Topology in constructive mathematics & computation

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What is topology?

- A mathematician is a machine for turning coffee into theorems.
  Alfred Renyi
- A topologist is a machine for turning coffee mugs into doughnuts.
- Topology studies properties of spaces that are invariant under continuous deformation.

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- A mathematician is a machine for turning coffee into theorems.
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- Topology studies properties of spaces that are invariant under continuous deformation.
- Open problem: what kind of machine is a theoretical computer scientist?

What is topology?

- Topology is geometry.
- Topology is about approximation.

  (epsilons & deltas, limits, open sets, continuity, ...)

- Topology generalizes analysis.

Topological spaces and continuous functions

These definitions distill centuries of mathematical work.

- A topological space is a set X together with a collection of subsets of X, called open, such that
  - an intersection of finitely many open sets is open,
  - a union of arbitrarily many open sets is open.
- A function $f : X \to Y$ of topological spaces is continuous iff for every open set $V \subseteq Y$, the set $f^{-1}(V) \subseteq X$ is open.

We'll discuss these definitions from a computational perspective later.

$\boxed{\text{computers are finite, but we want to compute with infinite objects}}$

- $\sqrt{2}$, π
- $\sin, \cos : \mathbb{R} \to \mathbb{R}$
- $\int_0^1 : (\mathbb{R} \xrightarrow{\text{cts}} \mathbb{R}) \to \mathbb{R}$

John Longley & Dag Normann.
Higher-Order Computability.
Springer 2015.

- Functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, $((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N}$, etc.
  in Gödel's system T or Scott-Plotkin PCF.
- Lazy lists in Haskell.

computers are finite, but we want to compute with infinite objects

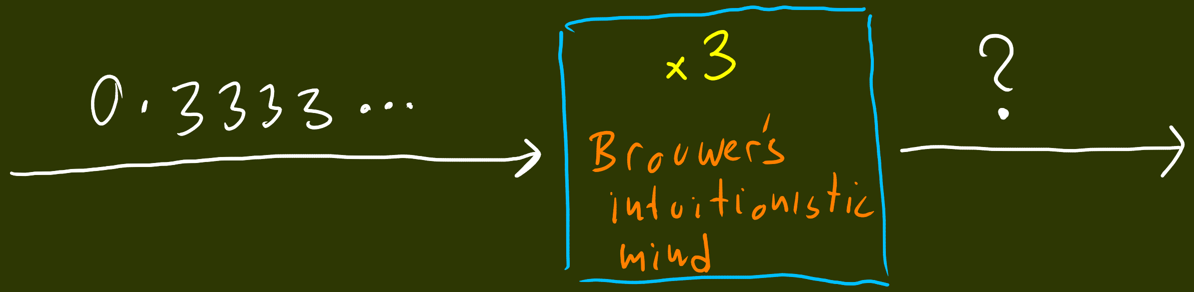
- Topology provides a bridge.
- Slogan: A function is continuous if finite amounts of output depend only on finite amounts of input.
- Topology makes this slogan rigorous.

Does every real number have a decimal expansion?

Brouwer 1921

Brower is regarded as a founder of modern topology.

- Decimal expansions are not suitable for computation.
  - All algebraic numbers have (computable) decimal expansions.
  - So does π, but not every real has a decimal expansion.
  - And we cannot even multiply by 3.



A non-existent Brouwer Machine.

(Modern) Topological view

- $A := \{0,1,2,3,4,5,6,7,8,9\}$
- Endow A with the discrete topology (every subset is open), and A^ω with the product topology (which has the usual universal property).
- Fact. A function $f : A^\omega \to A^\omega$ is (topologically) continuous iff

$$\forall \alpha : A^\omega \; \forall k : \mathbb{N} \; \exists m : \mathbb{N} \; \forall \beta : A^\omega . \; \alpha =_m \beta \to f(\alpha) =_k f(\beta)$$

$\forall i < m . \; \alpha_i = \beta_i$

Finite amounts of output depend only on finite amounts of input

Topological view

$$\begin{tikzcd} A^\omega \arrow[r, "\bar f"] \arrow[d, "q"'] & A^\omega \arrow[d, "q"] \\ {[0,1]} \arrow[r, "f"'] & {[0,1]} \end{tikzcd}$$

$$q(\alpha) := \sum_{i>0} \alpha_i 10^{-i-1} = \frac{\alpha_0}{10} + \frac{\alpha_1}{100} + \frac{\alpha_2}{1000} + \cdots$$

Fact (Brouwer)

For $f = (x \mapsto 3x)$, there is no continuous $\bar f$ making the diagram commute.

Fact. The function q is a topological quotient map:

- It is a continuous surjection.
- For any $V \subseteq [0,1]$, if $q^{-1}(V)$ is open then so is V.

Corollary. In a commutative diagram as above, if $\bar f$ is continuous then so is f.

Fixing the problem

There are many equivalent ways.

$$3 := \{-1, 0, 1\} \qquad q(\alpha) := \sum_{i \geq 0} \alpha_i 2^{-i-1}$$

$$\begin{tikzcd}
3^\omega \arrow[r, "\bar{f}"] \arrow[d, "q"'] & 3^\omega \arrow[d, "q"] \\
{[-1,1]} \arrow[r, "f"'] & {[-1,1]}
\end{tikzcd}$$

- If the diagram commutes,
  $\bar{f}$ continuous $\Rightarrow$ f continuous.
- If f is continuous, then there is a continuous $\bar{f}$ making the diagram commute.

This is an admissible quotient representation in the sense Weihrauch's school of computability. C.f. Matthias Schröder's work.

Any two admissible quotient representations are computably equivalent.

Brouwer 1927

All functions are continuous

Brouwer rejected the principle of excluded middle and accepted other principles that imply the above (and hence contradict excluded middle)

1955

Myhill & Shepherdson Thm

Now we leave intuitionism and work classically with computability theory.

A crude version is that a function

$$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$$

mapping sequences to sequences is computable iff it is effectively continuous.

(Make the condition "finite amounts of output depend only on finite amounts of input" effective.)

KLST theorem

Kreisel 1959
Lacombe
Shoenfield
Tseitin

This time the crude formulation is that for a wide variety of sets studied in computability theory

computable function = effectively continuous function.

Kleene-Kreisel continuous functionals 1959

- They wanted to make sense of computation with higher types.

  $\mathbb{N}$ $\quad \mathbb{N} \to \mathbb{N}$ $\quad (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ $\quad ((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N}$ $\quad \ldots$

- They came up with what people call "the continuous functionals".
- Martin Hyland 1970s characterized them as follows:
  - Work in the cartesian closed category of compactly generated spaces (which were originally introduced for the purposes of homotopy theory)
  - Give $\mathbb{N}$ the discrete topology. Then just keep taking exponentials.

Dag Normann. Recursion on the countable functionals. Springer LNM, 1980

Dana Scott

1969 LCF (Logic of Computable Functionals)
1971 Continuous lattices

- Considers partial continuous functionals of higher type.
- Uses domains with a partial information order.
- Domains become topological spaces under the Scott topology.
- Simply typed lambda calculus with base type for natural numbers, primitive recursion and fixed point recursion (called PCF by Plotkin).
- Types interpreted as domains.
- Can define total element of such domains by induction on types.
- Ershov 1970s proved total elements ≃ Kleene-Kreisel functionals.

Smyth

1983 Powerdomains and predicate transformers: A topological view.
1992 Topology (Handbook of Logic in Computer Science)

Smyth's dictionary

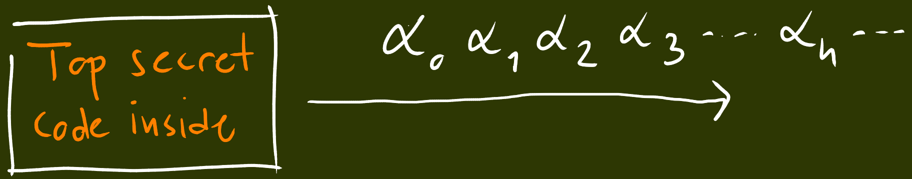
Computation		topology
data type	~	topological space
Piece of data	~	point
Semidecidable property	~	open set
computable function	~	continuous function
?	~	Compact set

observable property (Abramsky)
Affirmable property (Vickers)

"Harder to motivate ... Smyth"

semidecidable or Observable properties of A^ω

- Consider a machine computing for ever an element of A^ω, digit by digit:



- We can't e.g. observe that it is computing the decimal expansion of π, as this would take an infinite amount of time.

- But we can e.g. observe that it is computing something $\neq \pi$, if this is indeed the case (we just compute π ourselves and compare until $\neq$).

Observable properties of A^ω

- $U \subseteq A^\omega$ is observable iff for every $\alpha \in U$ there is a finite prefix $s \in A^*$ s.t. every infinite word $\beta \in A^\omega$ also having s as a prefix belongs to U.

- Fact. A property is observable in this sense iff it is open in the product topology of A^ω with A discrete.

Computational motivation for the axioms for open sets

- A <u>finite</u> conjunction of observable properties is observable.

  Just observe each of them.

- An <u>arbitrary</u> disjunction of observable properties is observable.

  Just observe one of them.

Compact spaces

- Perhaps the most important concept in topology.
- Intuitively, a set is compact iff it behaves, for topological purposes, as if it were finite.

- A space X is compact if whenever $X = \bigcup_{i\in I} U_i$ with $U_i \subseteq X$ open, there is $F \subseteq I$ finite s.t. already $X = \bigcup_{i\in F} U_i$.

Every open cover of the space has a finite subcover.

(Counter) examples

- $[0,1]$ and A^ω are compact.

  A subspace of a discrete space is compact iff it is finite.

- $\mathbb{N}^\omega$ ($\mathbb{N}$ with discrete topology and $\mathbb{N}^\omega$ with the product topology) is not compact. (Baire space.)

- $\mathbb{R}$ is not compact.

Asymmetric topology

The Sierpinski space

space of results of observations.

- $S = \{\top, \bot\}$

  $\top$: observation succeeds
  terminates

  $\bot$: observation is inconclusive
  doesn't terminate

  The open sets are $\emptyset$, $\{\top\}$, $\{\bot, \top\}$.

- A set $U \subseteq X$ is open iff its characteristic function $\chi_U : X \longrightarrow S$ is continuous.

Compactness via Sierpinski

M.E. Barbados notes 2004
Also arXiv 2001.06050

We work with compactly generated spaces so that S^X exists.

Theorem. X is compact iff the characteristic function $A_X : S^X \to S$ of its universal quantifier is continuous.

apply dictionary

$$A_X(p) = \top \iff \forall x:X.\ p\,x = \top$$

Definition. X is semisearchable iff A_X is computable.

Examples. A^ω and 3^ω are semisearchable.

Extended Smyth's dictionary

Computation		Topology
semisearchable data type	$\sim$	Compact space
semidecidable $\neq$	$\sim$	Hausdorff space
semidecidable $=$	$\sim$	Discrete space

Searchable sets

Preliminaries.

- $2 := \{0,1\}$ with discrete topology. Symmetric topology this time.
- Continuous functions $f : X \to 2$ correspond to clopen sets (sets that are open and whose complements are also open)
- Lemma. The Kleene-Kreisel spaces are totally separated.
- This means that for every $x_0 \neq x_1$ in X there is a continuous $f : X \to 2$ with $f(x_0) = 0$ and $f(x_1) = 1$.
- The clopens separate the points.
- There are plenty of clopens (decidable properties).

Searchable sets

This is possible only because there are plently of clopens in Kleene-Kreisel spaces.

- **Lemma** A subspace X of a Kleene-Kreisel space is compact iff the function $A_X : 2^X \to 2$ is continuous.

- **Definition.** X is searchable iff the function $A_X : 2^X \to 2$ is computable.

(Replace S by 2.)

Compact sets

- Finite spaces are compact.
- Products of compact spaces are compact (Tychonoff).

  So the Cantor space 2^ω is compact.
- Continuous images of compact spaces are compact.
- Every non-empty countably based compact Hausdorff space is a continuous image of the Cantor space.

- In particular, non-empty compact subspaces of Kleene-Kreisel spaces are continuous images of the Cantor space.

$\boxed{\text{~~Compact~~ searchable sets}}$

compact $\longmapsto$ searchable
continuous $\longmapsto$ computable

- Finite spaces are ~~compact~~. searchable
- countable Products of non-empty compact spaces are ~~compact~~ searchable (Tychonoff).

  So the Cantor space 2^ω is ~~compact~~. searchable
- ~~Continuous~~ computable images of ~~compact~~ searchable spaces are ~~compact~~. searchable
- Every non-empty countably based compact Hausdorff space is a continuous image of the Cantor space.

In particular, non-empty ~~compact~~ searchable subspaces of Kleene-Kreisel spaces are ~~continuous~~ computable images of the Cantor space.

M.E. LICS'2007
LMCS'2008

Searchable sets

Tychonoff is implemented by a form of <u>bar recursion</u> (product of selection functions)

- Finite spaces are searchable
- Countable products of non-empty searchable Kleene-Kreisel spaces are searchable.

  So the Cantor space 2^ω is searchable.
- Computable images of searchable spaces are searchable.
- Non-empty searchable subspaces of Kleene-Kreisel spaces are computable images of the Cantor space.

These facts are witnessed by PCF programs, some of which I also write in Haskell.

Searchable sets

We are still working with Kleene-Kreisel spaces.

Theorem. Every non-empty searchable set X has a selection function $\varepsilon : 2^X \to X$.

Moreover, there is a computable function

$$(2^X \to 2) \longrightarrow (2^X \to X)$$

that transforms quantification functionals into selection functions.

(This relies on the Kleene-Kreisel density theorem.)

Searchability in Gödel's System T

M.E. JSL' 2011.

The Cantor space itself is not system T searchable

- Simply typed λ-calculus with base type for natural numbers and higher type primitive recursion
- Theorem. Every subset of the Cantor space 2^ω of Cantor-Bendixon rank $< \varepsilon_0$ is system T searchable.
- Conjecture. This is the best possible.
- Proved by Dag Normann (J. Computability 2016).

  If a subset of 2^ω is T-searchable then its rank is $< \varepsilon_0$.
- Definition of CB-rank. Take the subspace of limit points. Keep repeating this. At limit ordinals, take intersections. We take the ordinal at which this process stops.

CB rank 1.

Simplest example

- $\mathbb{N}$ is of course not searchable.
- $\mathbb{N}_\infty := \{ \alpha : 2^\omega \mid \forall i.\ \alpha_i \geq \alpha_{i+1} \}$ (decreasing binary sequences)

  is T-searchable

One-point compactification of the discrete natural numbers.

$\infty = 1^\omega$

$\mathbb{N} \hookrightarrow \mathbb{N}_\infty$

$n \longmapsto 1^n 0^\omega$

Two applications.

- Normann & Tait. On the computability of the Fan Functional (Springer Nature 2017)
  (they use this to fill a gap in a widely circulated 1960's manuscript by Tait.)

- Brown & Pradic. Cantor-Bernstein implies excluded middle.
  (arxiv 2019, HAL open science 2023)

Searchable types in HoTT/UF

- Homotopy type theory / univalent foundations

  = a Martin-Löf type theory + univalence axiom + higher-inductive types.

- Many more searchable types with further interesting properties and much higher ranks.

  → Tom de Jong (domains)
  → Todd Ambridge (signed digits)
  → Ayberk Tosun (locales)

- Constructive development, written in Agda (github.com/martinescardo/TypeTopology).

- Hence we can run it as a program using Agda's cubical implementation of HoTT/UF.

- I gave a CSL'2022 talk just about this page.

  This talk can be considered as a prequel to that.

Thanks!