$$\begin{array}{rcl} X \times JY & \xrightarrow{t} & J(X \times Y) \\ (x, \varepsilon) & \longmapsto & \lambda p.(x, \varepsilon(\lambda y.p(x,y))). \end{array}$$

This produces a selection function for the above quantifier.

We have monoidal monad structures

Because we have strong monads on a cartesian closed category.

$$\begin{array}{rcl} TX \times TY & \stackrel{\otimes}{\longrightarrow} & T(X \times Y) \\ (u, v) & \longmapsto & (T(\lambda x.t(x, v)))(u). \end{array}$$

Warning. This is one way of getting this.

The other way of getting this gives a different $\otimes$.

| The monads are not commutative. | (And this is good!)

The above choice of $\otimes$ is what we need for our purposes.

("left-to-right" as opposed to "right-to-left".)

Monoidal monad structures

$$\begin{array}{rcl} TX \times TY & \xrightarrow{\otimes} & T(X \times Y) \\ (u, v) & \longmapsto & (T(\lambda x.t(x, v)))(u). \end{array}$$

1. Sequential games of length two.

2. Binary Tychonoff Theorem.

3. Bekic's Lemma. When $X = R$ and hence $JX = KX$, an element of $KX = ((X \to X) \to X)$ is a fixed point operator iff it is its own selection function. Then Bekic's Lemma follows.

Examples

$$\begin{array}{ccc} KX \times KY & \stackrel{\otimes}{\longrightarrow} & K(X \times Y) \\ (\exists_A, \exists_B) & \longmapsto & \exists_{A \times B}. \end{array}$$

$$\begin{array}{ccc} KX \times KY & \stackrel{\otimes}{\longrightarrow} & K(X \times Y) \\ (\forall_A, \exists_B) & \longmapsto & \lambda p.\forall x \in A.\exists y \in B.p(x, y). \end{array}$$

What about J?

Well, of course, its product $\otimes$ commutes with that of K.

(Because we have a monad morphism.)

This means:

if

$\varepsilon \in JX$ is a selection function for $\phi \in KX$

$\delta \in JY$ is a selection function for $\gamma \in KY$,

then

$\varepsilon \otimes \delta$ is a selection function for $\phi \otimes \gamma$.

This is good for optimally playing games, as we'll see.

Binary product of quantifiers and selection functions

In every pub there are a man a_0 and a woman a_1 such that if a_0 buys a drink to a_1 then every man buys a drink to some woman.

Binary product of quantifiers and selection functions

In every pub there are a man b and a woman c such that if b buys a drink to c then every man buys a drink to some woman.

If $X =$ set of men and $Y =$ set of women, and if we define $\phi = \forall \otimes \exists$ by

$$\phi(p) = (\forall x \in X \ \exists y \in Y \ p(x, y)),$$

then this amounts to saying that

$$\phi(p) = p(a)$$

for a suitable pair $a = (b, c) \in X \times Y$,

This is calculated as $a = (\varepsilon \otimes \delta)(p)$ where $\overline{\varepsilon} = \forall_X$ and $\overline{\delta} = \exists_Y$.

Infinitely iterated left-to-right monoidal monad structure.

In certain categories of interest

There is a countable monoidal-monad structure

$$\bigotimes\colon \prod_i JX_i \to J\prod_i X_i$$

uniquely determined by the equation

$$\bigotimes_i \varepsilon_i = \varepsilon_o \otimes \bigotimes_i \varepsilon_{i+1}.$$

(Which turns out to be an instance of bar recursion.)

Playing games

Products of selection functions compute optimal plays and strategies.

First example

Alternating, two-person game that finishes after exactly n moves.

1. Eloise plays first, against Abelard. One of them wins (no draw).

2. The i-th move is an element of the set X_i.

3. The game is defined by a predicate $p\colon \prod_{i<n} X_i \to \texttt{Bool}$
that tells whether Eloise wins wins a given play $x = (x_0, \ldots, x_{n-1})$.

4. Eloise has a winning strategy for the game p if and only if

$$\exists x_0 \in X \forall x_1 \in Y \exists x_2 \in X_2 \forall x_3 \in X_3 \cdots p(x_0, \ldots, x_{n-1}).$$

First example

4. Eloise has a winning strategy for the game p if and only if

$$\exists x_0 \in X \forall x_1 \in Y \exists x_2 \in X_2 \forall x_3 \in X_3 \cdots p(x_0, \ldots, x_{n-1}).$$

If we define

$$\phi_i = \begin{cases} \exists_{X_i} & \text{if } i \text{ is even,} \\ \forall_{X_i} & \text{if } i \text{ is odd,} \end{cases}$$

then this condition for Eloise having a winning strategy can be equivalently expressed as

$$\left(\bigotimes_{i<n} \phi_i \right) (p).$$

Calculating the optimal outcome of a game

More generally, the value

$$\left(\bigotimes_{i<n} \phi_i\right)(p)$$

gives the optimal outcome of the game.

This takes place when all players play as best as they can.

In the first example, the optimal outcome is `True` if Eloise has a winning strategy, and `False` if Abelard has a winning strategy.

Calculating an optimal play

Suppose each quantifier ϕ_i has a selection function ε_i (thought of as a policy function for the i-th move).

Theorem. The sequence

$$a = (a_0, \ldots, a_{n-1}) = \left(\bigotimes_{i<n} \varepsilon_i \right) (p)$$

is an optimal play.

This means that for every stage $i < n$ of the game, the move a_i is optimal given that the moves $a_0, \ldots, a_{i-1}$ have been played.

Finding an optimal strategy

Theorem. The function $f_k : \prod_{i<k} X_i \to X_k$ defined by

$$f_k(a) = \left(\left(\bigotimes_{i=k}^{n-1} \varepsilon_i \right) (\lambda x . p(a :: x)) \right)_0$$

is an optimal strategy for playing the game.

This means that given that the sequence of moves $a_0, \ldots, a_{k-1}$ have been played, the move $a_k = f_k(a_0, \ldots, a_{k-1})$ is optimal.

Second example

Choose $R = \{-1, 0, 1\}$ instead, with the convention that

$$\begin{cases} -1 = \text{Abelard wins}, \\ \quad 0 = \text{draw}, \\ \quad 1 = \text{Eloise wins}. \end{cases}$$

The existential and universal quantifiers get replaced by $\sup$ and $\inf$:

$$\phi_i = \begin{cases} \sup_{X_i} & \text{if } i \text{ is even}, \\ \inf_{X_i} & \text{if } i \text{ is odd}. \end{cases}$$

The optimal outcome is still calculated as $\bigotimes_{i<n} \phi_i$, which amounts to

$$\sup_{x_0 \in X!} \inf_{x_1 \in Y} \sup_{x_2 \in X_2} \inf_{x_3 \in X_3} \cdots p(x_0, \ldots, x_{n-1}).$$

Second example

The optimal outcome is

$$\begin{cases} -1 = \text{Abelard has a winning strategy,} \\ \;\;\, 0 = \text{the game is a draw,} \\ \;\;\, 1 = \text{Eloise has a winning strategy.} \end{cases}$$

Can compute optimal outcomes, plays and strategies with the same formulas.

Classical choice

Tychonoff Theorem

Conclusion

Selection functions everywhere.