Boolean compactness

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Smyth's dictionary (1983,1992)

computation		topology
data types	$\approx$	topological spaces
computable functions	$\approx$	continuous maps
semidecidable properties	$\approx$	open sets
decidable properties	$\approx$	clopen sets
decidable equality	$\approx$	discreteness
semidecidable apartness	$\approx$	Hausdorff property
	$\vdots$	

See also:

Abramsky, Domain Theory and the Logic of Observable Properties, 1987.

Vickers, Topology via Logic, 1989.

The Sierpiński space

The Sierpiński space $\mathcal{S}$ has a limit point $\bot$ and an isolated point $\top$.

The open sets of a space X correspond to the continuous maps $X \to \mathcal{S}$, by taking the characteristic function of an open.

Computationally, $\mathcal{S}$ is a type of results of semidecisions.

$\bot$ is a negative non-terminating answer, and $\top$ is a positive terminating answer.

Compactness

Assume the exponential $\mathcal{S}^X$ exists. We'll come back to this later.

X is compact $\iff$ the universal quantifier $\mathcal{S}^X \to \mathcal{S}$ is continuous.

This is a theorem. We make it into a definition.

X is computationally compact $\iff$ the universal quantifier $\mathcal{S}^X \to \mathcal{S}$ is computable.

We can semidecide whether a given semidecidable property holds everywhere.

The two-point discrete space

The two-point discrete space 2 has two isolated points 0 and 1.

The clopen sets of a space X correspond to the continuous maps $X \to 2$, by taking the characteristic function of a clopen.

Computationally, 2 is a type of booleans or results of decisions.

0 and 1 are negative and positive terminating answers.

Boolean compactness

We say that X is boolean compact if the universal quantifier $2^X \to 2$ is continuous.

Under the dictionary:

We say that X is computationally boolean compact if the universal quantifier $2^X \to 2$ is computable.

We can decide whether a given decidable property holds everywhere.

Boolean compactness can hold trivially

The real line $\mathbb{R}$ has only two clopens, the empty set and $\mathbb{R}$ itself.

Topologically, this means it is connected.

Computationally, this means it has no nontrivial decidable property.

Equivalently, all continuous/computable functions $\mathbb{R} \to 2$ are constant.

Then $\mathbb{R}$ is trivially (computationally) boolean compact.

To decide whether $p : \mathbb{R} \to 2$ holds everywhere, it is enough to evaluate e.g. $p(0)$.

For the notion of boolean compactness to be meaningful . . .

. . . we need to restrict to spaces/types with plenty of clopens/decidable properties.

Two possibilities in topology:

1. The space is totally separated. The clopens separate the points.
2. The space is zero dimensional. The opens are unions of clopens.

In general distinct, and zero dimensional implies totally separated.

For compact spaces the two notions agree.

Conjecture (Summer 2025)

A totally separated space is boolean compact $\iff$ it is compact.

Turned out to be wrong.

The answer is more complicated, but still involves total separatedness and zero dimensionality.

Totally separated spaces

A space is totally separated if and only if for any two distinct points, there is a clopen containing one but not the other.

Computationally, this is a boolean Leibniz principle.

If two elements of a type satisfy the same decidable properties, then they are equal.

This fails e.g. for a type of real numbers, but holds e.g. for all types of Gödel's system T. Both topologically and computationally.

Zero dimensional spaces

Topologically. Every open is a union of clopens.

Computationally. It doesn't have a name, but it is a theorem for many types.

E.g. every semidecidable subset of $\mathbb{N}$ is the union of a computably enumerable collection of decidable subsets of $\mathbb{N}$.

True also for e.g. the Cantor type $2^{\mathbb{N}}$ and the Baire type $\mathbb{N}^{\mathbb{N}}$.

But there are counterexamples.

Matthias Schröder. The type $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not zero dimensional, although it is totally separated.

The dictionary so far

data types	$\approx$	topological spaces
computable functions	$\approx$	continuous maps
semidecidable properties	$\approx$	open sets
decidable properties	$\approx$	clopen sets
decidable equality	$\approx$	discreteness
semidecidable apartness	$\approx$	Hausdorff property
computational compactness	$\approx$	compactness
computational boolean compactness	$\approx$	boolean compactness
no nontrivial decidable property	$\approx$	connectedness
boolean Leibniz principle	$\approx$	total separatedness
nameless, but it occurs	$\approx$	zero dimensionality

Problem

The exponential 2^X, used to define boolean compactness, doesn't exist in general.

The category of topological spaces is not cartesian closed.

But a number of full subcategories are.

Categories that show up in higher-type computation

- **Seq**: the sequential spaces.
- **QCB**: the quotients of countably based spaces, a full subcategory of **Seq**.
- **CGH**: the compactly generated Hausdorff spaces.

All three are cartesian closed.

QCB was introduced by Alex Simpson and collaborators as a convenient category for the semantics of programming languages.

Kleene–Kreisel spaces

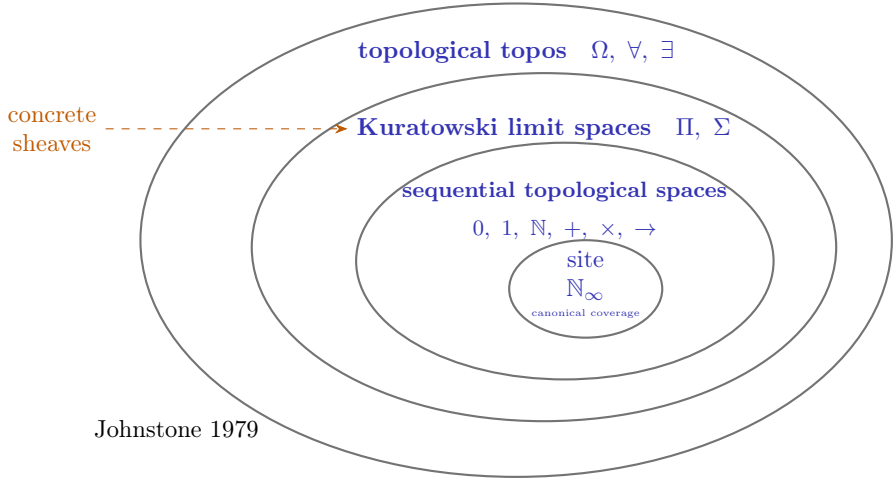
The Kleene–Kreisel spaces are obtained from the discrete space $\mathbb{N}$ by repeatedly forming exponentials.

Kleene and Kreisel defined them independently in 1959, by different methods, for the purpose of computation at higher types.

Hyland: We get the same result if we build them in **Seq**, **QCB** or **CGH**.

They form a fully abstract model of system T.

A topological model of dependent type theory



Johnstone 1979

Wanted

Characterizations of boolean compactness under reasonable assumptions.

(Boolean compactness is the continuity of the universal quantifier $2^X \to 2$.)

They should apply to the above models of computation, among others.

The exponential 2^X is calculated in different ways in the different categories, and does give different spaces.

So, in principle, the notion of boolean compactness depends on the cartesian closed category under consideration.

Partition compactness

We now introduce a new notion that avoids the exponential 2^X and applies to any topological space.

Definition

A space X is partition compact if every partition of X into open sets is finite.

Lemma

For every topological space X the following are equivalent.

(a) *X is partition compact.*

(b) *Every continuous map from X to a discrete space has finite image.*

(c) *Every continuous map $X \to \mathbb{N}$ has finite image.*

(d) *Every countable cover of X by clopen sets has a finite subcover.*

A characterization of boolean compactness

Let $\mathcal{C}$ be any of the three cartesian closed categories of spaces mentioned above.

Theorem

For a space X in $\mathcal{C}$ the following are equivalent.

(a) *X is boolean compact.*

(b) *The exponential 2^X is discrete.*

(c) *X is partition compact.*

The first two conditions mention the category, but the third doesn't.

Consequences

Corollary

Let X be a space in $\mathcal{C}$.

(a) *If X is sequentially compact, countably compact or compact, then it is boolean compact. So are connected spaces, such as $\mathbb{R}$.*

(b) *Boolean compactness depends only on the clopens of X, and hence not on which of the three categories is used.*

(c) *If X is Lindelöf, then X is boolean compact if and only if $\mathcal{Z}X$ is compact.*

Here $\mathcal{Z}X$ is X with the topology generated by its clopen sets, the reflection of X into the zero dimensional spaces.

Boolean compactness is weaker than compactness

Condition (d) of the lemma concerns countable covers, and covers by clopen sets.

The ordinal ω_1, with the order topology, is first countable, Hausdorff and zero dimensional, and it is boolean compact and not compact. It is not Lindelöf.

A refinement of the Cantor set K. Let $s_n \to s$ be distinct points of K, let $S = \{s_1, s_2, \ldots\}$, and let X be K with the topology generated by the open sets of K together with $K \setminus S$.

Then X is Hausdorff, countably based, Lindelöf and totally separated, and it has the same clopen sets as K, so that $\mathcal{Z}X = K$. It is boolean compact and not countably compact, and it is not zero dimensional.

Characterization via clopen projections

Kuratowski and Mrówka: A space X is compact $\iff$ the projection $X \times Y \to Y$ maps closed sets to closed sets for every space Y.

Theorem

For a space X in $\mathcal{C}$ the following are equivalent.

(a) *X is boolean compact.*

(b) *For every Y and every clopen $C \subseteq X \times Y$, the set $\{y : X \times \{y\} \subseteq C\}$ is clopen.*

(c) *Every projection $X \times Y \to Y$ maps clopen sets to clopen sets.*

(d) *The projection $X \times 2^X \to 2^X$ maps clopen sets to clopen sets.*

(e) *The projection $X \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ maps clopen sets to clopen sets.*

When does boolean compactness agree with compactness?

For a Lindelöf totally separated X, boolean compactness says that $\mathcal{Z}X$ is compact, so the question is when X inherits compactness from $\mathcal{Z}X$.

Definition

X is sequentially zero dimensional if X and $\mathcal{Z}X$ have the same convergent sequences with the same limits.

Theorem

Let X be a Lindelöf, sequentially zero dimensional sequential space such that

(K) *if $\mathcal{Z}X$ is compact, then $\mathcal{Z}X$ is sequentially compact.*

Then X is compact if and only if it is sequentially compact, if and only if it is boolean compact.

QCB spaces

QCB spaces are hereditarily Lindelöf and satisfy (K), so the previous theorem applies to them with no further hypotheses.

Theorem

For a totally separated QCB space X the following are equivalent.

(a) *X is compact.*

(b) *X is boolean compact and sequentially zero dimensional.*

So among totally separated boolean compact QCB spaces, the failure of sequential zero dimensionality is the only obstruction to compactness.

The refined Cantor space shows that it does occur.

Compactly generated Hausdorff spaces

Definition

A totally separated space X in **CGH** is compactly zero dimensional if X and $\mathcal{Z}X$ have the same compact subsets.

Theorem

Let X be a Lindelöf totally separated space in **CGH**. *Then X is compact if and only if it is boolean compact and compactly zero dimensional.*

Again ω_1 and the refined Cantor space show that neither hypothesis can be omitted.

Function spaces and Kleene–Kreisel spaces

Proposition

Let X and D be spaces in **Seq** *or* **QCB** *with D discrete. Then D^X is totally separated and sequentially zero dimensional.*

Both properties pass to finite products and to closed subspaces.

So the Kleene–Kreisel spaces are totally separated and sequentially zero dimensional, the space $\mathbb{N}$ trivially and the others because every Kleene–Kreisel space other than $\mathbb{N}$ is an exponential $\mathbb{N}^Y$.

They need not be zero dimensional: by a theorem of Schröder, the space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not zero dimensional.

So the conditions above are genuinely weaker than zero dimensionality.

Closed subspaces of Kleene–Kreisel spaces

Corollary

For a closed subspace F of a Kleene–Kreisel space the following are equivalent.

(a) *F is compact.*

(b) *F is countably compact.*

(c) *F is sequentially compact.*

(d) *F is boolean compact, in* **Seq**, **QCB** *or* **CGH**.

(e) *Every countable cover of F by clopen sets has a finite subcover.*

(f) *Every partition of F into clopen sets is finite.*

(g) *The space $\mathcal{Z}F$ is compact.*

If they hold, then F is zero dimensional and metrizable, and, if F is nonempty, it is a retract of the Cantor space $2^{\mathbb{N}}$.

I used this in my LMCS 2008 paper "Exhaustible sets in higher-type computation", to characterize the exhaustively searchable sets of the domain model of PCF.

Summary

- In **Seq**, **QCB** and **CGH**, boolean compactness, discreteness of 2^X and partition compactness coincide. So the notion depends only on the clopen sets of the space, and not on the category.
- It is characterized by projections that map clopen sets to clopen sets, in analogy with the Kuratowski–Mrówka theorem.
- For Lindelöf totally separated spaces it agrees with compactness under conditions weaker than zero dimensionality, which hold for the closed subspaces of the Kleene–Kreisel spaces.

Further work: compact totally separated types

- Constructively, in univalent type theory, a type X is compact if for every $p\colon X \to 2$ we can either find an x with $p(x) = 0$ or show that there is none. This is boolean compactness constructively, with the topological topos as model.
- The notion is informative only for the totally separated types, those whose maps into 2 separate the points.
- The natural numbers are not compact, but the conatural numbers $\mathbb{N}_\infty$ are, and much else is built from them.
- All the compact types so constructed carry well-orders. Two systems of ordinal notations, the Brouwer codes and an inductive-recursive universe generalizing them, read a single notation both as a discrete ordinal and as a compact one, the first embedding into the second with empty complement.
- The discrete ordinals are trichotomous, the compact ones have the least element property for complemented subsets, and constructively these cannot both hold.

Formalized in Agda, in `TypeTopology`. `arXiv:2609.10447`